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Progress in **Nonlinear Dynamics and Chaos**

Lie Group Analysis, Optimal System and Exact Solutions of a Class of Fifth-Order Nonlinear Evolution Equation

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Abstract. In this article, the Lie group analysis method is used to study a class of high-order nonlinear development equations. First of all, the point symmetries of the equation are obtained by employing the Lie group method. Secondly, the one-dimensional optimal system and symmetry reductions are constructed. The last but not least, a series of new exact solutions of the fifth-order nonlinear evolution equation have been obtained by using auxiliary function expansion methods and homogeneous balance method, including soliton solutions and trigonometric function solutions. For some solutions, this article has made corresponding images, which are of great significance to the study of the properties of the solutions.

Keywords: Higher-order nonlinear evolution equations; Lie group analysis; Lie algebra; Exact solution

AMS Mathematics Subject Classification (2010): 35R03, 35B06, 35E20, 35G50

1. Introduction

In the related fields of mathematics, physics, biology and engineering, nonlinear evolution equations are able to describe some specific complex phenomena. With the enhancement of people's cognitive level, we find increasingly more physical and engineering problems in continuous exploration on nature. These reality problems can be converted into a corresponding problem solving nonlinear partial differential equations [1-3]. Therefore, the research on the exact solutions of the PDEs is rather significant. Nowadays, there are many mature methods for solving nonlinear development equations. For example, classic Lie group method [4-8], (G'/G) expansion method [9-12], homogeneous balance method [13-18], F-expansion method [19-22] and hyperbolic function expansion method [23-26].

There is a long history of the relationship between Lie groups and differential equations, its founder first proposed the concept of Lie group when exploring the symmetry of differential equations. Nowadays, how to construct exact solutions of equations has become a very significant subject in the field of differential equations. Using the Lie group method, the Lie point symmetry [27-29] of the equation can be found. Finally, the partial differential equations are transformed into the mature ordinary differential equations. In this way, the group-invariant solution of the equation can be constructed.

In the engineering and design around us, there is a kind of important structural element beams and columns. The related mechanics problem has always been an important aspect of solid mechanics research. Since the ultimate bearing capacity of the beam can be impacted by many factors, as follows, the shape and size of the section, the mechanical properties of the material, the residual stress, the initial bending of the component, and the initial eccentricity of the load point. So the corresponding calculation formulas and methods are often based on the experimental. Most of these problems are simulated by establishing nonlinear evolution equations, so it is a great significance for the exact solutions and optimal systems of nonlinear evolution equations. This article will study the fourth-order nonlinear evolution equation of the following form

$$u_t + \alpha u^2 u_x + \beta u u_x + \kappa u_{xxxx} + \delta u_{xxxxx} = 0 \tag{1}$$

$u = u(x,t)$, $\alpha, \beta, \kappa, \delta$ are arbitrary constants. Equation (1) contains many famous equations, when $\beta = 0, \delta = 0$, equation (1) is a kind of equation used to describe the bending condition of the elastic beam and the stability of the solution [30]. Article [31] employs the power series method to find the power series solution. Article [32] combines the elliptic function expansion method with the power series expansion method to find various exact solutions, such as the trigonometric function solution and the elliptic function expansion solution when $\alpha = \alpha(t), \beta = 0, \delta = 0$. This article will study the exact solution when $\kappa = 0$,

$$u_t + \alpha u^2 u_x + \beta u u_x + \delta u_{xxxxx} = 0. \tag{2}$$

Compared with the equation in article [27], equation (2) adds more restrictive terms, it is transformed into a higher-order physical model. Probably, it is able to simulate the stability of the real elastic beam more realistical, so it is more practically significant.
This article will adopt two auxiliary function expansion methods to construct a new exact solution of equation (2), the method used have also been improved. These results obtained enrich the types of exact solutions of the equation, which are quite different from the previous solutions. These precise solutions will be helpful in future beam-column structural problems.

This article consists of the following parts. The first part finds the Lie point symmetry of the equation. The second part aims to construct the optimal system of one-dimensional Lie algebra. The third part adopts symmetris to reduce the original equation into ODEs. The fourth part, in order to construct a new exact solution of equation, combines the homogeneous balance method and the construction auxiliary function expansion method (2). The fifth part summarizes the whole context.

