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Progress in Nonlinear Dynamics and Chaos

Vector Valued Matrices and their Product

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Abstract. In this paper, we have generalized the idea of matrices of scalar field by define odd-dimensional vector valued matrices. Various type of vector valued matrices such that row (column), null, identity, diagonal, upper and lower triangular vector valued matrices are also defined. Various type of operations such as addition, subtraction, multiplication, etc. are also discussed. To implement multiplication on odd-dimensional vector valued matrices, we have defined two type of multiplication dot and cross-multiplication. Specially, for cross-multiplication of two vector valued matrices we first have used the generalized definition of vector cross product (VCP) in odd-dimensional space by extended the definition of VCP defined by Eckmann to an odd -dimensional space by introducing cross term which was proposed by Xiu-Lao Tian, Chao Yang, Yang Ho & Chao Tian in their papers. This proposed generalized definition can be reduced to Eckmann's definition in three and seven dimensional vector space. Based on these algorithms we defined cross-multiplication between two odd-dimensional vector valued matrices.

Keywords: Product of n-dimensional vectors, vector valued matrix, product of vector valued matrices

AMS Mathematics Subject Classification (2010): 15A60, 20H20

1. Introduction

A matrix is a very important mathematical tool. It has various application in the field of Mathematics, Physics etc. We already have used matrices of scalar field of different order and their various properties. In this article, an extension on conventional matrix is investigated called vector values matrices. In this matrix, each element of the matrix is a vector instead of a scalar.

In this paper, the idea of matrices with scalar field is generalised by defining 'odd-multi dimensional vector valued matrices' i.e. matrices with each element is a vector of odd multi-dimensional. Here various properties of this type of matrices, such as matrix addition, subtraction, multiplication, determinant etc, are also discussed.

Specially for the case of matrix multiplication, we have used generalised concept of odd multi-dimensional vector cross product by generalizing Eckmann [2] axioms which was given by Tian et al. [12]. Here, vector cross product in n-dimensional vector space is defined. Multi-dimensional vector product is defined by Silagadze [14]. For other works on vector product see [1,2,3,4,7,9,10,13] and multi-dimensional vector product are used in [5,6,8].

Several other types of matrices are available on fuzzy setup. There are some limitations in dealing with uncertainties by fuzzy set. Pal et al. defined intuitionistic fuzzy determinant in 2001 [29] and intuitionistic fuzzy matrices (IFMs) in 2002 [30]. Bhowmik and Pal [19] introduced some results on IFMs, intuitionistic circulant fuzzy matrix and generalized intuitionistic fuzzy matrix [19-25]. Shyamal and Pal [36-38] defined the distances between IFMs and hence defined a metric on IFMs. They also cited few applications of IFMs. In [28], the similarity relations, invertibility conditions and eigenvalues of IFMs are studied. Idempotent, regularity, permutation matrix and spectral radius of IFMs are also discussed. The parameterizations tool of IFM enhances the flexibility of its applications. For other works on IFMs see [16-18,27,33,34,37,38]. The concept of interval-valued fuzzy matrices (IVFMs) as a generalization of fuzzy matrix was introduced and developed in 2006 by Shaymal and Pal [39] by extending the max-min operation in fuzzy algebra. For more works on IVFMs see [32]. Combining IFMs and IVFMs, a new fuzzy matrix called interval-valued intuitionistic fuzzy matrices (IVIFMs) is defined [26]. For other works on IVIFMs, see [23,25]. For more recent works on fuzzy matrices see [40-43].

1.1. Definition of odd multi-dimensional vector valued matrix

A rectangular array of mn elements A_{ij} into m rows and n columns, where the elements A_{ij}'s are the vectors i.e. of the form $(a_1^{ij}, a_2^{ij}, \dots, a_k^{ij})$ where $a_m^{ij} \in F$(scalar field), belong to a vector space $V^K$of k dimension is called an odd multi-dimensional vector valued matrix for $k \geq 2$.

A $m \times n$ order k-dimensional vector valued matrix is exhibited in the form

$$\begin{bmatrix} (a_1^{11}, a_2^{11}, \dots a_k^{11}) & (a_1^{12}, a_2^{12}, \dots a_k^{12}) & \cdots & (a_1^{1n}, a_2^{1n}, \dots a_k^{1n}) \\ (a_1^{21}, a_2^{21}, \dots a_k^{21}) & (a_1^{22}, a_2^{22}, \dots a_k^{22}) & \cdots & (a_1^{2n}, a_2^{2n}, \dots a_k^{2n}) \\ \cdots & \cdots & & \cdots \\ (a_1^{n1}, a_2^{n1}, \dots a_k^{n1}) & (a_1^{n2}, a_2^{n2}, \dots a_k^{n2}) & \cdots & (a_1^{nn}, a_2^{nn}, \dots a_k^{nn}) \end{bmatrix}$$

1.3. Various type of vector valued matrices

Row and column vector valued matrix

In a $m \times n$ VVM (vector valued matrix (VVM) if $m = 1$,then the VVM is called r*ow VVM.*

e.g. $\begin{bmatrix} (1,2,3) & (0,0,1) & (1,0,1) \end{bmatrix}$ etc.

When $n = 1$, then the VVM is called column VVM.

e.g. $\begin{bmatrix} (1,4,5) \\ (2,0,1) \\ (0,0,4) \end{bmatrix}$ etc.

Null vector valued matrix

If each element of a VVM be zero vector then the VVM is called *Null VVM*. A $m \times n$ order k-dimensional Null VVM is denoted as $O_{m,n}$.

e.g. $\begin{bmatrix} (0,0.0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{bmatrix}$ etc.

