Progress in Nonlinear Dynamics and Chaos
Vol. 9, No. 1 & 2, 2021, 1-7
ISSN: 2321 – 9238 *(online)*
Published on 14 March 2021
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/pindac.v9n1a08401

Progress in
Nonlinear
Dynamics and Chaos

Approximate Solution of Damped Forced and Damped Oscillatory Motion of Differential Equation of Nonlinear Differential Systems with Varying Parameter

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Received 3 November 2020; accepted 10 March 2021

Abstract. Our aim is to develop and present a new approximate solution of motion of differential equations of nonlinear differential system with damping and external force .The resonance case is considered and second order time dependent nonlinear differential system is calculated. Then a new perturbation technique is developed and applied to find an approximate solution of nonlinear systems in presence of an external force. The process is illustrated by an example. Finally, outcomes are discussed and shown graphically by utilizing MATHEMATICA and MATLAB.

Keywords: Non-linear equations, KB method, Perturbation methods, Damped nonlinear system, Damping force and Varying parameter.

AMS Mathematics Subject Classification (2010): 34A34

1. Introduction

Nonlinear problems acquire a gradually increasing importance in various branches of applied science. Oscillations occur when a system is disturbed from a position of stable equilibrium. This displacement from equilibrium changes periodically over time. Thus, Oscillations are said to be periodic, and display periodic motion. Oscillations are very common in everyday life with familiar examples being the motion of a clock pendulum or the vibrations of strings on musical instruments. Oscillations are also important in many mechanical systems in the real world such as a car suspension. It is thus very important to be able to study and understand these mechanical systems in order to control them in critical situations. Krylov-Bogoliubov-Mitropolskii (KBM) [3,13,18] method is particularly convenient and is the widely used technique to obtain the approximate solutions and Meldelson [12] for damped nonlinear oscillations. Karim et al. [15] KB method for obtaining an approximate solution of slowly varying amplitude and phase of nonlinear differential systems with varying coefficients and also widely used to technique Karim [16] approximate solutions of damped non linear system with varying parameter and damping force. Arya and Bojadziev [1] studied a second order time dependent differential equation with damping, slowly varying coefficients and small time delay in which a non-periodic external force acted. Shamsul [19] has presented a unified method for solving an *n*-th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes. Finally we obtain an Approximate solution of motion and damped forced motion of differential equation of nonlinear differential systems with varying coefficient.

2. Methods and materials

We now consider an important special case of forced motion. That is, we not only consider the effect of damping upon the mass on the spring but also the effect upon it of a periodic external impressed force F defined by F(t) $=F_1 cos\omega t$ for all $t \geq 0$, where F_1 and ω are constants .Then the basic differential equation is

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = F_1 cos\omega t \tag{2.1}$$

Dividing through by m and letting

$$\frac{a}{m} = 2b, \ \ \frac{k}{m} = \lambda^2 \ \text{ and } \ \frac{F_1}{m} = E_1 \tag{2.2}$$

This takes the more convenient form

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \lambda^2 \ x = E_1 cos\omega t \tag{2.3}$$

We shall assume that the positive damping constant a is small enough so that the damping is less than critical .In other words we assume that $b < \lambda$ .Hence by Equation the complementary function of Equation (2.2) can be written

$$x_c = ce^{-bt}\cos(\sqrt{\lambda^2 \ - b^2 t} \ +\emptyset \tag{ 2. 4}$$

We shall now find a particular integral of (2.2) by the method of undetermined coefficients. Let

$$\mathrm{x_p} = \mathrm{Acos\omega t + Bsin\omega t} \tag{2.5}$$

Then $$\frac{dx_p}{dt} = -\omega A sin\omega t + \omega B cos\omega t \tag{2.6}$$

$$\frac{d^2x_p}{dt^2} = -\omega^2 A cos\omega t - \omega^2 B sin\omega t.$$

Substituting into Equation (2.2), we have

[-2bhωA+$(\lambda^2 - \omega^2)B]sin\omega t + [(\lambda^2 - \omega^2)A + 2b\omega B]cos\omega t = E_1 cos\omega t.$

