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Progress in Nonlinear Dynamics and Chaos

Fractional Calculus and Certain Spiral-Like Functions with Negative Coefficients

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Abstract. The main goal of this paper is to study an application of the fractional calculus techniques for the subclass of Spiral-Like functions $\mathcal{M}_{ŧ}(\tau,\eta,\varepsilon,u,\nu)$. Also, we obtain the coefficient estimates, Distortion theorems for the fractional derivative and fractional integration are obtained, extreme points, closure theorems, radii of starlikeness, convexity and close-to-convexity and partial sum.

Keywords: Fractional calculus, spiral- like function, convex function, partial sums, distortion theorem, radius of starlikeness

AMS Mathematics Subject Classification (2010): 26A33

1. Introduction

Let $\mathcal{M}$ denote the class of functions of the form:

$$f(z) = z - \sum_{\kappa=2}^{\infty} a_{\kappa} z^{\kappa}, \quad (a_{\kappa} \geq 0, \kappa \in N = \{1,2,3,\dots\}), \tag{1}$$

which are analytic and univalent in the unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. For τ real, $|\tau| < \frac{\pi}{2}$, a function f in the form (1) is said to be in $\mathcal{M}(\tau)$, the class of $\tau -$ spiral $-$ Like function if

$$\mathrm{Re}\left(e^{i\tau}\frac{zf'(z)}{f(z)}\right) > 0, z \in U. \tag{2}$$

For $\tau = 0, \mathcal{M}(0) \equiv \mathcal{M}$ is the well-known class of functions starlike with respect to the origin, for $\tau \neq 0$, it is know that $\mathcal{M}(\tau)$ is not contained in $\mathcal{M}$. In fact the class $\mathcal{M}(\tau)$ was introduced and shown to be a subfamily of $\mathcal{M}$ by Spaček [6]. Later, Zomorski [8] obtained sharp coefficient bounds for the class.

Definition 1. [7] The fractional integral of order $ŧ$ ($ŧ > 0$), is defined by

$$D_z^{-ŧ} f(z) = \frac{1}{\Gamma(ŧ)} \int_0^z \frac{f(\alpha)}{(z-\alpha)^{1-ŧ}} d\alpha, \tag{3}$$

where f is an analytic function in a simply – connected region of the z- plane containing the origin, and the multiplicity of $(z-\alpha)^{ŧ-1}$ is removed by requiring $\log(z-\alpha)$ to be real, when $\mathrm{Re}(z-\alpha) > 0$.

Definition 2. [7] The fractional derivative of order t $(0 \le t < 1)$, is defined by

$$D_z^{t} f(z) = \frac{1}{\Gamma(1-t)} \frac{d}{dz} \int_0^z \frac{f(\alpha)}{(z-\alpha)^{t}} d\alpha, \tag{4}$$

where $f(z)$ is as in Definition (1) and the multiplicity of $(z-\alpha)^{-t}$ is removed like Definition (1).

Definition 3. [7] [Under the condition of Definition (2)] the fractional derivative of order $\kappa + t, (\kappa = 0,1,2,\dots)$ is defined by

$$D_z^{\kappa+t} f(z) = \frac{d^{\kappa}}{dz^{\kappa}} D_z^{t} f(z).$$

From Definition (1) and Definition (2) by applying a simple calculation, we get

$$D_z^{-t} f(z) = \frac{1}{\Gamma(2+t)} z^{t+1} - \sum_{\kappa=2}^{\infty} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1+t)} a_\kappa z^{\kappa+t}, \tag{5}$$

$$D_z^{t} f(z) = \frac{1}{\Gamma(2-t)} z^{1-t} - \sum_{\kappa=2}^{\infty} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-t)} a_\kappa z^{\kappa+t}, \tag{6}$$

Definition 4. We introduce a new subclass of $M(\tau)$ as functions in the form (1) that satisfy the inequality:

$$\left| \frac{\frac{z^2 f''(z) + (u+\nu) z f'(z)}{z f'(z) + (u+\nu) f(z)}}{2\eta \left[\frac{z^2 f''(z) + (u+\nu) z f'(z)}{z f'(z) + (u+\nu) f(z)} + (1-\varepsilon) e^{-i\tau} \cos\tau \right] + \frac{z^2 f''(z) + (u+\nu) z f'(z)}{z f'(z) + (u+\nu) f(z)}} \right| < 1,$$

for $z \in U$, where $0 \le \varepsilon < 1, 0 < \eta \le 1, 0 < u < \nu \le 1$ $|\tau| < \frac{\pi}{2}$. We denote for our class by $M_t(\tau, \eta, \varepsilon, u, \nu)$.