Square vector valued matrix

A VVM is said to be a *square VVM* if the number of rows of it is equal to the number of column of it.

$$\text{e.g.} \begin{bmatrix} (1,5,9) & (0,0,0) \\ (1,0,8) & (0,4,3) \end{bmatrix}, \begin{bmatrix} (1,0,0) & (3,9,0) & (2,0,1) \\ (3,0,0) & (5,0,0) & (0,1,0) \\ (0,4,9) & (0,0,1) & (0,0,4) \end{bmatrix} \text{ etc.}$$

Diagonal vector valued matrix

A Square VVM is called *Diagonal matrix*,if all of its non-diagonal elements are zero vector.

$$\text{e.g.} \begin{bmatrix} (1,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,0) \\ (0,0,0,0,0) & (0,1,0,0,0) & (0,0,0,0,0) \\ (0,0,0,0,0) & (0,0,0,0,0) & (0,0,0,0,1) \end{bmatrix} \text{ etc.}$$

Identity or unit vector valued matrix

A Square VVM is said to be *Identity* or *Unit VVM*, if all diagonal elements of it are equal to unit vector and non-diagonal elements are all zero vectors. A $n \times n$ order k-dimensional VVM is denoted as I_n^k.

$$\text{e.g.} \begin{bmatrix} (1,1,1) & (0,0,0) \\ (0,0,0) & (1,1,1) \end{bmatrix}, \begin{bmatrix} (1,1,1,1,1) & (0,0,0,0,0) & (0,0,0,0,0) \\ (0,0,0,0,0) & (1,1,1,1,1) & (0,0,0,0,0) \\ (0,0,0,0,0) & (0,0,0,0,0) & (1,1,1,1,1) \end{bmatrix} \text{ etc.}$$

Upper and lower triangular vector valued matrix

A Square VVM is said to be *Upper Triangular VVM* if all elements of it below the leading diagonal are zero vectors and it is said to be *Lower Triangular VVM* if all elements of it above the leading diagonal are zero vectors.

e.g. of upper &lower triangular vector valued matrix

$$\begin{bmatrix} (1,3,4) & (4,4,5) & (1,9,8) \\ (0,0,0) & (9,0,0) & (0,7,0) \\ (0,0,0) & (0,0,0) & (1,0,1) \end{bmatrix}, \begin{bmatrix} (1,1,2,3,4) & (0,0,0,0,0) & (0,0,0,0,0) \\ (0,0,1,0,1) & (1,0,0,0,1,1)) & (0,0,0,0,0) \\ (0,5,4,0,1) & (5,0,5,0,7) & (0,9,8,7,1) \end{bmatrix} \text{etc.}$$

1.3. Various type of algebraic operations on vector valued matrices

We consider Vector Valued Matrices of same dimension.

Addition

Two vector valued matrices A and B are said be Conformal for Addition if they have same order.

If $A = (a_1^{ij}, a_k^{ij}, \dots a_k^{ij})_{m,n}$ and $B = (b_1^{ij}, b_k^{ij}, \dots b_k^{ij})_{m,n}$ be two k dimensional vector valued matrices of order $\times n$. Then their sum is a k dimensional vector valued matrix C of order $m \times n$ and it is defined as

$$C = (c_1^{ij}, c_2^{ij}, \dots, c_k^{ij})_{m,n} = (a_1^{ij} + b_1^{ij}, a_2^{ij} + b_2^{ij}, \dots, a_k^{ij} + b_k^{ij})_{m,n}$$

Example 1.1.

Let $A = \begin{bmatrix} (1,0,1) & (0,1,3) & (2,1,0) \\ (9,1,1) & (1,0,0) & (1,1,1) \\ (1,1,0) & (1,1,2) & (0,4,5) \end{bmatrix}$ and $B = \begin{bmatrix} (2,2,1) & (1,2,3) & (1,0,0) \\ (4,6,3) & (2,0,1) & (0,1,0) \\ (0,0,1) & (0,0,9) & (0,2,0) \end{bmatrix}$ then

$$C = A + B = \begin{bmatrix} (3,2,2) & (1,3,6) & (3,1,0) \\ (13,7,4) & (3,0,1) & (1,2,1) \\ (1,1,1) & (1,1,11) & (0,6,5) \end{bmatrix}.$$

If A and B two vector valued matrices of different order and different dimensions then Addition is not defined.

Lemma 1. The Addition of two VVM of same order is commutative.

Lemma 2. Addition of VVM is associative.
Proof of the above two lemmas are obvious.

Subtraction

Two vector valued matrices A and B are said be Conformal for Subtraction if they have same order.
If $A = (a_1^{ij}, a_k^{ij}, \dots a_k^{ij})_{m,n}$ and $B = (b_1^{ij}, b_k^{ij}, \dots b_k^{ij})_{m,n}$ be two k(odd) dimensional vector valued matrices of order $\times n$. Then their difference is an k(odd) dimensional vector valued matrix C of order $m \times n$ and it is defined as

$$C = (c_1^{ij}, c_2^{ij}, \dots, c_k^{ij})_{m,n} = (a_1^{ij} - b_1^{ij}, a_2^{ij} - b_2^{ij}, \dots, a_k^{ij} - b_k^{ij})_{m,n}$$

Example 1.2.

Let $A = \begin{bmatrix} (1,2,3,1,0) & (2,0,0,1,9) \\ (4,5,7,1,3) & (0,0,1,2,1) \end{bmatrix}$ and $B = \begin{bmatrix} (1,0,1,2,1) & (2,1,1,7,7) \\ (4,5,7,1,3) & (1,0,0,1,1) \end{bmatrix}$
Then $C = A - B = \begin{bmatrix} (0,2,2,-1,-1) & (0,-1,-1,-6,2) \\ (2,5,4,1,-2) & (-1,0,1,1,0) \end{bmatrix}$ etc.

