

## Fuzzy $\omega$ -Tree Automata

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**Abstract.** Fuzzy  $\omega$  - tree automata are defined as an accepting device for fuzzy  $\omega$  - tree languages. We study some relationships among the classes of fuzzy  $\omega$  - tree automata and some closure properties of the same. Furthermore, we have shown that a fuzzy  $\omega$  - tree language is Buchi recognizable if and only if it is  $\omega$  - rational.

**Keywords:** Fuzzy  $\omega$  - tree automata; Fuzzy  $\omega$  - tree language; Fuzzy Buchi tree automaton; Fuzzy Muller tree automaton; Fuzzy Rabin tree automaton

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### 1. Introduction

An automaton is a control mechanism designed to follow automatically a predetermined sequence of operations or respond to encoded instructions. It involves a sequential switching circuit with a finite number of states, with state changing when it is subjected to an input symbol. Zadeh introduced the concept of fuzzy sets and W. G. Wee introduced fuzzy automata. The theory of inverse monoids were introduced independently by Wagner and Preston via the study of partial one-one transformations of a set. Sebastian and Johnson [10] defined an inverse fuzzy automaton such that its transition monoid is an inverse monoid and study the structure of the automorphism group of an inverse fuzzy automaton. Samuel et al [9] introduced A Watson-Crick online tessellation automata which work on double-stranded arrays where the two strands relate to each other through a complementary relation inspired by the DNA complementarity. Chatrapathy and Ramanswamy [2] proved the equivalence between acceptance by empty stack and acceptance by final states for an intuitionistic fuzzy pushdown automata.

Fuzzy tree automata have been studied in [4] over ranked alphabet to recognize fuzzy tree language [1]. Fuzzy tree language is the fuzzy set of trees. Fuzzy tree automata is an extension of fuzzy automata [6] when the words are viewed as an element of unary term and fuzzy  $\omega$  tree automata is an extension of fuzzy  $\omega$  automata [3,5,6,8] when the words are viewed as an element of  $n$  - ary term. In this paper, a fuzzy tree automata is redefined to recognize  $n$  - ary tree. Further, we defined a fuzzy  $\omega$  tree automata as a generalization of the redefined fuzzy tree automata to recognize a fuzzy set of infinite trees. Fuzzy  $\omega$  tree language is the fuzzy set of infinite  $n$  - ary trees. In Section 2, we

redefined the fuzzy tree automata and studied some preliminaries. Section 3 studies the relationships among fuzzy  $\omega$  - tree automata and section 4 presents the conclusion of the paper.

## 2. Fuzzy finite tree automata

Let  $n$  be a positive integer and  $[n]$  denote the set  $\{0,1,2,\dots,n-1\}$ . Let  $[n]^*$  be a set of all strings over  $[n]$  including the null string  $\Lambda$ . A finite subset  $D$  of  $[n]^*$  is called a finite tree domain if the following conditions hold:

- (1)  $w \in D$  and  $w = uv$  implies  $u \in D$ , where  $u, v, w \in [n]^*$ ;
- (2)  $wn \in D$  and  $m \leq n$  implies  $wm \in D$ , where  $w \in [n]^*, m, n \in [n]$ .

Let  $A$  be a nonempty set of alphabets. A finite  $n$ -ary tree is a mapping  $t: D \rightarrow A$ . We denote by  $T$  the set of all  $n$ -ary trees on  $A$ . The elements of  $D$  are called the nodes of the tree. If  $x \in D$  is a node, any node of the form  $xy$  for  $y \in [n]$  is called a child of  $x$ . The height of a finite tree  $t$  is the maximal length of the elements of  $D$ .

The frontier of  $t$ , denoted  $Fr(t)$  is the set

$$Fr(t) = \{x \in D \mid x[n] \cap D = \emptyset\}$$

The elements of  $Fr(t)$  are usually called the leaves of the tree. The outer frontier of  $t$ , denoted by  $Fr^+(t)$  is the set

$$Fr^+(t) = D[n] - D$$

formed by all  $xy \in D$  such that  $x \in D$  and  $y \in [n]$  and the set  $D^+(t) = D \cup Fr^+(t)$ . A path  $P$  through the tree  $t$  is a sequence  $u_0, u_1, u_2, \dots, u_n$  of successive nodes string in the root  $u_0 = \Lambda$ .