2. Main results

In the following theorem, we obtain the coefficient estimates for the class $M_t(\tau, \eta, \varepsilon, u, \nu)$.

Theorem 1. Let $f(z) \in M(\tau)$. Then $f(z)$ is in the class $M_t(\tau, \eta, \varepsilon, u, \nu)$ if and only if

$$\sum_{\kappa=2}^{\infty} \left[\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon) \left| e^{-i\tau} \cos\tau \right| \right] a_\kappa \le (u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon) \left| e^{i\tau} \cos\tau \right|, \tag{7}$$

where $0 \le \varepsilon < 1, 0 < \eta \le 1, 0 < u < \nu \le 1$ $|\tau| < \frac{\pi}{2}$.

The result (7) is sharp for the function $f(z)$ given by:

$$f(z) = z - \frac{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon) \left| e^{-i\tau} \cos\tau \right|}{\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon) \left| e^{-i\tau} \cos\tau \right|} z^{\kappa}. \tag{8}$$

Proof: Let (7) holds true and $|z| = 1$, we have

$$\left| z^2 f''(z) + (u+\nu) z f'(z) \right| - \left| 2\eta \left[z^2 f''(z) + (u+\nu) z f'(z) + (1-\varepsilon) e^{-i\tau} \cos\tau \left(z f'(z) + (u+\nu) f(z) \right) \right] + z^2 f''(z) + (u+\nu) z f'(z) \right|$$

$$= \left|\sum_{\kappa=2}^{\infty} \kappa(\kappa+u+\nu-1)a_\kappa z^\kappa - (u+\nu)z\right|$$
$$- \left|[2\eta(1-\varepsilon)e^{-i\tau}\cos\tau\,(u+\nu+1) + (u+\nu)(2\eta+1)]z - \sum_{\kappa=2}^{\infty}[\kappa(\kappa+u+\nu-1)(2\eta+1) + 2\eta(1-\varepsilon)e^{-i\tau}\cos\tau\,(\kappa+u+\nu)]a_\kappa z^\kappa\right|$$
$$\le \sum_{\kappa=2}^{\infty}[\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]a_\kappa - (u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)|e^{-i\tau}\cos\tau| \le 0,$$

by hypothesis. Thus by Maximum modules theorem $f \in \mathcal{M}_t(\tau,\eta,\varepsilon,u,\nu)$.
Conversely, assume that

$$\left|\frac{\frac{z^2f''(z)+(u+\nu)zf'(z)}{zf'(z)+(u+\nu)f(z)}}{2\eta\left[\frac{z^2f''(z)+(u+\nu)zf'(z)}{zf'(z)+(u+\nu)f(z)} + (1-\varepsilon)e^{-i\tau}\cos\tau\right] + \frac{z^2f''(z)+(u+\nu)zf'(z)}{zf'(z)+(u+\nu)f(z)}}\right|$$
$$= \left|\frac{z^2f''(z)+(u+\nu)zf'(z)}{2\eta[z^2f''(z)+(u+\nu)zf'(z)+(1-\varepsilon)e^{-i\tau}\cos\tau\,(zf'(z)+(u+\nu)f(z))] + z^2f''(z)+(u+\nu)zf'(z)}\right|$$