Scalar multiplication

The product of a $m \times n$ order k-dimensional VVM , $A = (a_1^{ij}, a_k^{ij}, \dots a_k^{ij})_{m,n}$by a scalar c where $c \in F$, the field of scalars, is a $m \times n$ order k(odd)-dimensional VVM, $B = (b_1^{ij}, b_k^{ij}, \dots b_k^{ij})_{m,n}$ defined by

$\left(b_1^{ij}, b_k^{ij}, \ldots b_k^{ij}\right) = \left(c \cdot a_1^{ij}, c \cdot a_k^{ij}, \ldots c \cdot a_k^{ij}\right), i = 1,2, \cdots, m; j = 1,2, \cdots, n;$ and it can be written as $B = cA$.

Example 1.3.

Let $A = \begin{bmatrix} (1,0,1) & (0,1,1) & (2,0,1) \\ (5,3,2) & (0,2,0) & (1,1,2) \\ (4,5,6) & (1,9,3) & (0,5,6) \end{bmatrix}$ be a 3-dimensional 3×3 order VVM then scalar multiplication of A by 2 is given by $2A$ and $2A = \begin{bmatrix} (2,0,2) & (0,2,2) & (4,0,2) \\ (10,6,4) & (0,4,0) & (2,2,4) \\ (8,10,12) & (2,18,6) & (0,10,12) \end{bmatrix}$, etc.

Let A be a $m \times n$ order odd-dimensional VVM and c, d are scalars.Then the following results are obvious.

a) $c(dA) = (cd)A$,
b) $0A = O_{m,n}$; 0 being the zero element of$F$,
c) $cO_{m,n} = O_{m,n}$,
d) $1A = A$, 1 being the identity element of F.

Multiplication
Here the matrices are of vector valued. Hence we define two type of multiplication for such matrices. These are scalar multiplication (dot product) and vector multiplication (cross product).

Dot product between two VVM
Two VVM, A and B are said to be conformal for the dot product if they have the same dimensions and the number of columns of A is equal to the number of rows of B. If $A = (a_1^{ij}, a_k^{ij}, \ldots a_k^{ij})_{m,n}$and $B = (b_1^{ij}, b_k^{ij}, \ldots b_k^{ij})_{n,p}$ then the dot product between A and B (denoted as $A \cdot B$) is a scalar matrix C of order $m \times p$ defined as,

$A \cdot B = C = (c_1^{ij}, c_2^{ij}, \ldots, c_k^{ij})_{m,p}$ where $\left(c_1^{ij}, c_2^{ij}, \ldots, c_k^{ij}\right) = \sum_{t=1}^{n} (a_1^{ij}, a_2^{ij}, \ldots, a_k^{ij}) \times (b_1^{ij}, b_2^{ij}, \ldots, b_k^{ij}) = \left(a_1^{it} \cdot b_1^{tj}, a_2^{it} \cdot b_2^{tj}, \ldots, a_k^{it} . b_k^{tj}\right),\ i = 1,2, \cdots, m;\ j = 1,2, \cdots, p.$

Example 1.4.
Let us consider two 3-dimensional VVMs each of order 2 × 2 with each element are of the Euclidean space R^3 such that

A=$\begin{bmatrix} (1,2,3) & (1,0,1) \\ (0,1,2) & (5,2,9) \end{bmatrix}$ and B=$\begin{bmatrix} (4,2,3) & (9,1,1) \\ (1,2,8) & (6,7,9) \end{bmatrix}$

Then A·B = $\begin{bmatrix} (1,2,3) & (1,0,1) \\ (0,1,2) & (5,2,9) \end{bmatrix} \cdot \begin{bmatrix} (4,2,3) & (9,1,1) \\ (1,2,8) & (6,7,9) \end{bmatrix}$

$$= \begin{bmatrix} (1 \cdot 4 + 2 \cdot 2 + 3 \cdot 3 + 1 \cdot 1 + 0 \cdot 2 + 1 \cdot 8) & (1 \cdot 9 + 1 \cdot 2 + 1 \cdot 3 + 6 \cdot 1 + 0 \cdot 7 + 9 \cdot 1) \\ (0 \cdot 4 + 1 \cdot 2 + 2 \cdot 3 + 5 \cdot 1 + 2 \cdot 2 + 9 \cdot 8) & (9 \cdot 0 + 1 \cdot 1 + 2 \cdot 1 + 5 \cdot 6 + 2 \cdot 7 + 9 \cdot 9) \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 29 \\ 89 & 136 \end{bmatrix}$$

Here A·B is a 2 × 2 order scalar matrix.

Before define the cross product between two odd-dimensional VVMs, we have to discuss about generalised multi-dimensional cross product. So first we will Introduce Generalised multi-dimensional cross product for odd dimensions briefly in the next section.

2. Vector cross product in odd *n*-dimensional vectors

Let V denote the n-dimensional vector space over over the real numbers and $\langle , \rangle$
Denote the ordinary (positive definite) vector inner product . The **VCP** (**V**ector **C**ross **P**roduct) $\vec{A} \times \vec{B}$ of any two vectors on V has been defined by B. Eckmann satisfying the following axioms

$$\langle \vec{A} \times \vec{B}, \vec{A}\ or\ \vec{B} \rangle = 0 \tag{1}$$

$$\left\| \vec{A} \times \vec{B} \right\|^2 = \left\| \vec{A} \right\|^2 \left\| \vec{B} \right\|^2 - \langle \vec{A}, \vec{B} \rangle^2 \tag{2}$$

Considering arbitrary two vectors $\vec{A} = a^i \hat{e}_i$ and$\vec{B} = b^j \hat{e}_j$ in n-dimensional vector space , $\hat{e}_i$ and $\hat{e}_j$ are basis vectors from the given orthogonal coordinate system , $a^i$and b^i are vector components corresponding to $\vec{A}\ and\ \vec{B}$,
The VCP $\vec{A} \times \vec{B}$ can be expressed as,

$$\vec{A} \times \vec{B} = a^i \hat{e}_i \times b^j \hat{e}_j = a^i b^j \hat{e}_i \times \hat{e}_j \tag{3}$$

Obviously, the magnitude of the cross product is determined by these vector components a^i, b^i and the direction is determined by basis vectors $\hat{e}_i \times \hat{e}_j$.