**Definition 2.1.** A fuzzy finite tree automaton over the alphabet  $A$  is a 4 - tuple  $F = (S, f, I, F)$ , where

- $S$  is the finite set of states.
- $f: S \times A \times S^n \rightarrow [0,1]$  is a fuzzy transition function.
- $I: S \rightarrow [0,1]$  is a fuzzy set of initial states.
- $F: S \rightarrow [0,1]$  is a fuzzy set of final states.

A run of the fuzzy finite tree automaton  $F$  on a tree  $t$  is a finite tree  $R: D^+ \rightarrow S$  with  $I(R(\Lambda)) > 0$  such that  $f(R(x), t(x), R(x_0), R(x_1), \dots, R(x_{n-1}))) > 0$  for all  $x \in D$ . The run  $R$  is successful if  $F(R(u)) > 0$  for all  $u$  on the outer frontier  $Fr^+(t)$  of  $t$ .

Let  $\mathbf{R}$  denote the set of all different runs of  $F$  on a tree  $t$ . The weight of an accepted tree  $t$  is calculated as follows:

$$W(t) = \bigvee_{R \in \mathbf{R}} \{ \bigwedge \{ I(R(\Lambda)) \}, \bigwedge \{ f(R(x), t(x), R(x_0), R(x_1), \dots, R(x_{n-1}))) \}, \bigwedge \{ F(R(u)) \mid u \in Fr^+(t) \} \}$$

### Fuzzy $\omega$ -Tree Automata

The set  $T$  of finite tree languages recognized by a fuzzy finite tree automaton is formed by all trees  $t$  such that there is a successful run  $R$  on  $t$ . A fuzzy set  $\mu : T \rightarrow [0,1]$  of finite tree languages is recognizable if there is a fuzzy tree automaton  $F$  such that  $W(t) = \mu(t)$ .

#### 3. Classes of fuzzy $\omega$ - tree automata

An infinite  $n$ -ary tree on the alphabet  $A$  is a mapping  $t : [n]^* \rightarrow A$ . We denote by  $T^\omega$  the set of all  $n$ -ary infinite trees on  $A$ . An infinite tree is a generalization of an infinite word when  $[n] = \{0\}$ . A path  $P$  through the infinite tree  $t$  is an infinite sequence  $u_0, u_1, u_2, \dots$  of successive nodes string in the root  $u_0 = \Lambda$ .

**Definition 3.1.** A fuzzy  $\omega$ -tree automaton on the alphabet  $A$  is a 4-tuple  $F = (S, f, I, Acc)$ , where

- $S$  is the finite set of states.
- $f : S \times A \times S^n \rightarrow [0,1]$  is a fuzzy transition function.
- $I : S \rightarrow F$  is fuzzy set of initial states.
- $Acc$  is the acceptance component.

A run of the fuzzy infinite tree automaton  $F$  on a infinite tree  $t$  is a infinite tree  $R : [n]^* \rightarrow S$  with  $I(R(\Lambda)) > 0$  such that

$$f(R(x), t(x), R(x_0), R(x_1), \dots, R(x_{n-1}))) > 0 \text{ for all } x \in [n]^*.$$

**Definition 3.2.** A fuzzy Buchi tree automaton is a 5 - tuple  $F = (S, A, f, I, B)$ , where

- $(S, A, f, I)$  is a fuzzy  $\omega$  - tree automaton.
- $B : S \rightarrow F$  is the fuzzy set of infinitely repeated states.

A run  $R$  in  $F$  is successful if each path  $P$  visits  $B$  infinitely often, that is  $inf(R) \cap B \neq \emptyset$ . The weight of the accepted  $\omega$  - tree is calculated as follows

$$W(t) = \bigvee_{R \in R} \{ \bigwedge \{ I(R(\Lambda)) \}, \bigwedge \{ f(R(x), t(x), R(x_0), R(x_1), \dots, R(x_{n-1}))) \}, \\ \bigwedge \{ B(t) \mid t \in inf(R) \cap B \} \}$$

**Theorem 3.3.** Let  $L(F_1)$  and  $L(F_2)$  be the fuzzy  $\omega$  - tree languages recognized by fuzzy Buchi tree automata  $F_1$  and  $F_2$  respectively. Then there exists a fuzzy Buchi tree automaton  $F$  such that  $L(F) = L(F_1) \cup L(F_2)$ .