Let $A = \sum_{\kappa=2}^{\infty}\kappa(\kappa+u+\nu-1)a_\kappa z^\kappa - (u+\nu)z$ and

$$B = [2\eta(1-\varepsilon)e^{-i\tau}\cos\tau\,(u+\nu+1) + (u+\nu)(2\eta+1)]z - \sum_{\kappa=2}^{\infty}[\kappa(\kappa+u+\nu-1)(2\eta+1) + 2\eta(-\varepsilon)e^{-i\tau}\cos\tau\,(\kappa+u+\nu)]a_\kappa z^\kappa$$

Since $\operatorname{Re}(z) \le |z|$ for all z, we have

$$\operatorname{Re}\left(\frac{A}{B}\right) < 1, \tag{9}$$

we can choose value of z on the real axis so that $f(z)$ is real. Let $z \to 1^-$, through real values, so we write (9) as

$$\sum_{\kappa=2}^{\infty}[\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]a_\kappa \le (u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)|e^{-i\tau}\cos\tau|. \blacksquare$$

Corollary 1. Let $f(z) \in \mathcal{M}_t(\tau,\eta,\varepsilon,u,\nu)$. Then

$$a_\kappa \le \frac{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{-i\tau}\cos\tau|}{\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|},\ \kappa \ge 2.$$

Theorem 2. Let $f(z) \in \mathcal{M}_t(\tau,\eta,\varepsilon,u,\nu)$. Then

$$|D_z^{-t} f(\zeta)| \le \frac{1}{\Gamma(2+t)} |\zeta|^{t+1}$$

$$\left[1 + \frac{2\left[(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2+t)[2(u+\nu+1)(\eta+1) + \eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]} |\zeta|\right], \tag{10}$$

and

$$|D_z^{-t} f(\zeta)| \ge \frac{1}{\Gamma(2+t)} |\zeta|^{t+1}$$

$$\left[1 - \frac{2\left[(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2+t)[2(u+\nu+1)(\eta+1) + \eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]} |\zeta|\right]. \tag{11}$$

The inequalities in (10) and (11) are attained for the function $f(\zeta)$ given by:

$$f(\zeta) = \zeta - \frac{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{2(u+\nu+1)(\eta+1) + \eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|} \zeta^2. \tag{12}$$

Proof: By using Theorem (1), we have

$$\sum_{\kappa=2}^{\infty} a_\kappa \le \frac{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{2(u+\nu+1)(\eta+1) + \eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|}, \tag{13}$$

by Definition (3), we have

$$D_\zeta^{-t} f(\zeta) = \frac{1}{\Gamma(2+t)} \zeta^{t+1} - \sum_{\kappa=2}^{\infty} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1+t)} a_\kappa \zeta^{\kappa+t},$$

and

$$\begin{aligned}\Gamma(2+t)\zeta^{-t} D_\zeta^{-t} f(\zeta) &= \zeta - \sum_{\kappa=2}^{\infty} \frac{\Gamma(2+t)\Gamma(\kappa+1)}{\Gamma(\kappa+1+t)} a_\kappa \zeta^{\kappa} \\ &= \zeta - \sum_{\kappa=2}^{\infty} \phi(\kappa)\, a_\kappa \zeta^{\kappa},\end{aligned} \tag{14}$$

where

$$\phi(\kappa) = \frac{\Gamma(2+t)\Gamma(\kappa+1)}{\Gamma(\kappa+1+t)}.$$

We know that $\phi(\kappa)$ is a decreasing function of κ and

$$0 < \phi(\kappa) < \phi(2) = \frac{2}{2+t}.$$

Using (13) and (14), we have

$$|\Gamma(2+t) z^{-t} D_z^{-t} f(\zeta)| \le |\zeta| + \phi(2)|\zeta|^2 \sum_{\kappa=2}^{\infty} a_\kappa$$

$$\le |\zeta| + \frac{2\left[(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2+t)[2(u+\nu+1)(\eta+1) + \eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]} |\zeta|^2,$$

which gives (10), we also have

$$|\Gamma(2+t) z^{-t} D_z^{-t} f(\zeta)| \le |\zeta| - \phi(2)|\zeta|^2 \sum_{\kappa=2}^{\infty} a_\kappa$$

