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Fuzzy Measure: A Different View

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Abstract. In this paper the existing fuzzy measure "m" is defined in a different manner so as to ensure and enhance the effectiveness of fuzzy measure. Some examples relating to various contexts where "m" describes a fuzzy measure and where "m" fails to be a fuzzy measure are stated.

Keywords: Fuzzy measure, fuzzy relation

AMS Mathematics Subject Classification (2010): 05C12, 03E72, 05C72

1. Introduction

Fuzzy sets are defined as a model, fit for vague concepts and subjective judgment. Such a model is used where the deterministic or probabilistic models fail to describe the system.

Fuzzy set theory is used in various domains as decision making under uncertainty, information retrieval, machine fault diagnosis in large scale industries and image processing etc.

Various definitions of fuzzy measure exist. For the definition the reader is directed to refer [1, 4, 5, 6].

We use fuzzy measure theory for decision making. Our new fuzzy measure increases the efficiency so as to make decision easy. This measure is monotonic.

2. A new definition of fuzzy measure

A relation R:A→B is a subset of A ×B where A and B are any two sets. Let A, B be any two sets. A relation $\rho : A \to B$ is said to be a fuzzy relation if

1) D(ρ) = A where $D(\rho)$ is the domain of ρ and
2) there exists a membership function $\mu : \rho \to [0,1]$

Let us consider an example

Let $A = \{a_1, a_2, a_3\}$ $B = \{b_1, b_2\}$

Let $\rho = \left\{\frac{(a_1,b_1)}{1}, \frac{(a_2,b_1)}{1}, \frac{(a_3,b_1)}{0.3}, \frac{(a_3,b_2)}{0.7}\right\}$

Then D(ρ) = A

The membership function $\mu : \rho \to [0,1]$ is

$\mu[(a_1, b_1)] = \mu[(a_2, b_1)] = 1$

$\mu[(a_3, b_1)] = 0.3$, $\mu[(a_3, b_2)] = 0.7$

Remarks:

1. For the sake of simplicity $\mu_\rho[(a,b)]$ is simply denoted by $\mu_\rho(a,b)$.
2. If (a,b) ∈ ρ we write it as $\rho(a) = b$.
3. By $\mu_\rho(a,b) = x$ we mean that x is the membership value (in [0,1]) of the case that $\rho(a) = b$. That is $\mu[\rho(a) = b] = x$ is simply denoted by $\mu_\rho(a,b) = x$.

The new definition of fuzzy measure

Suppose X is any set, $\mathcal{A}$ be a class of sub sets of X and $(X,\mathcal{A})$ be a measurable space. A fuzzy relation $m : \mathcal{A} \to R$ is said to be a fuzzy measure if the following conditions are satisfied

1) $\mu_m(\phi, 0) = 1$
2) (i) $A \subseteq B \Rightarrow \sup\limits_{m(A)=x} x \le \sup\limits_{m(B)=y} y$

   (ii) $A \subseteq B$; $A \ne \phi \Rightarrow \mu_m(A, \operatorname{Sup} x) \le \mu_m(B, \operatorname{Sup} y)$

Making use of this definition, various instances where 'm' forms a measure are given below.

Example 2.1. Let X={1,2,3}and $\mathcal{A} = P(X) = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\phi\}$
$= \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ (say)

Define a measure $m : A \to R$ as $m(A_i) = |A_i|$

$m(A_1) = m(A_2) = m(A_3) = 1$; $m(A_4) = m(A_5) = m(A_6) = 2$;
$m(A_7) = 3$, $m(A_8) = 0$

Let the membership function be defined as

$$\mu_m(A_i, x) = \begin{cases} \dfrac{x}{x+1} & when\ x \ne 0 \\ 1 & when\ x = 0 \end{cases}$$

Then the corresponding membership values are

$\mu_m(A_1, 1) = \mu_m(A_2, 1) = \mu_m(A_3, 1) = \dfrac{1}{2}$; $\mu_m(A_4, 2) = \mu_m(A_6, 2) = \mu_m(A_5, 2) = \dfrac{2}{3}$