Thus, we have the following two equations from which to determine A and B:

-2bhωA+$(\lambda^2 - \omega^2)B = 0,$

$(\lambda^2 - \omega^2)A + 2b\omega B = E_1,$

Solving these ,we obtain

$A = \frac{E_1(\lambda^2-\omega^2)}{(\lambda^2-\omega^2)^2+4b^2\omega^2}$

$B = \frac{2b\omega E_1}{(\lambda^2-\omega^2)^2+4b^2\omega^2}$

Substituting these values into equation (2.4), we obtain a particular integral in the form

$x_p = \frac{E_1}{(\lambda^2-\omega^2)^2+4b^2\omega^2} \ [\ (\lambda^2 - \omega^2)cos\omega t + 2b\omega sin\omega t \ ]$

We now put this in the alternative "phase angle" form; we write

$$(\lambda^2 - \omega^2)cos\omega t + 2b\omega sin\omega t$$
$$= \sqrt{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}[\frac{(\lambda^2-\omega^2)}{\sqrt{(\lambda^2-\omega^2)^2+4b^2\omega^2}}cos\omega t + \frac{2b\omega}{\sqrt{(\lambda^2-\omega^2)^2+4b^2\omega^2}}sin\omega t]$$

$= \sqrt{(\lambda^2-\omega^2)^2+4b^2\omega^2}$ [cosωtcosθ + sinωtsinθ]
where

$cos\ \theta = \frac{(\lambda^2-\omega^2)}{\sqrt{(\lambda^2-\omega^2)^2+4b^2\omega^2}}$

$sin\ \theta = \frac{2b\omega}{\sqrt{(\lambda^2-\omega^2)^2+4b^2\omega^2}}$

Thus the particular integral in the form

$$x_p = \frac{E_1}{(\lambda^2-\omega^2)^2+4b^2\omega^2}\cos(\omega t-\theta), \tag{2.7}$$

Where θ is determined from Equations (2.6) . Using (2.3) and (2.7) the general solution of Equation (2.2) is

$x = x_c + x_p$

$$= ce^{-bt}\cos(\sqrt{\lambda^2\ -b^2t}\ +\emptyset\ ) + \frac{E_1}{(\lambda^2-\omega^2)^2+4b^2\omega^2}\cos(\omega t-\theta), \tag{2.8}$$

3. Example

The basic differential equation for the motion is

$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = F(t)$

(2.9)

Here $m = \frac{w}{g} = \frac{16}{32} = \frac{1}{2}(slug),\ \ a = 2, k = 10, and\ F(t) = 5cos2t.$

Thus Equation (2.9) becomes

$\frac{1}{2}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 5cos2t$

Or

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 20x = 10cos2t \tag{2.10}$$

The initial conditions are

$$x(0) = 0$$
$$x'(0) = 0. \tag{2.11}$$

The auxiliary equation of the homogeneous equation corresponding to (3.10) is $r^2 + 4r + 20 = 0$; its root are $-2 \pm 4i$.Thus the complementary function of Equation (3.10) is $x_c = e^{-2t}(c_1sin4t + c_2cos4t)$.

where c_1 and c_2 are arbitrary constants .Using the method of undetermined coefficients to obtain a particular integral ,we let ,$x_p = Acos2t + Bsin2t.$

Upon differentiating and substituting into 2.10, we find the following equation for the determination of A and B

$-8A + 16B = 0,$

$16A + 8B = 10.$Solving these , we find $A = \frac{1}{2}$ $, B = \frac{1}{4}$

Thus a particular integral is

$x_p = \frac{1}{2}\cos 2t + \frac{1}{4}sin2t$

And the general solution of 2.10 is

$x = x_c + x_p$

$$= e^{-2t}(c_1sin4t + c_2cos4t) + \frac{1}{2}\cos 2t + \frac{1}{4}sin2t \tag{2.12.}$$