$$\ge |\zeta| - \frac{2\left[(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2+t)[2(u+\nu+1)(\eta+1) + \eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]} |\zeta|^2,$$

which gives (11).■

Theorem 3. Let $f(z) \in M_t(\tau,\eta,\varepsilon,u,\nu)$. Then

$$\left|D_z^t f(z)\right| \le \frac{1}{\Gamma(2-t)}|z|^{1-t}\left[1+\frac{2\left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2-t)[2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]}|z|\right], \quad (15)$$

and

$$\left|D_z^t f(z)\right| \ge \frac{1}{\Gamma(2-t)}|z|^{1-t}\left[1-\frac{2\left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2-t)[2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]}|z|\right]. \quad (16)$$

The inequalities in (15) and (16) are attained for the function $f(z)$ given by (12).

Proof: From Definition (3), we have

$$D_z^t f(z) = \frac{1}{\Gamma(2-t)}z^{1-t} - \sum_{\kappa=2}^{\infty}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-t)}a_\kappa z^{\kappa+t}$$

and

$$\Gamma(2-t)z^t D_z^t f(z) = z - \sum_{\kappa=2}^{\infty}\frac{\Gamma(2-t)\Gamma(\kappa+1)}{\Gamma(\kappa+1-t)}a_\kappa z^\kappa$$
$$= z - \sum_{\kappa=2}^{\infty}\Psi(\kappa)\, a_\kappa z^\kappa$$

where $\Psi(\kappa) = \frac{\Gamma(2-t)\Gamma(\kappa+1)}{\Gamma(\kappa+1-t)}$ for $\kappa \ge 2$, $\Psi(\kappa)$ is a decreasing of κ, then

$$\Psi(\kappa) \le \Psi(2) = \frac{2}{2-t}.$$

Also by using Theorem (1), we have

$$\sum_{\kappa=2}^{\infty} a_\kappa \le \frac{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|},$$

thus

$$\left|\Gamma(2-t)z^t D_z^t f(z)\right| \le |z| - \Psi(2)|z|^2\sum_{\kappa=2}^{\infty}a_\kappa$$
$$\le |z| - \frac{2\left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2-t)[2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]}|z|^2.$$

Then

$$\left|D_z^t f(z)\right| \le \frac{1}{\Gamma(2-t)}|z|^{1-t}\left[1+\frac{2\left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2-t)\left[2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}|z|\right],$$

and by the same way, we obtain

$$\left|D_z^t f(z)\right| \ge \frac{1}{\Gamma(2-t)}|z|^{1-t}\left[1-\frac{2\left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2-t)[2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]}|z|\right].$$ ■

Corollary 2. For every $f(z) \in M_{t}(\tau,\eta,\varepsilon,u,\nu)$, we have

$$\frac{|z|^2}{2}\left[1-\frac{2\left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{3[2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]}|z|\right] \le \left|\int_0^z f(\alpha)\,d\alpha\right|$$
$$\le \frac{|z|^2}{2}\left[1+\frac{2\left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{3[2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]}|z|\right] \quad (17)$$

and

$$|z|\left[1-\frac{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|}|z|\right] \le |f(z)|$$
$$\le |z|\left[1+\frac{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|}|z|\right] \quad (18)$$

Proof: i) By Definition (1) and Theorem (2) for $t = 1$, we have

$$D_z^{-1}f(z) = \int_0^z f(\alpha)\,d\alpha,$$

the result is true.

ii) By Definition (2) and Theorem (3) for $t = 0$, we have

$$D_z^0 f(z) = \frac{d}{dz}\int_0^z f(\alpha)\,d\alpha = f(z),$$

the result is true.■

Corollary 3. $D_z^{-t}f(z)$ and $D_z^{t}f(z)$ are included in the disk with center at origin and radii

$$\frac{1}{\Gamma(2+t)}\left[1-\frac{2\left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2+t)[2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]}\right],$$
$$\frac{1}{\Gamma(2-t)}\left[1-\frac{2\left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(2-t)[2(u+\nu+1)(\eta+1)+\eta(2+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|]}\right].$$

In the following theorem, we obtain the extreme points of the class $M_{t}(\tau,\eta,\varepsilon,u,\nu)$.