2. Symmetry of equation (2)

Suppose the vector field of equation (2) is

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u}, \tag{3}$$

where $\xi(x,t,u), \tau(x,t,u), \phi(x,t,u)$ are undetermined functions, and the following equation must be satisfied,

$$pr^{(5)}V(\Delta)|_{\Delta=0}=0, \tag{4}$$

where $\Delta=u_t+\alpha u^2u_x+\beta uu_x+\delta u_{xxxxx}$, the fifth-order extension of equation (2) can be obtained as,

$$pr^{(5)}V=\phi^t+(\alpha u^2+\beta u)\phi^x+\delta\phi^{xxxxx}=0, \tag{5}$$

the $\phi^t,\phi^x,\phi^{xxxxx}$ are determined by the differential term of $\xi(x,t,u),\tau(x,t,u),\phi(x,t,u)$,

$$\phi^t=D_t(\phi-\xi u_x-\tau u_t)+\xi u_{xt}+\tau u_{tt}, \tag{6}$$

$$\phi^x=D_x(\phi-\xi u_x-\tau u_t)+\xi u_{xx}+\tau u_{xt}, \tag{7}$$

$$\phi^{xxxxx}=D_{xxxxx}(\phi-\xi u_x-\tau u_t)+\xi u_{xxxxxx}+\tau u_{xxxxxt}, \tag{8}$$

here, D_x,D_t,D_{xxxxx} in (6)(7)(8) are the total differential operator of t,x.

Substituting (6) (7) (8) into (5). Let the coefficients of all derivatives containing u be equal to zero. Determining equations can be obtained on the ξ,τ,ϕ. After solving the equations, the Lie point symmetry of equation (2) can be obtained

$$\phi=-\frac{C_1(2\alpha u+\beta)}{5\alpha},\tau=C_1t+C_2,\xi=\frac{C_1x}{5}-\frac{C_1\beta^2t}{5\alpha}+C_3, \tag{9}$$

where C_1,C_2,C_3 are non-zero arbitrary constant. These results will be discussed in the following categories,

(a) When $C_1=1,C_2=C_3=0$,

$$\phi=-\frac{2}{5}u-\frac{\beta}{5\alpha},\tau=t,\xi=\frac{1}{5}x-\frac{\beta^2}{5\alpha}t. \tag{10}$$

Substituting (10) into (3), get

$$V_1=(\frac{\beta^2}{5\alpha}t-\frac{1}{5}x)\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}+(-\frac{2}{5}u-\frac{\beta}{5\alpha})\frac{\partial}{\partial u}. \tag{11}$$

(b) When $C_2=1,C_1=C_3=0$,

$$\phi=0,\tau=1,\xi=0. \tag{12}$$

Substituting (12) into (3), get

$$V_2=\frac{\partial}{\partial t}. \tag{13}$$

(c) When $C_3=1,C_1=C_2=0$,

$$\phi=0,\tau=0,\xi=1. \tag{14}$$

Substituting (14) into (3), get

$$V_3=\frac{\partial}{\partial x}. \tag{15}$$

In summary, the three Lie point symmetries of equation (2) are obtained. In the next part, the one-dimensional optimal system of equation (2) will be constructed with symmetry. In the next part, the three symmetries obtained in the first part will also be used to transform equation (2) into an ordinary differential equation.

3. Lie Algebra and Optimal System

In the second part, the Lie point symmetry vector field of equation (2) has been obtained,

$$V_1=(\frac{\beta^2}{5\alpha}t-\frac{1}{5}x)\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}+(-\frac{2}{5}u-\frac{\beta}{5\alpha})\frac{\partial}{\partial u},V_2=\frac{\partial}{\partial t},V_3=\frac{\partial}{\partial x}. \tag{16}$$

Next, we will use the conclusions of the first part to calculate the optimal system. Before that, we obtain the Lie algebra commutator table 2. 1 and the Lie algebra adjoint function table 2. 2 from the definition of Lie bracket operation $[V_i,V_j]=V_iV_j-V_jV_i$ and the adjoint expression $Ad_{\exp(\varepsilon V_i)}V_j=V_j-\varepsilon[V_i,V_j]+\frac{\varepsilon}{2!}[V_i,[V_i,V_j]]-\cdots$ as follows,

Table 2.1: Lie algebra commutator

	V_1	V_2	V_3
V_1	0	$-hV_3-V_2$	$\frac{1}{5}V_3$
V_2	hV_3+V_2	0	0
V_3	$-\frac{1}{5}V_3$	0	0

Table 2.2: Lie Algebra adjoint action

	V_1	V_2	V_3
V_1	V_1	$2V_2+hV_3$	$\frac{4}{5}V_3$
V_2	$V_1-hV_3-V_2$	V_2	V_3
V_3	$V_1+\frac{1}{5}V_3$	V_2	V_3

Note, $h=\frac{\beta^2}{5\alpha}$.