In the following a generalized definition of VCP is presented based on the orthogonal completeness, magnitude of VCP and all kinds of combinations of basis vector.

Firstly, orthogonal completeness of VCP requires the VCP only exist in an odd n-dimensional space.

As $\langle \vec{A} \times \vec{B}, \vec{A}\ or\ \vec{B} \rangle = 0$, the cross product of any two vector is always perpendicular to both of the vectors being multiplied and a plane containing them(orthogonality of VCP) and $\hat{e}_i \times \hat{e}_j$ of any two basis vectors must be equal to another basis vector $\hat{e}_k$,i.e. $\hat{e}_i \times \hat{e}_j = \pm \hat{e}_k$ (completeness of the VCP).

Based on the definition of $\hat{e}_i \times \hat{e}_j = \hat{e}_k$, a cross product $\hat{e}_i \times \hat{e}_j$ of any two basis vector is Is equivalent to a 2-conbination of two basis vector $\hat{e}_i$ and $\hat{e}_j$. There are n basis vectors in n-dimensional vector space, the number of 2-combination is that the number of combinations of n basis vectors taken 2 vectors at a time without repetitions. The number of 2-combinations of arbitary two basis vectors in an n-dimensional vector space is $C_2^n = \frac{1}{2} n(n-1)$.

Taking the equality of each basis vector into account, so C_2^n should be averagely distributed to each one basis vector. The number K is denoted by

$$K = \frac{C_2^n}{n} = \frac{1}{2}(n-1) \quad (4)$$

Then $$n = 2K + 1. \quad (5)$$

So the VCP of two vectors there only exist in an odd n-dimensional space.
Secondly, the definition of magnitude of the VCP in Eq. (2) can be generalized to

$$\left\|\vec{A} \times \vec{B}\right\|^2 = \left\|\vec{A}\right\|^2 \left\|\vec{B}\right\|^2 - \langle \vec{A}, \vec{B}\rangle^2 + X_{AB} \quad (6)$$

where X_{AB} is called cross item and it is expressed as

$$X_{AB} = a_i b_j a^l b^m \chi_{lm}^{ij} = a_i b_j a^l b^m [T_{lm}^{ij} + \delta_m^i \delta_l^j - \delta_m^j \delta_l^i] \quad (7)$$

$$T_{ij}^{lm} = \langle (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k \rangle \langle (\hat{e}_l \times \hat{e}_m) \cdot \hat{e}_k \rangle \quad (8)$$

where T_{ij}^{lm} is a sign function and $\langle (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k \rangle$denote the vector inner product of $(\hat{e}_i \times \hat{e}_j)$ and $\hat{e}_k$.
Subsequently, the generalised definition and calculation formula of the VCP in an odd n-dimensional space will be presented.

Proposition 1. The VCP $\vec{A} \times \vec{B}$ of any two vectors on an odd-dimensional space satisfy the following generalized axioms:

$$\langle \vec{A} \times \vec{B}, \vec{A}\ or\ \vec{B} \rangle = 0 \quad (9)$$

$$\left\|\vec{A} \times \vec{B}\right\|^2 = \left\|\vec{A}\right\|^2 \left\|\vec{B}\right\|^2 - \langle \vec{A}, \vec{B}\rangle^2 + X_{AB} \quad (10)$$

2.1. Algorithm of VCP in an odd n-dimensional space

As we know, there is only an algorithm of VCP in 3-dimensional space. Although the definition of VCP has been extended to an odd n-dimensional space (n>3), the algorithm of the VCP is not unique. Owing to the diversity of the combination of basis vector, there are many kind of algorithm of the VCP in odd n-dimensional space.
So -called an algorithm depend on the calculation rule. In the following section various type of algorithms are discussed.

2.1.1. Algorithm for the VCP in 3-dimensional space

Obviously, there are 3 basis vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ in 3-dimensional vector space, one of the basis vector can be expressed by the cross product of other two basis vectors. From Eq. (4), one can find k=1, so one of the basis vector can be expressed by one 2-combination of basis vector uniquely, which is demonstrate in Table 1.

$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$
$\hat{e}_2 \times \hat{e}_3$	$\hat{e}_3 \times \hat{e}_1$	$\hat{e}_1 \times \hat{e}_2$

Table 1: One kind of distributive combination form of $\hat{e}_i$ in 3-dimensional vector space

Here right handed rotation rule is adopted in constructing algorithm.So for any two 3-dimensional vector (x_1, x_2, x_3) and $(y_1, y_2, y_3)$the cross product is another 3-dimensional vector defined as $(x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$.

2.1.2. Algorithm for the VCP in 5-dimensional space

There are 5 basis vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5)$ in 5-dimensional vector space, one of the basis vector can be expressed by the cross product of other two basis vectors. From Eq. (4), one can find k=2, so one of the basis vector can be expressed by two 2-combination of basis vector, which is demonstrate in Table 2.