Theorem 4. Let

$$f_1(z) = z,\ f_\kappa(z) = z-\frac{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|}{\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|}z^\kappa, \quad (19)$$
$$(\kappa \ge 2),$$

where $\left(\kappa \in N, 0 \le \varepsilon < 1, 0 < \eta \le 1, 0 < u < \nu \le 1\ |\tau| < \frac{\pi}{2}\right)$. Then the function $f \in M_{t}(\tau,\eta,\varepsilon,u,\nu)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{\kappa=1}^{\infty}\xi_\kappa f_\kappa(z), \quad (20)$$

where $\xi_\kappa \ge 0$, $\sum_{\kappa=1}^{\infty}\xi_\kappa = 1$ or $1 = \xi_1 + \sum_{\kappa=2}^{\infty}\xi_\kappa$.

Proof: Let $f(z)$ can be expressed as in (20). Then

$$f(z) = \sum_{\kappa=1}^{\infty}\xi_\kappa f_\kappa(z) = \xi_1 f_1(z) + \sum_{\kappa=2}^{\infty}\xi_\kappa f_\kappa(z)$$

$$= \xi_1 z + \sum_{\kappa=2}^{\infty} \xi_\kappa \left(z + \frac{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{i\tau}\cos\tau\right|}{\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|} z^\kappa \right)$$

$$= z\left(\xi_1 + \sum_{\kappa=2}^{\infty}\xi_\kappa\right) + \sum_{\kappa=2}^{\infty} \frac{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{i\tau}\cos\tau\right|}{\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|} \xi_\kappa z^\kappa$$

$$= z + \sum_{\kappa=2}^{\infty} \frac{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{i\tau}\cos\tau\right|}{\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|} \xi_\kappa z^\kappa$$

$$= z - \sum_{\kappa=2}^{\infty} \beta_\kappa z^\kappa,$$

where

$$\beta_\kappa = \sum_{\kappa=2}^{\infty} \frac{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{i\tau}\cos\tau\right|}{\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|} \xi_\kappa.$$

Thus

$$\sum_{\kappa=2}^{\infty} \beta_\kappa \frac{\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{i\tau}\cos\tau\right|}$$

$$= \sum_{\kappa=2}^{\infty} \frac{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{i\tau}\cos\tau\right|}{\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|} \xi_\kappa \times \frac{\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{i\tau}\cos\tau\right|}$$

$$= \sum_{\kappa=2}^{\infty} \xi_\kappa = 1 - \xi_1 < 1.$$

Therefore, we have $f \in \mathcal{M}_{t}(\tau, \eta, \varepsilon, u, \nu)$.
Conversely, suppose that $f \in \mathcal{M}_{t}(\tau, \eta, \varepsilon, u, \nu)$. Then by (7), we have

$$a_\kappa \le \frac{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}, \quad (\kappa \ge 2)$$

we may set,

$$\xi_\kappa = \frac{\kappa(\kappa+u+\nu-1)(\eta+1) + \eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{(u+\nu)(\eta+1) + \eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|} a_\kappa, \quad (\kappa \ge 2)$$

and

$$\xi_1 = 1 - \sum_{\kappa=2}^{\infty} \xi_\kappa.$$

Then

$$f(z) = z - \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa,$$

$$= z - \sum_{\kappa=2}^{\infty} \frac{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{-i\tau}\cos\tau|}{\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|}\xi_\kappa z^\kappa$$

$$= z - \sum_{\kappa=2}^{\infty}\xi_\kappa\left(z - f_\kappa(z)\right) = \left(1-\sum_{\kappa=2}^{\infty}\xi_\kappa\right)z + \sum_{\kappa=2}^{\infty}\xi_\kappa f_\kappa(z)$$

$$= z\xi_1 + \sum_{\kappa=2}^{\infty}\xi_\kappa f_\kappa(z) = \sum_{\kappa=2}^{\infty}\xi_\kappa f_\kappa(z).\ \blacksquare$$

In the following Theorem, we prove the class $\mathcal{M}_{\flat}(\tau,\eta,\varepsilon,u,\nu)$ is closed under linear combination.