According to the method of finding the one-dimensional optimal system. Set up a non-zero $V\in L_3, L_3$ is composed Lie algebra.

$$V=a_1V_1+a_2V_2+a_3V_3,$$

a_1,a_2,a_3 are constant.

$$\begin{aligned}(\text{a})\tilde{V}_1&=Ad_{(\exp(\varepsilon V_2)}(V)=e^{-\varepsilon V_2}Ve^{\varepsilon V_2}=V-\varepsilon[V_2,V]+\frac{1}{2!}\varepsilon^2[V_2,[V_2,V]]-\cdots\\&=a_1V_1+a_2V_2+a_3V_3-\varepsilon[a_1[V_2,V_1]+a_2[V_2,V_2]+a_3[V_2,V_3]]+\cdots\\&=a_1V_1+a_2V_2+a_3V_3-a_1\varepsilon hV_3-a_1\varepsilon V_2\\&=a_1V_1+(a_2-a_1\varepsilon)V_2+(a_3-a_1\varepsilon h)V_3.\end{aligned}$$

When $a_2 = a_1\varepsilon$, $\tilde{V}_1 = a_1V_1 + (a_3 - a_1\varepsilon h)V_3$.

When $a_3 = a_1\varepsilon h$, $\tilde{V}_1 = a_1V_1 + (a_2 - a_1\varepsilon)V_2$.

$$(b)\tilde{V}_2 = Ad_{(\exp(\varepsilon V_3))}(V) = e^{-\varepsilon V_3}Ve^{\varepsilon V_3} = V - \varepsilon[V_3,V] + \frac{1}{2!}\varepsilon^2[V_3,[V_3,V]] - \cdots$$
$$= a_1V_1 + a_2V_2 + a_3V_3 - \varepsilon[a_1[V_3,V_1] + a_2[V_3,V_2] + a_3[V_3,V_3]] + \cdots$$
$$= a_1V_1 + a_2V_2 + a_3V_3 - \frac{1}{5}a_1\varepsilon V_3$$
$$= a_1V_1 + a_2V_2 + (a_3 - \frac{1}{5}a_1\varepsilon)V_3.$$

When $a_3 = \frac{1}{5}a_1\varepsilon$, $\tilde{V}_2 = a_1V_1 + a_2V_2$.

In summary, the one-dimensional optimal system of equation (2) is obtained as

$$\{V_1, V_2, V_3, V_1 + \lambda_1 V_2, V_1 + \lambda_2 V_3\},$$

note, λ_1, λ_2 are arbitrary constants.

4. Symmetry reduction

Symmetry reduction is one of the commonly used reduction methods when dealing with nonlinear development equations. The vector field obtained in the first part can be used to symmetrically reduce the equation (2), so that the equation (2) can be transformed into an ordinary differential equation. Next, equation (2) will be symmetrically reduced based on equations (11) (13) (15) in the first part.

(a) For the vector field $V_1 = (\frac{\beta^2}{5\alpha}t - \frac{1}{5}x)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + (-\frac{2}{5}u - \frac{\beta}{5\alpha})\frac{\partial}{\partial u}$,

the corresponding characteristic equation is,

$$\frac{\mathrm{d}x}{\frac{\beta^2}{5\alpha}t - \frac{1}{5}x} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}u}{-\frac{2}{5}u - \frac{\beta}{5\alpha}}, \tag{17}$$

the invariant ξ_1 is obtained from the characteristic equation (17), and the group-invariant solution(18)of V_1 is derived from $\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}u}{-\frac{2}{5}u - \frac{\beta}{5\alpha}}$. $\xi_1 = \frac{\beta^2 t + 4\alpha x}{4\alpha t^{\frac{2}{5}}}$