$\mu_m(A_7, 3) = \dfrac{3}{4}$; $\mu_m(A_8, 0) = 1$

Then the following cases arise

$A_1 \subseteq A_4$	$m(A_1) = 1 < m(A_4) = 2$	and	$\mu_m(A_1, 1) = \frac{1}{2} < \mu_m(A_4, 2) = \frac{2}{3}$
$A_1 \subseteq A_5$	$m(A_1) = 1 < m(A_5) = 2$	and	$\mu_m(A_1, 1) = \frac{1}{2} \le \mu_m(A_5, 2) = \frac{2}{3}$
$A_1 \subseteq A_7$	$m(A_1) = 1 < m(A_7) = 3$	and	$\mu_m(A_1, 1) = \frac{1}{2} \le \mu_m(A_7, 3) = \frac{3}{4}$
$A_2 \subset A_4$	$m(A_2) = 1 < m(A_4) = 2$	and	$\mu_m(A_2, 1) = \frac{1}{2} < \mu_m(A_4, 2) = \frac{2}{3}$
$A_2 \subset A_6$	$m(A_2) = 1 < m(A_6) = 2$	and	$\mu_m(A_2, 1) = \frac{1}{2} < \mu_m(A_6, 2) = \frac{2}{3}$
$A_2 \subset A_7$	$m(A_2) = 1 < m(A_7) = 3$	and	$\mu_m(A_2, 1) = \frac{1}{2} < \mu_m(A_7, 3) = \frac{3}{4}$
$A_3 \subset A_5$	$m(A_3) = 1 < m(A_5) = 2$	and	$\mu_m(A_3, 1) = \frac{1}{2} < \mu_m(A_5, 2) = \frac{2}{3}$
$A_3 \subset A_6$	$m(A_3) = 1 < m(A_6) = 2$	and	$\mu_m(A_3, 1) = \frac{1}{2} < \mu_m(A_6, 2) = \frac{2}{3}$

$A_3 \subset A_7$ $m(A_3) = 1 < m(A_7) = 3$ and $\mu_m(A_3, 1) = \frac{1}{2} < \mu_m(A_7, 3) = \frac{3}{4}$

$A_4 \subset A_7$ $m(A_4)=2<m(A_7)=3$ and $\mu_m(A_4, 2) = \frac{2}{3} < \mu_m(A_7, 3) = \frac{3}{4}$

$A_5 \subset A_7$ $m(A_5) = 2<m(A_7)=3$ and $\mu_m(A_5, 2) = \frac{2}{3} < \mu_m(A_7, 3) = \frac{3}{4}$

$A_6 \subset A_7$ $m(A_6)=2<m(A_7)=3$ and $\mu_m(A_6, 2) = \frac{2}{3} < \mu_m(A_7, 3) = \frac{3}{4}$

and $m(A_8) = 0$ but $\mu_m(A_8, 0) = 1$.

In all the above cases "m" satisfies the conditions laid down in the definition 2.1 "m " is a fuzzy measure.

Example 2.2.(Definition of atom) If $\mathcal{A}$ is any non empty class of subsets of sets of X and for any point $x \in X$, the set $\cap \{E | x \in E \in \mathcal{A}\}$ is called **atom** of $\mathcal{A}$ ` at x and is denoted by $A(x|\mathcal{A})$. $X = \{x_1, x_2, x_3\}$

$P(X)=\{\{x_1,x_2,x_3\},\{x_1,x_2\},\{x_2,x_3\},\{x_1,x_3\},\{x_1\},\{x_2\},\{x_3\},\phi\}$

$=\{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$ (say)

$A(x_1/P(X))$ denotes the atom of $P(X)$ at x_1

$A(x_1/P(X)) = S_1 \cap S_2 \cap S_4 \cap S_5 = S_5$; $A(x_2/P(X)) = S_2 \cap S_3 \cap S_4 \cap S_6 = S_6$

$A(x_3/P(X)) = S_7$

Let the measure be defined by $m : P(X) \to R$ *where*

$$m(S_i) = \sum_{x_i \in S_i} |A(x_i/P(X))|$$

$m(S_1) = 3$; $m(S_2) = m(S_3) = m(S_4) = 2$; $m(S_5) = m(S_6) = m(S_7) = 1$;$m(S_8) = 0$.

Let$\mu : m \to [0,1]$be defined as

$$\mu_m(S_i, x) = \begin{cases} \frac{1-x}{1+x^2} & when\ x \neq 0 \\ 1 & when\ x = 0 \end{cases}$$

$\mu_m(S_1, 3) = 0.7$, $\mu_m(S_2, 2) = \mu_m(S_3, 2) = \mu_m(S_4, 2) = 0.6$

$\mu_m(S_5, 1) = \mu_m(S_6, 1) = \mu_m(S_7, 1) = 0.5$; $\mu_m(S_8, 0) = 1$.

Clearly 'm' satisfies the conditions of a fuzzy measure.

Example 2. 3. Let X={a,b,c}

$A = \{\{a,b\},\{b\},\{a,b,c\},\phi\}$. Here A is a subclass of P(X).

$= \{A_1, A_2, A_3, A_4\}$ (say)

$m : A \to R$ is defined as $m(A_i) = \frac{|A_i|}{|X|}$

Then $m(A_1) = \frac{2}{3}$; $(A_2) = \frac{1}{3}$; $m(A_3) = 1$; $m(A_4) = 0$

$\mu : m \to [0,1]$ be defined as

$\mu_m(A_i, x) = 1 - \left(1/(1+x)^2\right)$;$\mu_m\left(A_1, \frac{2}{3}\right) = \frac{16}{25} = 0.64$;$\mu_m\left(A_2, \frac{1}{3}\right) = \frac{7}{16} = 0.437$,

$\mu_m(A_3, 1) = 0.75$.