Kind	$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$	$\hat{e}_4$	$\hat{e}_5$
1.	23,45	15,34	12,45	13,25	14,23
2.	24,35	14,35	14,25	12,35	12,34
3.	25,34	13,45	15,24	15,23	13,24

Table 2: Three kinds of distributive combination form of $\hat{e}_i$ in 5-dimensional vector space

From the above Table 2, e.g. the basis vector $\hat{e}_1$ can be expressed by $\hat{e}_2 \times \hat{e}_3$ (23) and $\hat{e}_4 \times \hat{e}_5$ (45). Here, a double-digit is used to denote the cross product of two basis vector for simplicity. The (23,45) of two 2-combination of basis vector is called one kind of distributive combination form,

There are three kinds of different distributive combination forms under each basis vector $\hat{e}_i$. For example, there are 3 kinds of distributive combination forms (23,45) , (24,35) , (25,34) under $\hat{e}_1$ in the Table 2.

Furthermore, a pair of double-digit from each column of Table 2 is taken to constitute a kind of calculation rule, and the calculation rule demands all double-digit of five paif of double-digit which be extracted is different. Then, let these different double-digit arrange a row to represents a calculation rule. Accordingly, six kinds of different calculation of basis vector obtain in the Table 3 below.

algorithm	$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$	$\hat{e}_4$	$\hat{e}_5$
1.	23,45	14,35	15,24	13,25	12,34
2.	23,45	15,34	14,25	12,35	13,24
3.	**24,35**	**13 45**	**14,25**	**15,23**	**12,34**
4.	24,35	15,34	12,45	13,25	14,23
5.	25,34	13,45	15,24	12,35	14,23
6.	25,34	14,35	12,45	15,23	13,24

Table 3: Six kinds of algorithms of cross product in 5-dimensional space

That is to say, the VCP in 5-dimensional space has six sorts of different algorithms. Thus, one can find a kind of cross product algorithm of basis vector by selecting combination form form a row double-digit from Table 3. For example, the relation of cross product of basis vector from 3rd row in Table 3 can be expressed as

$$\hat{e}_2 \times \hat{e}_4 = \hat{e}_1, \hat{e}_3 \times \hat{e}_5 = \hat{e}_1, \hat{e}_3 \times \hat{e}_1 = \hat{e}_2, \hat{e}_4 \times \hat{e}_5 = \hat{e}_2, \hat{e}_4 \times \hat{e}_1 = \hat{e}_3$$
$$\hat{e}_5 \times \hat{e}_2 = \hat{e}_3, \hat{e}_5 \times \hat{e}_1 = \hat{e}_4, \hat{e}_2 \times \hat{e}_3 = \hat{e}_4, \hat{e}_1 \times \hat{e}_2 = \hat{e}_5, \hat{e}_3 \times \hat{e}_4 = \hat{e}_5.$$

The above relation of cross product of basis vector shows a kind of algorithm of the VCP . Under right-handed orthogonal coordinate frame ,the cross product of basis vector has to satisfy right-handed rotation rule. The relation of cross product of basis vector can be expressed as

$$\hat{e}_i \times \hat{e}_j = L_{ij\,k}\hat{e}_k\,, \tag{13}$$

where $L_{ij\,k}$ is the generalized Levi-Civita symbol.

$$L_{ij\,k} = \begin{cases} 1 & when\ (ij\ k)\ is\ even\ permutation \\ -1 & when\ (ij\ k)\ is\ odd\ permutation \end{cases}$$

In terms of the relation of basis vectors from Eq. (13) , the VCP $\vec{A} \times \vec{B}$ can be computed by the following

$$\vec{A} \times \vec{B} = a^i b^j \hat{e}_i \times \hat{e}_j$$

$$= \left[\begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} + \begin{vmatrix} a_3 & a_5 \\ b_3 & b_5 \end{vmatrix}\right]\hat{e}_1 + \left[\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \begin{vmatrix} a_4 & a_5 \\ b_4 & b_5 \end{vmatrix}\right]\hat{e}_2 + \left[\begin{vmatrix} a_4 & a_1 \\ b_4 & b_1 \end{vmatrix} + \begin{vmatrix} a_5 & a_2 \\ b_5 & b_2 \end{vmatrix}\right]\hat{e}_3 + \left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_5 & a_1 \\ b_5 & b_1 \end{vmatrix}\right]\hat{e}_4 + \left[\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}\right]\hat{e}_5 \tag{14}$$

Let $X_{\alpha\beta} = \begin{vmatrix} a_\alpha & a_\beta \\ b_\alpha & b_\beta \end{vmatrix}$, then $\vec{A} \times \vec{B}$ can be expressed as

$$\vec{A} \times \vec{B} = [X_{24} + X_{35}]\hat{e}_1 + [X_{31} + X_{45}]\hat{e}_2 + [X_{41} + X_{52}]\hat{e}_3 + [X_{23} + X_{51}]\hat{e}_4 + [X_{12} + X_{34}]\hat{e}_5 \tag{15}$$

Moreover, we can prove $\|\vec{A} \times \vec{B}\|^2 = \|\vec{A}\|^2\|\vec{B}\|^2 - \langle\vec{A},\vec{B}\rangle^2 + X_{AB}$. Where $X_{AB} = a_i b_j a^l b^m \chi_{lm}^{ij}$ is as follows

$$X_{AB} = a_i b_j a^l b^m \chi_{lm}^{ij} = 2([X_{24}X_{35}] + [X_{31}X_{45}] + [X_{41}X_{52}] + [X_{23}X_{51}] + [X_{12}X_{34}]) \neq 0$$