**Proof:** Let  $F_1 = (S_1, f_1, I_1, B_1)$  and  $F_2 = (S_2, f_2, I_2, B_2)$  be fuzzy Buchi tree automata. Construct a fuzzy Buchi tree automaton  $F = (S, A, f, I, B)$ , where

$$\begin{aligned} S &= S_1 \cup S_2, \\ f &= f_1 \cup f_2, \\ I &= I_1 \cup I_2, \\ B &= B_1 \cup B_2 \end{aligned}$$

**Definition 3.4.** A fuzzy Muller tree automaton is a 4 - tuple  $\mathbf{F} = (S, f, I, \mathbf{M})$ , where

- $(S, A, f, I)$  is a fuzzy  $\omega$  - tree automaton.
- $\mathbf{M}$  is a set of fuzzy sets of S, that is  $\mathbf{M} \subset F(S)$ .

A run  $R$  of  $\mathbf{F}$  on a infinite tree  $t$  is successful if for every path  $P$  of  $R$  the set of infinitely often states occurs is a member of  $\mathbf{M}$ , that is  $\text{inf}(R) \in \mathbf{M}$ .

The weight of the accepted word  $\omega$  is calculated as follows

$$W(t) = \bigvee_{R \in \mathbf{R}} \{ \wedge \{ \{ I(R(\Lambda)) \}, \wedge \{ f(R(x), t(x), R(x_0), R(x_1), \dots, R(x_{n-1})) \}, \wedge \{ \mathbf{M}(t) \mid t \in \text{inf}(R) \} \} \}$$

**Theorem 3.5.** Let  $L(\mathbf{F}_1)$  and  $L(\mathbf{F}_2)$  be the fuzzy  $\omega$  - tree languages recognized by fuzzy Muller tree automata  $\mathbf{F}_1$  and  $\mathbf{F}_2$  respectively. Then there exists a fuzzy Muller tree automaton  $\mathbf{F}$  such that  $L(\mathbf{F}) = L(\mathbf{F}_1) \cup L(\mathbf{F}_2)$ .

The idea of constructing the desired  $\mathbf{F}$  is very similar to the one for fuzzy Buchi tree automaton.

**Definition 3.6.** A fuzzy Rabin tree automaton is a 4 - tuple  $\mathbf{F} = (S, f, I, \mathbf{R})$ , where

- $(S, f, I)$  is a fuzzy  $\omega$  - automaton.
- $\mathbf{R} = \{ (E_1, F_1), (E_2, F_2), \dots, (E_n, F_n) \}$  is a pairs of fuzzy set of states. That

is

$$\mathbf{R} = \{ (E_i : Q \rightarrow F, F_i : Q \rightarrow F) \mid 1 \leq i \leq n \}$$

A run  $R$  in  $\mathbf{F}$  is successful if each path  $P$  visits  $F_i$  infinitely often and visits  $E_i$  finitely often, that is

$$\bigvee_{i=1}^n (\text{inf}(R) \cap E_i = \phi \wedge \text{inf}(R) \cap F_i \neq \phi)$$

$$\bigvee_i \{ \wedge \{ E_i(s), F_i(t) \mid s \in \text{fin}(R) \cap E_i, t \in \text{inf}(R) \cap F_i \} \}$$

The weight of the accepted  $\omega$  - tree is calculated as follows

$$W(t) = \bigvee_{R \in \mathbf{R}} \{ \wedge \{ \{ I(R(\Lambda)) \}, \wedge \{ f(R(x), t(x), R(x_0), R(x_1), \dots, R(x_{n-1})) \} \}$$

**Theorem 3.7.** A fuzzy  $\omega$  - tree language recognized by a fuzzy Buchi tree automaton is also recognized by a fuzzy Muller automaton.

**Proof:** Let  $\mathbf{F} = (S, f, I, \mathbf{B})$  be a fuzzy Buchi tree automaton recognizes the fuzzy  $\omega$  - tree language  $L(\mathbf{F})$ . Construct a fuzzy Muller tree automaton  $\mathbf{F}' = (S, f, I, \mathbf{M})$  where  $\mathbf{M} = \{ \mu \in F(S) \mid \mu \cap \mathbf{B} \neq \phi \}$ . It can be easily verify that  $L(\mathbf{F}) = L(\mathbf{F}')$  and the weight of the accepted infinite tree remains the same.