Theorem 5. Let the function $f_j(z) \in \mathcal{M}_{\flat}(\tau,\eta,\varepsilon,u,\nu)$ defined by

$$f_j(z) = z - \sum_{\kappa=2}^{\infty} a_{\kappa,j} z^\kappa, \qquad \left(a_{\kappa,j} \geq 0, j = 1,2,3,\dots,l\right).$$

Then the function $\beta(z)$ defined by

$$\beta(z) = \sum_{j=1}^{l} c_j f_j(z),$$

is in the class $\mathcal{M}_{\flat}(\tau,\eta,\varepsilon,u,\nu)$, where $\sum_{j=1}^{l} c_j = 1,\ c_j \geq 0$.

Proof: By definition of (z) , we have

$$\beta(z) = \left[\sum_{j=1}^{l} c_j\right] z - \sum_{\kappa=2}^{\infty}\left[\sum_{j=1}^{l} c_j a_{\kappa,j}\right] z^\kappa. \tag{21}$$

Further, since $f_j(z)$ are in the class $\mathcal{M}_{\flat}(\tau,\eta,\varepsilon,u,\nu)$ for every $j = 1,2,3,\dots,l$.

Hence, we can see that

$$\sum_{\kappa=2}^{\infty}\left(\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|\right)\left[\sum_{j=1}^{l} c_j a_{\kappa,j}\right]$$

$$= \sum_{j=1}^{l} c_j\left[\sum_{\kappa=2}^{\infty}\left[\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|\right]a_{\kappa,j}\right]$$

$$\leq \left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|\right]\sum_{j=1}^{l} c_j$$

$$= (u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|.\ \blacksquare$$

In the following theorems, we obtain radii of starlikness, convexity and close-to-convexity of the class $\mathcal{M}_{\flat}(\tau,\eta,\varepsilon,u,\nu)$.

Theorem 6. If $f \in \mathcal{M}_{\flat}(\tau,\eta,\varepsilon,u,\nu)$, then f is starlike of order $\flat(0 \leq \flat < 1)$ in the disk $|z| < r_1(\tau,\eta,\varepsilon,u,\nu,\flat)$, where

$$r_1(\tau,\eta,\varepsilon,u,\nu,ŧ) = \inf_\kappa \left\{\frac{(1-ŧ)\left[\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(\kappa-ŧ)[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|]}\right\}^{\frac{1}{\kappa-1}},$$

$\kappa \ge 2.$ (22)

The result is sharp with extremal function f given by

$$f(z) = z - \frac{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|} z^\kappa. \quad (23)$$

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1 - ŧ, \quad (0 \le ŧ < 1)$$

for $|z| < r_1(\tau,\eta,\varepsilon,u,\nu,ŧ)$. We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{zf'(z) - f(z)}{f(z)}\right| = \left|\frac{\sum_{\kappa=2}^{\infty} a_\kappa(\kappa-1)\, z^{\kappa-1}}{1-\sum_{\kappa=2}^{\infty} a_\kappa\, z^{\kappa-1}}\right| \le \frac{\sum_{\kappa=2}^{\infty} a_\kappa(\kappa-1)\, |z|^{\kappa-1}}{1-\sum_{\kappa=2}^{\infty} a_\kappa\, |z|^{\kappa-1}}.$$

The last expression above is bounded by $(1-ŧ)$ if

$$\sum_{\kappa=2}^{\infty} \frac{(\kappa-ŧ)a_\kappa|z|^{\kappa-1}}{(1-ŧ)} \le 1. \quad (24)$$

Hence, by Theorem (1), (24) will be true if

$$\frac{(\kappa-ŧ)}{(1-ŧ)}|z|^{\kappa-1} \le \frac{\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|},$$

or equivalently

$$|z| \le \left(\frac{(1-ŧ)\left[\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{(\kappa-ŧ)[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|]}\right)^{\frac{1}{\kappa-1}}.$$

