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The Abelian Subgroup : $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}$, p is Prime and $n \geq 1$

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Abstract. In this paper, the classification of finite p-groups is extended by computing an explicit formular for the number of distinct fuzzy subgroups for the abelian group of the form: $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}$, p is a prime and $n \geq 1$.

Keywords: Finite p-groups, nilpotent group, fuzzy subgroups, dihedral group, inclusion-exclusion principle, maximal subgroups, explicit formulae, non-cyclic subgroup, prime.

AMS Mathematics Subject Classification (2010): 20D15, 20E28, 20F18, 20N25, 20K27

1. Introduction

From [1] (See also [2] as well) equation (1) is applied for our computation:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\leq r_1<r_2\leq t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right) \tag{1}$$

Theorem 1. [3] The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:

1. $h(\mathbb{Z}_{p^n}) = 2^n$ (ii) $h(D_{2^n}) = 2^{2n-1}$(iii) $h(\varphi_{2^n}) = 2^{2n-2}$(iv) $h(S_{2^n}) = 3.2^{2n-3}$(v) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$

Theorem 2. [1] Let $G = D_{2^n} \times C_2$, the nilpotent group formed by the cartesian product of the dihedral group of order 2^n and a cyclic group of order 2. Then,the number of distinct fuzzy subgroups of G is given by : $h(G) = 2^{2n}(2n+1) - 2^{n+1}$.

Recall that the case for $p = 2$ We have that $h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^n}) = 2[6h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_{2^{n-1}}) - 6h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) - 8h(\mathbb{Z}_{2^n})]$.
The case for $p > 2$ is treated as follows. Let $p = 3$ and $n = 1$. Then, we have $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Theorem 3. (Berkovich) (i) Let G be a group of order p^n. If A is a subgroup of G of order p^k and $k < m < n$, then, the number of subgroups of G of order p^m containing $A \equiv 1(\mathrm{mod} p)$. (ii) If G is a noncyclic group of order p^n, $1 < m < n-1$, then, $S_m(G) \in \{1+p, 1+p+p^2\}$, where $S_m(G)$ is the number of subgroups of order p^m in G.

By the theorem above, let $\mathcal{M}$ be the collection of all the maximal subgroups of G. Then set $|\mathcal{M}| = 1+p+p^2$. This was true for

$$p = 2 \Longrightarrow |\mathcal{M}| = 1+2+2^2 = 7.$$

For $p = 3$, we have $|\mathcal{M}| = 1+3+3^2 = 13$. Therefore, by equation (1), we have:

$$\frac{1}{2}h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) = 13h(\mathbb{Z}_3 \times \mathbb{Z}_3) - 39h(\mathbb{Z}_3) + 27h(\mathbb{Z}_1) = 79$$

$$\therefore \quad h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) = 2 \times 79 = 158.$$

2. Determination of the number of fuzzy subgroups for $(\mathbb{Z}_3, \times\ \mathbb{Z}_3 \times\ \mathbb{Z}_{3^2})$

More advanced analysis shows that one of the 13 maximal subgroups is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, while each of the other 12 are isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{3^2}$. By this analysis, we have, by equation (#), we have that:

$$\frac{1}{2}h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^2}) = 12h(\mathbb{Z}_3 \times \mathbb{Z}_{3^2}) + 158 - 27h(\mathbb{Z}_{3^2}) - 12h(\mathbb{Z}_3) + 27 = 437$$

$$\therefore \quad h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}) = 2 \times 437 = 874.$$

2.1. Determination of $h(\mathbb{Z}_3, \times\ \mathbb{Z}_3 \times\ \mathbb{Z}_{3^n})$, *n* is positive integer

Following a similar trend as given above, we have

$$\therefore \quad h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}) = 2^{n+1}[18n^2 + 9n + 26] - 54$$

Similarly, for $p = 5$, using equation (c), we have

$$h(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{5^n}) = 2[30h(\mathbb{Z}_5 \times \mathbb{Z}_{5^n}) + h(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{5^{n-1}}) - p^3h(\mathbb{Z}_{5^n}) - 30h(\mathbb{Z}_{5^{n-1}}) + 125]$$

And for $p = 7$,

$$h(\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{7^n}) = 2[56h(\mathbb{Z}_7 \times \mathbb{Z}_{7^n}) + h(\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{7^{n-1}}) - 343h(\mathbb{Z}_{7^n}) - 56h(\mathbb{Z}_{7^{n-1}}) + 343].$$

We have, in general,

$$h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) = 2^{n-2}[4 + (3n-5)p + (n^2-5)p^2 + (n^2-5n+8)p^3] - 2p^2$$

Lemma 1. Let G be an abelian p-group of type $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}$, where p is a prime and $n \geq 1$. The number of distinct fuzzy subgroups of G is

$$h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}) = 2^n p(p+1)(n-1)(3+np+2p) + (2^n-2)p^3 - 2^{n+1}(n-1)p^3 + 2^n[p^3 + 4(1+p+p^2)].$$

Proof: There exist exactly $1+p+p^2$ maximal subgroups for the abelian type $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}$, [Berkovich(2008)]. One of them is isomorphic to

$\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$, while each of the remaining $p+p^2$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$. Thus, by the application of the Inclusion-Exclusion Principle,we have as follows:

$$h\left(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}\right) = 2^n p(p+1)(n-1)(3+np+2p) + (2^n-2)p^3 - 2^{n+1}(n-1)p^3 + 2^n[p^3 + 4(1+p+p^2)].$$

Thus, $h\left(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}\right) = 2^{n-2}[4 + (3n-5)p + (n^2-5)p^2 + (n^2-5n +$

$8)p^3] - 2p^2$. □

Theorem 4. (see[2]) Suppose that $G = D_{2^n} \times C_4$. Then, the number of distinct fuzzy subgroups of G is given by :

$$2^{2(n-2)}(64n + 173) + 3\sum_{j=1}^{n-3} 2^{(n-1+j)}(2n + 1 - 2j).$$

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