2.1.3. Algorithm for the VCP in 7-dimensional space

Similarly, we can find K=3 in 7-dimensional space using Eq. (4). So one of the basis vector can be expressed by three 2-combination of basis vectors. In 7-dimension there are 15 kinds of different different distributive combination forms and there are 6240 kinds of non-repeating algorithms. Out of them three kinds of distributive combinational forms of basis vector $\hat{e}_i$ and three different kinds of algorithms for the cross product in 7-dimension is illustrated by Table 4 and Table 5 respectively below:

Kind	$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$	$\hat{e}_4$	$\hat{e}_5$	$\hat{e}_6$	$\hat{e}_7$
1.	23,47,56	13,56,47	16,27,45	13,27,56	17,26,34	17,25,34	14,26,35
2.	24,37,56	14,35,67	17,25,46	12,36,57	16,23,47	15,27,34	13,26,45
3.	25,34,67	15,36,47	15,26,47	15,23,67	14,23,67	14,23,57	15,23,46

Table 4: Three kinds of distributive combination form of $\hat{e}_i$ in 7-dimensional vector space

algorithm	$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$	$\hat{e}_4$	$\hat{e}_5$	$\hat{e}_6$	$\hat{e}_7$
1.	23,47,56	13,45,67	12,46,57	17,25,36	16,24,37	15,27,34	14,26,35
2.	**24,37,56**	**14,35,67**	**17,25,46**	**12,36,57**	**16,23,47**	**15,27,34**	**13,26,45**
3.	25,34,67	15,36,47	14,26,57	13,27,56	12,37,46	17,23,45	16,24,35

Table 5: Three kinds of algorithms of cross product in 7-dimensional space

Now using the 2nd algorithm of Table 5 and using Eq. (13) we can define the VCP for any two vectors $\vec{A} = a^i \hat{e}_i$ and $\vec{B} = b^j \hat{e}_j$ in 7-dimensional space. The $\vec{A} \times \vec{B}$
Can be expressed as

$$\vec{A} \times \vec{B} = [X_{24} + X_{37} + X_{56}]\hat{e}_1 + [X_{41} + X_{35} + X_{67}]\hat{e}_2 + [X_{71} + X_{52} + X_{46}]\hat{e}_3 + [X_{12} + X_{63} + X_{57}]\hat{e}_4 + [X_{61} + X_{23} + X_{74}]\hat{e}_5 + [X_{15} + X_{72} + X_{34}]\hat{e}_6 + [X_{13} + X_{26} + X_{45}]\hat{e}_7$$

Certainly, one can verify

$$X_{AB} = a_i b_j a^l b^m \chi^{ij}_{lm} = 2\{[X_{24}X_{37} + X_{24}X_{56} + X_{37}X_{56}] + [X_{41}X_{35} + X_{41}X_{67} + X_{35}X_{67}] + [X_{71}X_{52} + X_{71}X_{46} + X_{52}X_{46}] + [X_{12}X_{63} + X_{12}X_{57} + X_{63}X_{57}] + [X_{61}X_{23} + X_{61}X_{74} + X_{23}X_{74}] + [X_{72}X_{15} + X_{34}X_{15} + X_{72}X_{34}] + [X_{13}X_{26} + X_{13}X_{45} + X_{26}X_{45}]\} = 0$$

Remark 1. For other odd-dimensions, aforesaid algorithms can be easily applied after generalizing those into higher dimensions.

Now we are able to define cross multiplication between two odd- dimensional VVMs by using algorithms of VCP for odd multi-dimensional vector spaces which are illustrated by examples below:

3. Cross product between two odd-dimensional VVM

Definition 3.1. Two odd-dimensional VVM, A and B are said to be conformal for the cross product if they have the same dimensions and the number of columns of *A* is equal to the number of rows of *B*. If , $A = (a^{ij}_1, a^{ij}_k, \dots a^{ij}_k)_{m,n}$ and $B = (b^{ij}_1, b^{ij}_k, \dots b^{ij}_k)_{n,p}$then the dot product between *A* and *B* (denoted as $A \times B$) is a k-dimensional VVM matrix *C* of order m × p defined as

$$A \times B = C = (c^{ij}_1, c^{ij}_2, \dots, c^{ij}_k)_{m,p}$$

where $$\left(c_1^{ij}, c_2^{ij}, \dots, c_k^{ij}\right) = \sum_{t=1}^{n}(a_1^{ij}, a_2^{ij}, \dots, a_k^{ij}) \times (b_1^{ij}, b_2^{ij}, \dots, b_k^{ij})$$

Example 3.1. (cross multiplication of two 3-dimensional VVMs)
Let us consider two 2×2 order 3-dimensional real VVMs,

$$A = \begin{bmatrix} (1,2,3) & (1,1,0) \\ (3,2,0) & (4,0,0) \end{bmatrix} \text{and } B = \begin{bmatrix} (1,0,1) & (2,3,1) \\ (2,3,6) & (0,0,1) \end{bmatrix}$$

then $A \times B =$

$$\begin{bmatrix} (1,2,3) \times (1,0,1) + (1,1,0) \times (2,3,6) & (1,2,3) \times (2,3,1) + (1,1,0) \times (0,0,1) \\ (3,2,0) \times (1,0,1) + (4,0,0) \times (2,3,6) & (3,2,0) \times (2,3,1) + (4,0,0) \times (0,0,1) \end{bmatrix}$$