Setting $|z| = r_1(\tau,\eta,\varepsilon,u,\nu,ŧ)$, we get the desired result. ■

Theorem 7. If $f \in M_ŧ(\tau,\eta,\varepsilon,u,\nu)$, then f is convex of order $ŧ(0 \le ŧ < 1)$ in the disk $|z| < r_2(\tau,\eta,\varepsilon,u,\nu,ŧ)$, where

$$r_2(\tau,\eta,\varepsilon,u,\nu,ŧ) = \inf_n \left\{\frac{(1-ŧ)\left[\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{\kappa(\kappa-ŧ)[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|]}\right\}^{\frac{1}{\kappa-1}}, \kappa$$

$\ge 2.$ (25)

The result is sharp with extremal function f given by (23).

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1 - ŧ, \quad (0 \le ŧ < 1)$$

for $|z| < r_2(\tau,\eta,\varepsilon,u,\nu,ŧ)$. We have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{\kappa=2}^{\infty} \kappa(\kappa-1)a_\kappa\, z^{\kappa-1}}{1-\sum_{\kappa=2}^{\infty} \kappa a_\kappa\, z^{\kappa-1}}\right| \le \frac{\sum_{\kappa=2}^{\infty} \kappa(\kappa-1)a_\kappa\, |z|^{\kappa-1}}{1-\sum_{\kappa=2}^{\infty} \kappa a_\kappa\, |z|^{\kappa-1}}.$$

The last expression above is bounded by $(1-ŧ)$ if

$$\sum_{\kappa=2}^{\infty} \frac{\kappa(\kappa-ŧ)a_\kappa|z|^{\kappa-1}}{(1-ŧ)} \le 1. \quad (26)$$

Hence, by Theorem (1), (26) will be true if

$$\frac{\kappa(\kappa-ŧ)}{(1-ŧ)}|z|^{\kappa-1} \le \frac{\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|},$$

or equivalently,

$$|z| \le \left(\frac{(1-ŧ)\left[\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{\kappa(\kappa-ŧ)[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|]}\right)^{\frac{1}{\kappa-1}}.$$

Setting $|z| = r_2(\tau,\eta,\varepsilon,u,\nu,ŧ)$, we get the desired result. ■

Theorem 8. If $f \in M_{ŧ}(\tau,\eta,\varepsilon,u,\nu)$, then f is close-to-convex of order $ŧ(0 \le ŧ < 1)$ in the disk $|z| < r_3(\tau,\eta,\varepsilon,u,\nu,ŧ)$, where

$$r_3(\tau,\eta,\varepsilon,u,\nu,ŧ) = \inf_\kappa \left\{\frac{(1-ŧ)\left[\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{\kappa\left[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)\left|e^{i\tau}\cos\tau\right|\right]}\right\}^{\frac{1}{\kappa-1}}, \quad \kappa \ge 2. \tag{27}$$

The result is sharp with extremal function f given by (23).

Proof: It is sufficient to show that

$$|f'(z)-1| \le 1-ŧ, \quad (0 \le ŧ < 1)$$

for $|z| < r_3(\tau,\eta,\varepsilon,u,\nu,ŧ)$. We have

$$|f'(z)-1| = \left|\sum_{\kappa=2}^{\infty} \kappa a_\kappa z^{\kappa-1}\right| \le \sum_{\kappa=2}^{\infty} \kappa a_\kappa |z|^{\kappa-1}.$$

The last expression above is bounded by $(1-ŧ)$ if

$$\sum_{\kappa=2}^{\infty} \frac{\kappa a_\kappa |z|^{\kappa-1}}{(1-ŧ)} \le 1. \tag{28}$$

Hence, by Theorem (1), (28) will be true if

$$\frac{\kappa}{(1-ŧ)}|z|^{\kappa-1} \le \frac{\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|}{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|},$$

or equivalently,

$$|z| \le \left(\frac{(1-ŧ)\left[\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)\left|e^{-i\tau}\cos\tau\right|\right]}{\kappa[(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|]}\right)^{\frac{1}{\kappa-1}}.$$