**Theorem 3.8.** A fuzzy  $\omega$  - tree language recognized by a fuzzy Rabin tree automaton is also recognized by a fuzzy Muller automaton.

### Fuzzy $\omega$ -Tree Automata

**Proof:** Let  $F = (S, f, I, R)$  be a fuzzy Rabin tree automaton. Construct a fuzzy Muller tree automaton  $F' = (S, f, I, M)$  where

$M = \{\mu \in F(S) \mid (E, F) \in R, \mu \cap E = \phi \wedge \mu \cap F \neq \phi\}$ . It can be easily verify that  $L(F) = L(F')$  and the weight of the accepted infinite tree remains the same.

**Theorem 3.9.** A fuzzy  $\omega$  - tree language recognized by a fuzzy Muller tree automaton is also recognized by a fuzzy Rabin tree automaton.

**Proof:** Let  $F = (S, f, I, M)$  be a fuzzy Muller tree automaton with  $M = \{M_1, M_2, \dots, M_n\}$ ,  $M_i \in F(S), i = 1, 2, \dots, n$ . Construct a fuzzy Rabin tree automaton  $F' = (S, f, I, R)$  where

$R = \{(S - M_1, M_1), (S - M_2, M_2), \dots, (S - M_n, M_n)\}$ . It can be easily verify that  $L(F) = L(F')$  and the weight of the accepted infinite tree remains the same.

**Theorem 3.10.** Let  $L(F_1)$  and  $L(F_2)$  be the fuzzy  $\omega$  - tree languages recognized by fuzzy Rabin tree automata  $F_1$  and  $F_2$  respectively. Then there exists a fuzzy Rabin tree automaton  $F$  such that  $L(F) = L(F_1) \cup L(F_2)$ .

By theorem 3.8 and 3.9, the fuzzy language recognized by a fuzzy Muller tree automaton is recognized by a fuzzy Rabin tree automaton and conversely. Since the class of languages recognized by fuzzy Muller tree automata is closed under union. Hence the same is true for the fuzzy language recognized by fuzzy Rabin tree automata.

**Theorem 3.11.** A fuzzy  $\omega$  - tree language is recognized by fuzzy Buchi tree automaton if and only if it is  $\omega$  - rational.

**Proof:** Let  $F = (S, f, I, B)$  be a fuzzy Buchi tree automaton with  $B = \{s_1, s_2, \dots, s_m\}$ .

For each  $s \in S$ , let  $\mu_s$  be the fuzzy set of finite trees  $t$  on  $A \cup S$  such that

- (1) they have values in  $B$  on their frontier but elsewhere in  $A$ ,
- (2) there is a run  $R$  starting at  $s$  on the tree obtained by deleting the frontier of  $t$  and  $R$  is such that  $R(x) = t(x)$  for all  $x \in Fr(t)$ .

Each fuzzy set  $\mu_s$  is recognizable. For each  $s \in S$ , let  $\mu_t$  be the fuzzy set recognized by the fuzzy Buchi tree automaton  $(S, f, I, B)$ . We can write each  $\mu_t$  as

$$\mu_t = \mu_s \bullet_p (\mu_{s_1}, \mu_{s_2}, \dots, \mu_{s_m})^{\omega, p}$$

where  $p = (s_1, s_2, \dots, s_m)$ . Indeed, for any tree  $t$  which belongs to the right-hand side of the formula, there is a run of  $F$  on  $t$  starting at  $s$  and passing infinitely often in  $B$ . Thus  $t \in \mu_t$ . Conversely, if  $t \in \mu_s$ , let  $R$  be a successful run of  $(S, f, s, B)$  on  $t$ . Any path  $P$  in  $R$  passes infinitely often in  $B$ . Thus, there is a tree  $t$  such that  $s \in \mu_t$ . Continuing the above procedure repeatedly leads to the desired decomposition.

#### 4. Conclusion

In this paper, we have proved that a fuzzy tree language recognized by a fuzzy Buchi tree automaton is also recognized by a fuzzy Muller tree automaton. Further, it is proved that a fuzzy tree language recognized by a fuzzy Rabin tree automaton is also recognized by a fuzzy Muller tree automaton and conversely. Also, we studied the closure properties of fuzzy tree languages recognized by fuzzy  $\omega$  - tree automata. Finally, we proved that a fuzzy  $\omega$  - tree language is recognized by fuzzy Buchi tree automaton if and only if it is  $\omega$  - rational.

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