Then by the definition m exhibits all the properties of fuzzy measure and hence it is a fuzzy measure.

Example 2.4. Let A be a σ-algebra given by
X={a,b,c} ; $A=\{\{a\},\{b,c\},\{a,b,c\},\phi\}$
$=\{A_1,A_2,A_3,A_4\}$ (say)
Define measure $m:A\to R$ as $m(A_i)=|A_i|$
Then $m(A_1)=1$; $m(A_2)=2$; $m(A_3)=3$; $m(A_4)=0$
$\mu: m\to[0,1]$ be defined as
$\mu_m(A_i,x)=(x^2-x+1)/1+x^2$; $\mu(A_1,1)=\frac{1}{2}$, $\mu(A_2,2)=\frac{3}{5}$, $\mu(A_3,3)=\frac{7}{10}$
Then m is a fuzzy measure.

Example 2.5. Let X={1,2,3,4,5}, $A=\{\{1,2\},\{3,4\},\{5\}\}$
The σ-algebra generated by A is denoted by $\sigma(A)$ and is given by
$\sigma(A)=\{\{1,2\},\{3,4,5\},\{3,4\},\{1,2,5\},\{5\},\{1,2,3,4\},\{1,2,3,4,5\},\phi\}$
$=\{A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8\}$ (say)
Let $m:\sigma(A)\to R$ be defined as $m(A_i)=|A_i|$
Then $m(A_1)=2$; $m(A_2)=3$; $m(A_3)=2$; $m(A_4)=3$; $m(A_5)=1$;
$m(A_6)=4$ and $m(A_7)=5$ and $m(A_8)=0$
The membership function $\mu: m\to[0,1]$ is defined as

$$\mu_m(A_i,x)=1-\frac{x}{1+x^2}$$

$\mu_m(A_1,2)=3/5$; $\mu_m(A_2,3)=\mu_m(A_4,3)=7/10$; $\mu_m(A_3,2)=\frac{3}{5}$; $\mu_m(A_5,1)=1/2$; $\mu_m(A_6,4)=13/17$;
$\mu_m(A_7,5)=21/26$; $\mu_m(A_8,0)=1$
Then m is a fuzzy measure.

Example 2.6. Let X ={1,2,3}
P(X) is ordered in descending order of cardinalities.
$P(X)=\{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\phi\}$
$=\{A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8\}$ (say)
Let $m: P(X)\to R$ be defined as $(A_i)=\frac{|A_i|}{i}$. Then
$m(A_1)=3$; $m(A_2)=1$; $m(A_3)=\frac{2}{3}$; $m(A_4)=\frac{2}{4}$
$m(A_5)=\frac{1}{5}$; $m(A_6)=\frac{1}{6}$; $m(A_7)=\frac{1}{7}$; $m(A_8)=0$
$\mu: m\to[0,1]$ is defined as $\mu_m(A_i,x)=\begin{cases}\frac{x^2}{1+x^2} & A_i\neq\phi\\ 1 & A_i=\phi\end{cases}$
$\mu_m(A_1,3)=\frac{9}{10}$;$\mu_m(A_2,1)=\frac{1}{2}$;$\mu_m\left(A_3,\frac{2}{3}\right)=\frac{4}{13}$;$\mu_m\left(A_4,\frac{2}{4}\right)=\frac{1}{5}$;$\mu_m\left(A_5,\frac{1}{5}\right)=\frac{1}{26}$
$\mu_m\left(A_6,\frac{1}{6}\right)=\frac{1}{37}$; $\mu_m\left(A_7,\frac{1}{7}\right)=\frac{1}{50}$; $\mu_m(A_8,0)=1$
Then m is a fuzzy measure.

Example 2.7. Consider **X,** P(X) and the measure 'm' be as in Example 2.6. If the membership function be defined as $\mu_m(A_i, x) = 1 - |A_i| / (1+x^2)$. We see that

$A_2 \subset A_5$ and $m(A_2) = 1/5 < m(A_2) = 1$ but

$\mu_m(A_5, \frac{1}{5}) = \frac{1}{26} > 0 = \mu_m(A_2, 1)$

Therefore m is not a fuzzy measure.

3. Conclusion

We have given different contexts where 'm'has proved to be a fuzzy measure by means of examples. The domain of the measure 'm' has been the ordinary power set, the subclass of the power set, the sigma-algebra, and the sigma algebra generated by the subclass of the power set. Measure 'm' is also defined in a variety of ways and is proved to be a fuzzy measure according to the new definition.

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