Setting $|z| = r_3(\tau,\eta,\varepsilon,u,\nu,ŧ)$, we get the desired result. ■

Let $f \in M$ be a function of the form (1). Motivated by Silverman [2] and Silvia [4], see also [3], [5], we define the partial sums f_m defined by

$$f_m(z) = z - \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa, \quad (m \in N). \tag{29}$$

Theorem 9. Let $f \in M$ be given by (1) and define the partial sums $f_1(z)$ and $f_m(z)$ as follows: $f_1(z) = z$ and

$$f_m(z) = z - \sum_{\kappa=2}^{m} a_\kappa z^\kappa, \qquad (m > 2). \tag{30}$$

Also suppose that

$$\sum_{\kappa=2}^{\infty} d_\kappa a_\kappa \le 1,$$

$$\left(d_\kappa = \frac{\kappa(\kappa+u+\nu-1)(\eta+1)+\eta(\kappa+u+\nu)(1-\varepsilon)|e^{-i\tau}\cos\tau|}{(u+\nu)(\eta+1)+\eta(u+\nu+1)(1-\varepsilon)|e^{i\tau}\cos\tau|}\right) \tag{31}$$

Then, we have

$$\operatorname{Re}\left\{\frac{f(z)}{d_m(z)}\right\} > 1 - \frac{1}{d_m}, \tag{32}$$

and

$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\} > \frac{d_{m+1}}{1+d_{m+1}}. \tag{33}$$

Each of the bounds in (32) and (33) is the best possible for $m \in N$.

Proof: For the coefficients d_κ given by (31), it is not difficult to verify that $d_{\kappa+1} > d_\kappa > 1, \kappa = 2,3 \dots$. Therefore, we have

$$\sum_{\kappa=2}^{m} a_\kappa + d_m \sum_{\kappa=m+1}^{\infty} a_\kappa \le \sum_{\kappa=2}^{\infty} d_\kappa a_\kappa \le 1. \tag{34}$$

By setting

$$g_1(z) = d_m\left[\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_m}\right)\right] = 1 + \frac{d_m \sum_{\kappa=m+1}^{\infty} a_\kappa z^{\kappa-1}}{1 - \sum_{\kappa=2}^{m} a_\kappa z^{\kappa-1}}, \tag{35}$$

and applying (34), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \le \frac{d_k \sum_{n=k}^{\infty} a_n}{2 - 2\sum_{\kappa=2}^{m} a_\kappa - d_m \sum_{\kappa=m+1}^{\infty} a_\kappa} \le 1, \qquad (z \in U),$$

which readily yields the left assertion (32). If we take

$$f(z) = z - \frac{z^m}{d_m}, \tag{36}$$

then

$$\frac{f(z)}{f_m(z)} = 1 - \frac{z^m}{d_m} \longrightarrow 1 - \frac{1}{d_m} (z \longrightarrow 1^-),$$

Similarly, if we take

$$g_2(z) = (1+d_m)\left[\frac{f_m(z)}{f(z)} - \frac{d_m}{1+d_m}\right]$$

and making use of (34), we deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \le \frac{(1+d_\kappa)\sum_{\kappa=m+1}^{\infty} a_\kappa}{2 - 2\sum_{\kappa=2}^{m} a_\kappa + (1-d_m)\sum_{\kappa=m+1}^{\infty} a_\kappa} \le 1, \tag{37}$$

which leads us to the assertion (33). The bound in (33) is sharp for each $m \in \mathbb{N}$ with the function given by (36).

3. Conclusion

In this work, we have studied an application of the fractional calculus techniques for the subclass of Spiral-Like functions $M_t(\tau, \eta, \varepsilon, u, \nu)$. Also, we have obtained the coefficient estimates, Distortion theorems for the fractional derivative and fractional integration are obtained, extreme points, closure theorems, radii of starlikeness, convexity and close-to-convexity and partial sum.For future studies, one can use the result in this work to find an analytical solution of the higher order ordinary differential equations. Also, one can study the result in this work in the fuzzy theory and then extend these result for solving fuzzy ordinary differential equation.

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