Now elementary cross products are calculated with the help of (*) as follows:

$$(1,2,3) \times (1,0,1) = (2 \cdot 1 - 0 \cdot 3\,, 3 \cdot 1 - 1 \cdot 1\,, 1 \cdot 0 - 2 \cdot 1) = (2,2,-2)$$
$$(1,1,0) \times (2,3,6) = (6 \cdot 1 - 3 \cdot 0\,, 0 \cdot 2 - 1 \cdot 6\,, 1 \cdot 3 - 1 \cdot 2) = (6,-6,1)$$
$$(1,2,3) \times (2,3,1) = (2 \cdot 1 - 3 \cdot 3\,, 3 \cdot 2 - 1 \cdot 1\,, 1 \cdot 3 - 2 \cdot 2) = (-7,5,-1)$$
$$(1,1,0) \times (0,0,1) = (1 \cdot 1 - 0 \cdot 0\,, 0 \cdot 0 - 1 \cdot 1\,, 1 \cdot 0 - 1 \cdot 0) = (1,-1,0)$$
$$(3,2,0) \times (1,0,1) = (2 \cdot 1 - 0 \cdot 0\,, 0 \cdot 1 - 3 \cdot 1\,, 3 \cdot 0 - 2 \cdot 1) = (2,-3,-2)$$
$$(4,0,0) \times (2,3,6) = (0 \cdot 6 - 0 \cdot 3\,, 0 \cdot 2 - 4 \cdot 6\,, 4 \cdot 3 - 0 \cdot 2) = (0,-24,12)$$
$$(3,2,0) \times (2,3,1) = (2 \cdot 1 - 3 \cdot 0\,, 0 \cdot 2 - 3 \cdot 1\,, 3 \cdot 3 - 2 \cdot 2) = (2,-3,5)$$
$$(4,0,0) \times (0,0,1) = (0 \cdot 1 - 0 \cdot 0\,, 0 \cdot 0 - 4 \cdot 1\,, 4 \cdot 0 - 0 \cdot 0) = (0,-4,0)$$

Hence $A \times B = \begin{bmatrix} (8,-4,-1) & (-6,4,-1) \\ (2,-27,10) & (2,-7,5) \end{bmatrix}$

Example 3.2. (cross multiplication of two 5-dimensional VVMs)
Let us consider two real VVMs, one of which is of order 2×2 and another of order 2×1 with each element are belongs to the space R^5 such that

$$A = \begin{bmatrix} (1,0,0,1,1) & (1,0,1,1,0) \\ (2,1,0,1,3) & (2,3,4,1,2) \end{bmatrix} \text{ and } B = \begin{bmatrix} (1,3,2,1,0) \\ (2,0,2,0,1) \end{bmatrix} \text{then}$$

$$A \times B = \begin{bmatrix} (1,0,0,1,1) \times (1,3,2,1,0) + (1,0,1,1,0) \times (2,0,2,0,1) \\ (2,1,0,1,3) \times (1,3,2,1,0) + (2,3,4,1,2) \times (2,0,2,0,1) \end{bmatrix}$$

Now elementary cross products in 5-dimension, are calculated with the help of (14) as follows:

$$(1,0,0,1,1) \times (1,3,2,1,0)$$
$$= \left\{\left[\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}\right], \left[\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right], \left[\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}\right], \left[\begin{vmatrix} 0 & 0 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\right], \left[\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\right]\right\} = (-5,-3,3,1,1)$$

$$(1,0,1,1,0) \times (2,0,2,0,1)$$
$$= \left\{\left[\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}\right], \left[\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right], \left[\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\right], \left[\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}\right], \left[\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}\right]\right\} = (1,1,2,-1,-2)$$

$(2,1,0,1,3) \times (1,3,2,1,0)$
$$= \left\{ \left[\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} \right], \left[\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} \right] \right\} = (-8,-7,8,5,3)$$

$(2,3,4,1,2) \times (2,0,2,0,1)$
$$= \left\{ \left[\begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} \right], \left[\begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} \right], \left[\begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} \right] \right\} = (0,5,-1,8,-8)$$

Hence $A \times B = \begin{bmatrix} (-4,-2,5,0,-1) \\ (-8,-2,7,13,-5) \end{bmatrix}$ which is a 2×1 order 5-dimensional VVM.

4. Conclusion

In this paper, vector valued fuzzy matrices are defined and several types of such matrices are defined. The basic arithmetic operations are defined. Two types of products, viz. dot and cross products are defined. The cross product is defined for the odd dimensional vectors only. And it is very difficult to find out cross product for even dimensional vectors.

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REFERENCES

1. A.Gray, Vector cross products on manifolds, *Trans. Am. Math. Soc.*, 141 (1969) 465-504.
2. B.Eckmann, Stetige Losungen linear Gleichungs system, *Comm. Math. Helv.*, 15 (1943) 318-339.
3. D.B.Fairlie and T.Uneo, Higher-dimensional generalizations of the Euler top equations, hepth/9710079.
4. M.Pal, Numerical analysis for scientists and engineers, 4th Edition, Narosa Publishing House, Delhi, 2012.
5. M.L.Hu and H.Fan, Competition between quantum correlations in the quantum-memory-assisted entropic uncertainty relation, *Phys. Rev. A,* 87 (2013) 022314.
6. M.L.Hu and H.Fan, Robustness of quantum correlations against decoherence, *Ann. Phys.* (NY) 327 (2012) 851.
7. M.Rost, On the dimension of composition algebra, *Doc. Math. J. DMV,* 1 (1996) 209-214.
8. D.P.O'Leary and G.W.Stewart, Computing the eigenvalues and eigenvectors of symmetric arrowhead matrices, *Journal of Computational Physics*, 90 (2) (1990) 497–505.
9. R.LBrown and A.Gray, Vector cross products, *Comm. Math. Helv.*, 42 (1967) 222-236.
10. S.H.Friedberg, A.J.Insel, L.E.Spence, 4th Edition, PHI Learning Private Limited, Delhi, 2014.