$$u = -\frac{\beta}{2\alpha} + \frac{f(\xi_1)}{t^{\frac{2}{5}}}, \tag{18}$$

Substituting (18) into (2), the reduced equation is obtained as,

$$5\alpha f_1' f_1^2 - f_1'\xi_1 + 5\delta f_1''''' - 2f_1 = 0 \tag{19}$$

(b) For the vector field $V_2=\frac{\partial}{\partial t}$, $\xi_2=x$ the corresponding group invariant solution is

$$u=f(\xi_2), \tag{20}$$

substituting (20) into (2), the reduced equation is obtained as,

$$\alpha f_2^{\,2} f_2'+\beta f_2 f_2'+\delta f_2''''' =0 \tag{21}$$

(c) For the vector field $V_3=\frac{\partial}{\partial x}$, $\xi_3=t$ the corresponding group invariant solution is

$$u=f(\xi_3), \tag{22}$$

substituting (22) into (2), the reduced equation is obtained as,

$$f_3'=0 \tag{23}$$

So far, the symmetry reduction of equation (2) is completed and three different forms of reduced equations are obtained, successfully transformed complex partial differential equations into ordinary differential equations are relatively easy to study. Therefore, the results obtained are of great significance for the in-depth study of equation (2).

5. Exact solutions of equation (2)

In this part, we will use the results of the reduction in the fourth part to construct a variety of exact solutions to equation (2). Equation (2) has been successfully transformed into the form of ODEs in the previous section, and we choose the case of (21)，but it may lead to the lack of comprehensiveness of the original equation. So we will choose the case of $C_1=0$ in the second part, constructing new exact solutions of equation (2) by using homogeneous balance method and two different auxiliary function expansion methods.

Let $u(x,t)=u(\psi),\psi=x-qt,$ where q is an arbitrary constant. Substitute the transformation into (2) get

$$-qu'+\alpha u^2u'+\beta uu'+\delta u'''''=0, \tag{24}$$

integrate (24) to get

$$C-qu+\frac{1}{3}\alpha u^3+\frac{1}{2}\beta u^2+\delta u''''=0. \tag{25}$$

(a) Suppose equation (25) has a solution of the following form,

$$u(\psi)=\sum_{m=-\infty}^{\infty}\alpha_m(\omega)^m, \tag{26}$$

where $\omega=\omega(\psi)$ and satisfies the equation,

$$\omega^2-\omega'+\mu=0, \tag{27}$$

where α_m,μ are arbitrary constant.

From the principle of homogeneous balance, $3m=m+4, m=2$, then the equation has a solution of the form,

$$u(\psi)=\alpha_0+\alpha_1\omega+\alpha_2\omega+\frac{\alpha_{-1}}{\omega}+\frac{\alpha_{-2}}{\omega^2}, \tag{28}$$

substituting(27)(28)into(25)to get a formula then extract the coefficients of ω^m to get the overdetermined equations,