Differentiating (2.12) with respect to t, we obtain

$$\frac{dx}{dt} = e^{-2t}[(-2c_1 - 4c_2)sin4t + (-2c_1 + 4c_2)cos4t - sin2t + \frac{1}{2}cos2t \qquad (2.13)$$

Applying the initial conditions (2.11) to Equations (2.12) and (2.13),we see that

$c_2 + \frac{1}{2} = 0,$

$4c_1$-$2c_2$+$\frac{1}{2} = 0$

From those equations we find that$c_1 = -\frac{3}{8}$, $c_2 = -\frac{1}{2}$

Hence the solution is

$$x = e^{-2t}\left(-\frac{3}{8}sin4t - \frac{1}{2}cos4t\right) + \frac{1}{2}\text{cos2t} + \frac{1}{4}sin2t \qquad (2.14)$$

Let us write this in the "phase angle" form. We have first

$3sin4t + 4cos4t = 5\left(\frac{3}{5}sin4t + \frac{4}{5}cos4t\right) \quad = 5\cos(4t - \emptyset),$

where

$$cos\emptyset = \ 4/5 \ , \qquad sin\emptyset = 3/5 \qquad (2.15)$$

$2cos2t + sin2t = \sqrt{5}\ (\frac{2}{\sqrt{5}}cos2t + \frac{1}{\sqrt{5}}sin2t) = \sqrt{5}\cos(2t - \theta)$

where

$$cos\theta = \frac{2}{\sqrt{5}}, \qquad sin\theta = \frac{1}{\sqrt{5}} \qquad (2.16)$$

Thus we may write the solution (3.14) as

$$x = -\frac{5e^{-2t}}{8}\cos(\ t - \emptyset) + \frac{\sqrt{5}}{4}\cos(2t - \theta), \qquad (2.17)$$

where are determined by Equations, respectively. We find that $\emptyset \approx 0.65$ (rad) and $\theta \approx 0.46$ (rad).

Thus the solution (2.17) is given a approximately by

$$x = -0.63e^{-2t}\cos(4t - 0.64) + 0.56\cos(2t - 0.46)$$

4. Program

<u>Figure under damped and undamped</u>

```
t=[0:0.001:15];
x=exp(-.15*t)*5.*cos(t)*(.45^2-.2^2)^(1/2);
y=exp(-.15*t)*3.*sin(t)*(.45^2-.2^2)^(1/2);
plot(t,x,'r-',t,y,'g-');
```

Figure 1

```
t=[0:0.001:10];
A=1;
f=4;
f1=2;
y1= -0.63 * exp(-2*t).*cos((f*t)-0.64);
plot(t,y1);
```

Figure 2

```
t=[0:0.001:10];
A=1;
f=4;
f1=2;
y2= -0.63 * exp(-2*A)*cos((f*t)-0.64) + 0.56*cos((f1*t)-0.46) ;
plot(t,y2);
```

Figure 3

```
t=[0:0.001:10];
A=1;
f=4;
f1=2;
y3= -0.63 * exp(-2*t).*cos((f*t)-0.64) + 0.56*cos((f1*t)-0.46) ;
plot (t,y3);
```

5. Results and discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one can easily compare the approximate solution to the numerical solution (considered to be exact). The term $-\frac{5e^{-2t}}{8}\cos(4t-\emptyset) \approx -0.63e^{-2t}\cos(4t-0.64)$ is the transient term, representing a damped oscillatory motion. It becomes negligible in a sort time ;for example, for t > 3, its numerical value is less than 0.002. The term$\frac{\sqrt{5}}{4}\cos(2t-\theta) \approx 0.56\cos(2t-0.46)$.Is the steady-state term, representing a simple harmonic motion of amplitude $\frac{\sqrt{5}}{4} \approx 0.56$ and period π. Its graph appears in the following fig. 1. The graph in fig. 2 is that of the complete solution (2.17). It is clear from this that the effect of the transient term soon becomes negligible, and that after a short time the contribution of the steady-state term is essentially all that remains.