11. T.Ueno, General solution of 7D octonionic top equation, *Phys. Lett. A*, 245 (1998) 373-381.
12. Xiu-Lao Tian, Chao Yang, Yang Ho, Chao Tian, Vector cross product in n-dimensional vector space. arXiv:1310.5197v1, *Math-Phys.*, 2013.
13. M.Yasuda, A spectral characterization of Hermitian centrosymmetric and Hermitian skew-centrosymmetric K-Matrices, *SIAM J. Matrix Anal. Appl.*, 25 (3) (2003) 601–605.
14. Z.K.Silagadze, Multi-dimensional vector product, *J. Phys. A: Math. Gen.*, 35 (2002) 4949.
16. A.K.Adak, M.Bhowmik and M.Pal, Some properties of generalized intuitionistic fuzzy nilpotent matrices over distributive lattice, *Fuzzy Inf. and Eng.*, 4(4) (2012) 371-387.
17. A.K.Adak, M.Bhowmik and M.Pal, Intuitionistic fuzzy block matrix and its some properties, *Annals of Pure and Applied Mathematics*, 1(1) (2012) 13-31.
18. A.K.Adak, M.Pal and M.Bhowmik, Distributive lattice over intuitionistic fuzzy matrices, *The Journal of Fuzzy Mathematics*, 21(2) (2013) 401-416.
19. M.Bhowmik and M.Pal, Generalized intuitionistic fuzzy matrices, *Far-East Journal of Mathematical Sciences,* 29(3) (2008) 533-554.
20. M.Bhowmik and M.Pal, Some results on intuitionistic fuzzy matrices and circulant intuitionistic fuzzy matrices, *International Journal of Mathematical Sciences*, 7(1-2) (2008) 81-96.
21. M.Bhowmik, M.Pal and A.Pal, Circulant triangular fuzzy number matrices, *Journal of Physical Sciences*, 12 (2008) 141-154.
22. M.Bhowmik and M.Pal, Intuitionistic neutrosophic set, *Journal of Information and Computing Science*, 4(2) (2009) 142-152.
23. M.Bhowmik and M.Pal, Intuitionistic neutrosophic set relations and some of its properties, *Journal of Information and Computing Science,* 5(3) (2010) 183-192,
24. M.Bhowmik and M.Pal, Generalized interval-valued intuitionistic fuzzy sets, *The Journal of Fuzzy Mathematics*, 18(2) (2010) 357-371.
25. M.Bhowmik and M.Pal, Some results on generalized intervalvalued intuitionistic fuzzy sets, *International Journal of Fuzzy Systems*, 14(2) (2012) 193-203.
26. S.K.Khan and M.Pal, Interval-valued intuitionistic fuzzy matrices, *Notes on Intuitionistic Fuzzy Sets*, 11(1) (2005) 16-27.
27. S.Mondal and M.Pal, Intuitionistic fuzzy incline matrix and determinant, *Annals of Fuzzy Mathematics and Informatics*, 8(1) (2014) 19-32.
28. S.Mondal and M.Pal, Similarity relations, invertibility and eigenvalues of intuitoinistic fuzzy matrix, *Fuzzy Inf. Eng.*, 4 (2013) 431-443.
29. M.Pal, Intuitionistic fuzzy determinant, *V.U.J. Physical Sciences*, 7 (2001) 87-93.
30. M.Pal, S.K.Khan and A.K.Shyamal, Intuitionistic fuzzy matrices, *Notes on Intuitionistic Fuzzy Sets*, 8(2) (2002) 51-62.
31. M.Pal, Interval-valued fuzzy matrices with interval-valued fuzzy rows and columns, *Fuzzy Engineering and Information*, 7(3) (2015) 335-368.
32. M.Pal, Fuzzy matrices with fuzzy rows and columns, *Journal of Intelligent & Fuzzy Systems*, 30 (1) (2016) 561 – 573.
33. R.Pradhan and M.Pal, Intuitionistic fuzzy linear transformations, *Annals of Pure and Applied Mathematics*, 1(1) (2012) 57-68.

34. R.Pradhan and M.Pal, Generalized inverse of block intuitionistic fuzzy matrices, *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, 3 (2013) 23-38.
35. R.Pradhan and M.Pal, Convergence of maxgeneralized meanmingeneralized mean powers of intuitionistic fuzzy matrices, *The Journal of Fuzzy Mathematics*, 22(2) (2013) 477-492.
36. A.K.Shyamal and M.Pal, Distances between intuitionistic fuzzy matrices, *V.U.J. Physical Sciences*, 8 (2002) 81-91.
37. A.K.Shyamal and M.Pal, Two new operations on fuzzy matrices, *Journal of Applied Mathematics and Computing*, 15(1-2) (2004) 91-107.
38. A.K.Shyamal and M.Pal, Distance between intuitionistic fuzzy matrices and its applications, *Natural and Physical Sciences*, 19(1) (2005) 39-58.
39. A.K.Shyamal and M.Pal, Interval-valued fuzzy matrices, *The Journal of Fuzzy Mathematics,* 14(3) (2006) 583-604.
40. A.K. Shyamal and M.Pal, Triangular fuzzy matrices, *Iranian Journal of Fuzzy Systems,* 4 (1) (2007) 75-87.
41. M.Pal, An introduction to fuzzy matrices, Chapter 1, *Handbook of Research on Emerging Applications of Fuzzy Algebraic Structures*, Eds. Jana, Senapati and Pal, IGI Global, USA, (2020).DOI: 10.4018/978-1-7998-0190-0.ch001
42. M. Pal and S. Mondal, Bipolar fuzzy matrices, *Soft Computing*, 23 (20) (2019) 9885-9897.
43. S.Dogra and M.Pal, Picture fuzzy matrix and its application. *Soft Comput.*, 24 (2020) 9413–9428. https://doi.org/10.1007/s00500-020-05021-4