$$\begin{cases}
\frac{1}{3}\alpha\alpha_2^3+120\delta\alpha_2=0\\
\alpha\alpha_1\alpha_2^2+24\delta\alpha_1=0\\
\alpha\alpha_0\alpha_2^2+\alpha\alpha_1^2\alpha_2+240\delta\mu\alpha_2+\frac{1}{2}\beta\alpha_2^2=0\\
\alpha\alpha_{-1}\alpha_2^2+2\alpha\alpha_0\alpha_1\alpha_2+\frac{1}{3}\alpha\alpha_1^3+40\delta\mu\alpha_1+\beta\alpha_1\alpha_2=0\\
120\delta\mu^4\alpha_{-2}+\frac{1}{3}\alpha\alpha_{-2}^3=0\\
136\delta\mu^2\alpha_2+\alpha\alpha_{-2}\alpha_2^2+2\alpha\alpha_{-1}\alpha_1\alpha_2+\alpha\alpha_0^2\alpha_2+\\
\alpha\alpha_0\alpha_1^2+\beta\alpha_0\alpha_2+\frac{1}{2}\beta\alpha_1-q\alpha_2=0\\
2\alpha\alpha_{-2}\alpha_1\alpha_2+2\alpha\alpha_{-1}\alpha_0\alpha_2+\alpha\alpha_{-1}\alpha_1^2+\alpha\alpha_0^2\alpha_1+\\
16\delta\mu^2\alpha_1+\beta\alpha_{-1}\alpha_2+\beta\alpha_0\alpha_1-q\alpha_1=0\\
2\alpha\alpha_{-2}\alpha_{-1}\alpha_2+2\alpha\alpha_{-2}\alpha_0\alpha_1+\alpha\alpha_{-1}^2\alpha_1+\alpha\alpha_{-1}\alpha_0^2+\\
16\delta\mu^2\alpha_{-1}+\beta\alpha_{-2}\alpha_1+\beta\alpha_{-1}\alpha_0-q\alpha_{-1}=0\\
136\delta\mu^2\alpha_{-2}+\alpha\alpha_{-2}^2\alpha_2+2\alpha\alpha_{-2}\alpha_{-1}\alpha_1+\alpha\alpha_{-2}\alpha_0^2+\\
\alpha\alpha_{-1}^2\alpha_0+\beta\alpha_{-2}\alpha_0+\frac{1}{2}\beta\alpha_{-1}^2-q\alpha_{-2}=0\\
40\delta\mu^3\alpha_{-1}+\alpha\alpha_{-2}^2\alpha_1+2\alpha\alpha_{-2}\alpha_{-1}\alpha_0+\frac{1}{3}\alpha\alpha_{-1}^3+\beta\alpha_{-2}\alpha_{-1}=0\\
24\delta\mu^4\alpha_{-1}+\alpha\alpha_{-2}^2\alpha_{-1}=0\\
240\delta\mu^3\alpha_{-2}+\alpha\alpha_{-2}^2\alpha_0+\alpha\alpha_{-2}\alpha_{-1}^2+\frac{1}{2}\beta\alpha_{-2}^2=0\\
16\delta\mu^3\alpha_2+2\alpha\alpha_{-2}\alpha_0\alpha_2+\alpha\alpha_{-2}\alpha_1^2+\alpha\alpha_{-1}^2\alpha_2+\\
+\beta\alpha_{-2}\alpha_2+\beta\alpha_{-1}\alpha_1+\frac{1}{2}\beta\alpha_0^2-q\alpha_0+\\
2\alpha\alpha_{-1}\alpha_0\alpha_1++\frac{1}{3}\alpha\alpha_0^3+16\delta\mu\alpha_{-2}=0
\end{cases}$$

when the coefficient $\alpha = -\frac{\beta^2}{2560\delta\mu^2}$ of the restriction equation (2), we can find the values of $\alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, q$, Select one set of solutions as,

$$q = 256\delta\mu^2, \alpha_{-2} = \frac{960\delta\mu^3}{\beta}, \alpha_{-1} = 0, \alpha_0 = \frac{1920\delta\mu^2}{\beta}, \alpha_1 = 0, \alpha_2 = \frac{960\delta\mu}{\beta}. \quad (29)$$

Substituting (29) into (28), when $\mu > 0$ the exact solution is

$$u_1(\psi) = \frac{960\delta\mu^2 \tan(C_\alpha\sqrt{\mu} + \psi\sqrt{\mu})^4 + 1920\delta\mu^2 \tan(C_\alpha\sqrt{\mu} + \psi\sqrt{\mu})^2 + 960\delta\mu^2}{\beta \tan(C_\alpha\sqrt{\mu} + \psi\sqrt{\mu})^2}. \quad (30)$$

When $\mu < 0$, the exact solution is

$$u_2(\psi) = \frac{960\delta\mu^2 \tan(C_\beta\sqrt{-\mu} + \psi\sqrt{-\mu})^4}{\beta \tan(C_\beta\sqrt{-\mu} + \psi\sqrt{-\mu})^2} + \frac{1920\delta\mu^2 \tan(C_\beta\sqrt{-\mu} + \psi\sqrt{-\mu})^2 + 960\delta\mu^2}{\beta \tan(C_\beta\sqrt{-\mu} + \psi\sqrt{-\mu})^2} \quad (31)$$

note, $\psi = x - (256\delta\mu^2)t$, C_α, C_β are arbitrary constants.