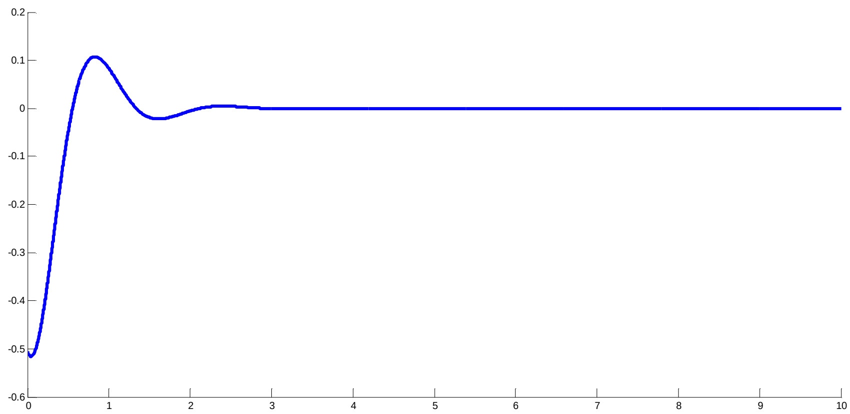

Figure 1: Damped oscillatory motion

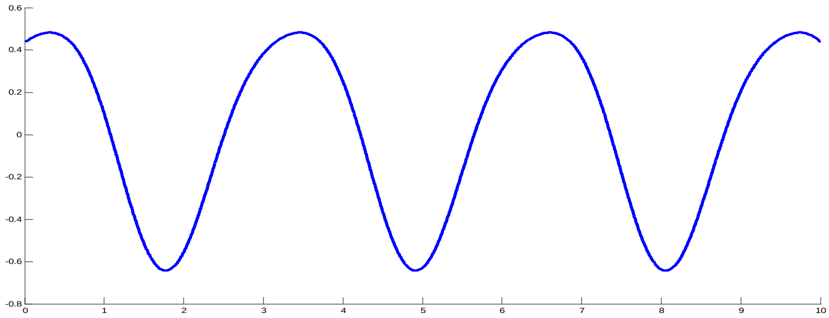

Figure 2: Simple Harmonic motion

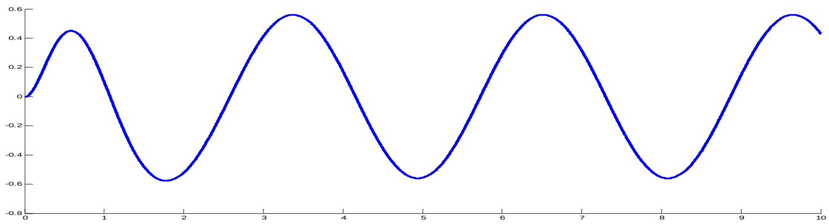

Figure 3: Complete solution of (2.17)

5. Conclusion

In this Chapter a technique is developed for obtaining the solution of nonlinear differential systems under the action of external force. In general, the variational equations for the amplitudes and phase are solved by numerically. In this case, the perturbation method facilitates the numerical method. The method is applied to nonlinear differential systems in presence of external forces. Applying this method in an example and we find a solution. The solutions are obtained for initial conditions. Figures are plotted.

Acknowledgements. The Authors gratefully acknowledge the support provided by Department of Mathematics, Mawlana Bhashani Science and Technology University and also support Department of CSE, Ranada Prasad Shaha University, Narayanganj, Bangladesh. Also, the authors are thankful to the reviewers for their valuable comments for improvement of the paper.

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