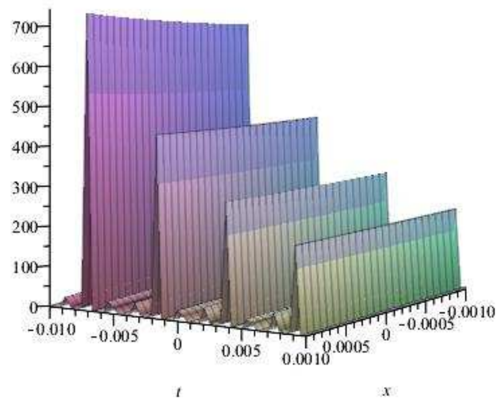

Figure 1: Wave profile of the solution for (30), when $\mu = 2, \delta = 960, \beta = 2, C_\alpha = 1$, $x \in [-0.001, 0.001], t \in [-0.01, 0.01]$

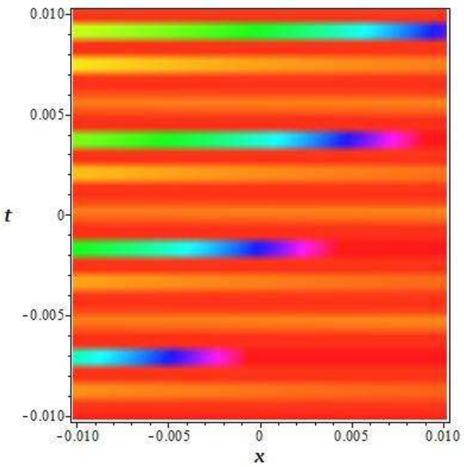

Figure 2: Density plot of the solution for(30), when $x \in [-0.01, 0.01], t \in [-0.01, 0.01]$

When the coefficient $\alpha = -\dfrac{\beta^2(9\delta + Z)}{8192\delta^2\mu^2}$ of the restriction equation (2), can find the values of $\alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, q$. Select one set of solutions as,

$$\begin{aligned} &q = 128\mu^2(\delta - Z), \alpha_{-2} = \frac{192Z\mu^3}{\beta}, \alpha_{-1} = \alpha_1 = 0, \\ &\alpha_0 = \frac{128\mu^2(9\delta + Z)}{\beta}, \alpha_2 = \frac{192Z\mu}{\beta}. \end{aligned} \tag{32}$$

Where Z is the real root of the following equation: $Z^2 + 5Z\delta - 20\delta^2 = 0.$
Substituting (32) into (28), when $\mu > 0$ the exact solution is

$$\begin{aligned} u_3(\psi) = &\frac{192Z\mu^2 \tan(C_\gamma\sqrt{\mu} + \psi\sqrt{\mu})^4 + 192Z\mu^2}{\beta\tan(C_\gamma\sqrt{\mu} + \psi\sqrt{\mu})^2} + \\ &\frac{128(8\delta - Z)\tan(C_\gamma\sqrt{\mu} + \psi\sqrt{\mu})^2}{\beta\tan(C_\gamma\sqrt{\mu} + \psi\sqrt{\mu})^2}. \end{aligned} \tag{33}$$

When $\mu < 0$, the exact solution is

$$\begin{aligned} u_4(\psi) = &\frac{192Z\mu^2 \tan(C_\chi\sqrt{-\mu} + \psi\sqrt{-\mu})^4 + 192Z\mu^2}{\beta\tan(C_\chi\sqrt{-\mu} + \psi\sqrt{-\mu})^2} + \\ &\frac{128(8\delta - Z)\tan(C_\chi\sqrt{-\mu} + \psi\sqrt{-\mu})^2}{\beta\tan(C_\chi\sqrt{-\mu} + \psi\sqrt{-\mu})^2}, \end{aligned} \tag{34}$$

note, $\psi = x - (128\delta\mu^2 - 128\mu^2 Z)t$, C_χ, C_γ are arbitrary constants.

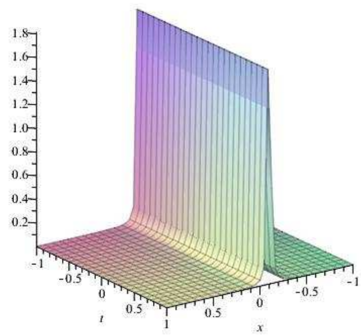

Figure 3: Wave profile of the solution for (33), when $\mu = 10^{-7}, \delta = 1, \beta = C_\gamma = 0.1,$ $x, t \in [-1, 1]$

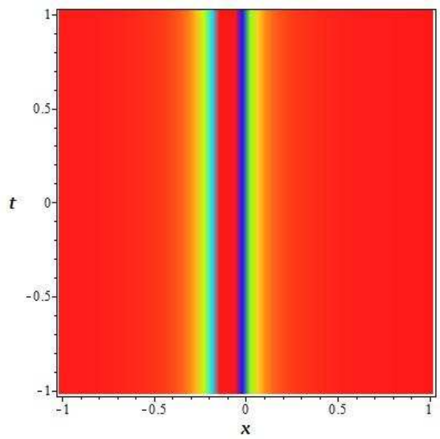

Figure 4: Density plot of the solution for (33), when $x \in [-1,1], t \in [-1,1]$

(b) Suppose equation (24) has a solution of the following form ,

$$u(\psi) = \sum_{m=0}^{\infty} \alpha_m \left(\frac{\omega'}{\omega}\right)^m, \tag{35}$$

where $\omega = \omega(\psi)$ and satisfies the equation,

$$\omega'' + \gamma\omega' + \rho\omega = 0, \tag{36}$$

where α_m, γ, ρ are arbitrary constant.

From the principle of homogeneous balance, $3m = m + 4, m = 2$, then the equation has a solution of the form,

$$u(\psi) = \alpha_0 + \alpha_1 \frac{\omega'}{\omega} + \alpha_2 \left(\frac{\omega'}{\omega}\right)^2, \tag{37}$$

Substituting (36) (37) into (25) to get a formula then extract the coefficients of $\left(\frac{\omega'}{\omega}\right)^m$ to get the overdetermined equations,

$$\begin{cases} \delta\gamma^4\alpha_1+30\delta\gamma^3\alpha_2+22\delta\gamma^2\rho\alpha_1+120\delta\gamma\rho^2\alpha_2+\alpha\alpha_0^2\alpha_1+16\delta\rho^2\alpha_1+\beta\alpha_0\alpha_1-q\alpha_1=0 \\ \beta\alpha_0\alpha_1+440\delta\gamma\rho\alpha_2+130\delta\gamma^3\alpha_2+50\delta\gamma^2\alpha_1+40\delta\rho\alpha_1+\frac{1}{3}\alpha\alpha_1^3+2\alpha\alpha_0\alpha_1\alpha_2=0 \\ \alpha\alpha_0^2\alpha_2+\alpha\alpha_0\alpha_1^2+\beta\alpha_0\alpha_2+232\delta\gamma^2\rho\alpha_2+60\delta\gamma\rho\alpha_1+16\delta\gamma^4\alpha_2+15\delta\gamma^3\alpha_1-q\alpha_2+ \\ \delta\gamma^3\rho\alpha_1+14\delta\gamma^2\rho^2\alpha_2+8\delta\gamma\rho^2\alpha_1+16\delta\rho^3\alpha_2\frac{1}{3}\alpha\alpha_0^3-q\alpha_0+\frac{1}{2}\beta\alpha_0^2=0 \\ \alpha\alpha_0\alpha_2^2+\alpha\alpha_1^2\alpha_2+330\delta\gamma^2\alpha_2+60\delta\gamma\alpha_1+240\delta\rho\alpha_2+\frac{1}{2}\beta\alpha_2^2=0 \\ 136\delta\rho^2\alpha_2+\frac{1}{2}\beta\alpha_1^2=0 \\ \alpha\alpha_1\alpha_2^2+336\delta\gamma\alpha_2+24\delta\alpha_1=0 \\ 120\delta\alpha_2+\frac{1}{3}\alpha\alpha_2^3=0 \end{cases}$$

When the coefficient $\alpha=-\frac{360\delta\rho^2}{\alpha_0^2},\beta=-\frac{60\delta\gamma^2\rho-240\delta\rho^2}{\alpha_0}$ of the restriction equation (2) can find the values of α_1,α_2,q, Select one set of solutions as,

$$q=\delta\gamma^4-8\delta\gamma^2\rho+16\delta\rho^2,\alpha_1=\frac{\gamma\alpha_0}{\rho},\alpha_2=\frac{\alpha_0}{\rho} \tag{38}$$

substituting (38) into (37), when $\gamma^2-4\rho>0$, the exact solution is

$$u_5(\psi)=-\frac{\frac{\mathrm{e}^{-\gamma\psi}C_\varepsilon C_\phi\gamma^2\alpha_0}{\rho}-4\mathrm{e}^{-\gamma\psi}C_\varepsilon C_\phi\alpha_0}{(C_\varepsilon \mathrm{e}^{\frac{1}{2}(-\gamma+\sqrt{\gamma^2-4\rho})\psi}+C_\phi \mathrm{e}^{-\frac{1}{2}(-\gamma+\sqrt{\gamma^2-4\rho})\psi})^2}. \tag{39}$$

When $\gamma^2-4\rho<0$, the exact solution is

$$u_6(\psi)=-\frac{\frac{\mathrm{e}^{-\gamma\psi}C_\varepsilon C_\phi\gamma^2\alpha_0}{\rho}-4\mathrm{e}^{-\gamma\psi}C_\varepsilon C_\phi\alpha_0}{(C_\varepsilon \mathrm{e}^{\frac{1}{2}(-\gamma+\sqrt{-\gamma^2+4\rho})\psi}+C_\phi \mathrm{e}^{-\frac{1}{2}(-\gamma+\sqrt{-\gamma^2+4\rho})\psi})^2}. \tag{40}$$

Note, $\psi=x-(\delta\gamma^4-8\delta\gamma^2\rho+16\delta\rho^2)t$, C_ε,C_φ are arbitrary constants.

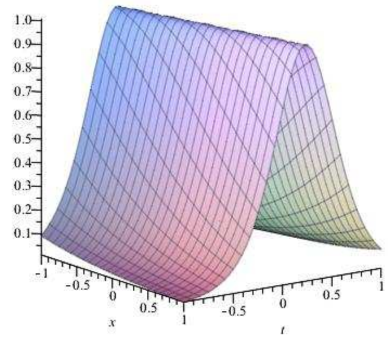

Figure 5: Wave profile of the solution for (39), when
$\gamma = 0.1, \rho = -0.3, \delta = 10, \alpha_0 = 1,\ C_\varepsilon = C_\varphi = 1, x \in [-1,1], t \in [-1,1]$

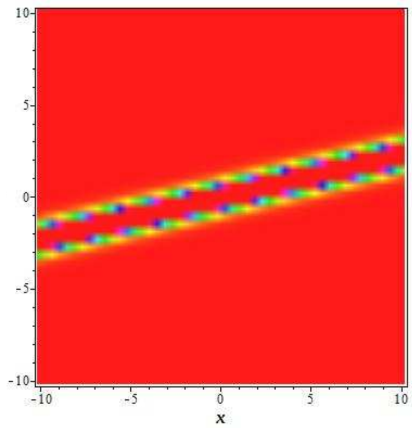

Figure 6: Density plot of the solution for (39) $x \in [-1,1], t \in [-1,1]$

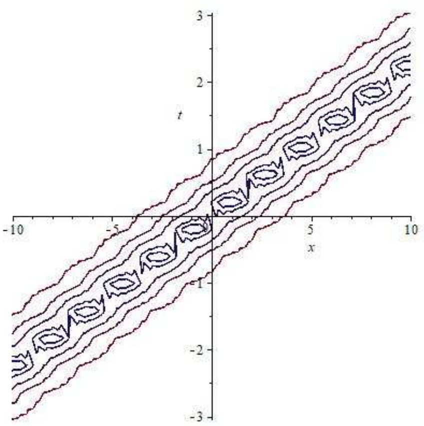

Figure 7: Contour plot of the solution for (38) $x \in [-1,1], t \in [-1,1]$

6. Conclusion

This paper employs the Lie group method to study a class of fifth-order nonlinear evolution equations. The symmetries of the equation are obtained, in the meantime, according to the symmetries, the optimal system of one-dimensional Lie algebra is constructed. Furthermore, symmetry is adopted to get the simplified equation. And, more importantly, two auxiliary function expansion methods are employed to get a series of new exact solutions of equation (2). In the study of beam-column structures, these extraordinary solutions will be of great practical significance. For the time to come, the equation will be further studied, as well as higher order and time coefficients will be considered. As a result, this kind of research is of great physical significance. Such results will be reported in following publications. It is hoped that the results in this article can be helpful for the future research of the field of beam-column structural stability.

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