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Progress in Nonlinear Dynamics and Chaos

Star-Critical Ramsey Numbers for Cycles Versus the Complete Graph on 5 Vertices

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Abstract. Let G, H and K represent three graphs without loops or parallel edges and n represent an integer. If any red/blue coloring of the edges of K there exists a red copy of G or a blue copy of H, we say that $K \rightarrow (G, H)$. Let K_n represent a complete graph on n vertices, C_n a cycle on n vertices and $S_n = K_{1,n}$ a star on $n + 1$ vertices. The Ramsey number $r(G, H)$ is defined as $\min\{n \mid K_n \rightarrow (G, H)\}$. Star-critical Ramsey number $r_*(G, H)$ is defined as $\min\{k \mid K_{r(G,H)-1} \sqcup K_{1,k} \rightarrow (G, H)\}$. We show that $r_*(C_4, K_5) = 13$ and for $n > 4$, $r_*(C_n, K_5) = 3n - 1$.

Keywords: Ramsey numbers, Star-critical Ramsey numbers

AMS Mathematics Subject Classification (2010):05C55, 05D10, 05C38

1. Introduction

In its classical form, Ramsey's Theorem ensures the existence of the Ramsey numbers $r(n, m)$ defined $r(n, m) = r(Kn, Km)$. One new branch of classical Ramsey number, introduced by Hook and Isaak in 2010 (see [6]), is the Star-critical Ramsey number. Star -critical Ramsey number is defined as the smallest positive integer k such that $K_{n-1} \sqcup K_{1,k} \rightarrow (G, H)$, where $n = r(G, H)$. Several authors have studied $r_*(G, H)$ for special pairs of graphs, such as trees versus complete graphs, stripes versus stripes, fans versus complete graphs (see [2, 4, 5, 6, 7]).In this paper, we find $r_*(C_n, K_m)$ for $m = 5$, related to the Bondy and Erdős conjecture introduced in 1976 (see [12]). The author would also like to acknowledge research work carried out independently by M. Ferreri et al (see [5]), related to critical graphs and star-critical Ramsey numbers for cycles versus K_5.

2. Notation

Suppose that a graph G contains an n cycle $(u_1, u_2, ..., u_n, u_1)$ and a vertex (say y_1) outside of the cycle such that y_1 is adjacent to exactly two vertices (say u_i and u_j) of the n cycle. In such a situation, we say that y_1 is adjacent to two vertices of the $\{u_1, u_2, ..., u_n\}$ which are length k apart where $k = \min\{(i-j) \bmod n, (j - i) \bmod n\}$ (also see [11,12]). Throughout the paper, in any graph which is colored by red and blue, we will denote the red edges by an unbroken line and the blue edges by a broken line

3. Properties of (C_4, K_5) Ramsey critical graphs

In order to prove the main result of this paper, namely finding $r_*(C_n, K_5)$, we try to utilize the critical graphs of $r(C_n, K_4)$ (see [7]) and the fact that $r(C_n, K_5) = 4n-3$ for $n \geq 4$ (see [3, 10]). In addition, we use the following lemmas to arrive at the main result. The first four lemmas are a direct consequence of [7,8,9,10], written by Jayawardene et al.

Lemma 1. *([10], Lemma 2;[8]). If G is a graph of order N that contains no C_m and the independent number is less than or equal to $n-1$ then the minimal degree is greater than or equal to $N - r(C_m, K_{n-1})$.*

Lemma 2. *([9], Lemma 5). Any C_5-free graph of order 11 with no independent set of 4 vertices is isomorphic to one of 19 possible graphs denoted by graphs $R_{11,1}$, ..., $R_{11,18}$(see [9]) or$R_{11,19}=2K_4 \cup K_3$.*

Lemma 3. *([9], Lemma 4). Any C_5-free graph of order 12 with no independent set of 4 vertices is isomorphic to one of the graphs $R_{12,1}$, $R_{12,2}$, $R_{12,3}$, $R_{12,4}$, $R_{12,5}$(see [4]) or $R_{12,6} \cong 3K_4$.*

Lemma 4. *([7], Lemma 7). Any C_6-free graph of order 15 with no independent set of 4 vertices is isomorphic to one of the five critical graphs denoted by $R_{15,1}$, $R_{15,2}$, $R_{15,3}$, $R_{15,4}$ or $R_{15,5}$,where $R_{15,4} \cong 3K_5 + e$ and $R_{15,5} \cong 3K_5$.The graphs $R_{15,1}$, $R_{15,2}$, $R_{15,3}$* are illustrated in *[7].*

The following lemmas are a direct consequence of [3], written by Bollobás et al.

Lemma 5. *Suppose G contains the cycle ($u_1, u_2, ..., u_{n-1}, u_1$) of length $n-1$ but no cycle of length n. Let $Y = V(G) \setminus \{u_1, u_2, ..., u_{n-1}\}$. Then,*

a) *No vertex $x \in Y$ is adjacent to two consecutive vertices on the cycle.*
b) *If $x \in Y$ is adjacent to u_i and u_j then $u_{i+1}u_{j+1} \notin E(G)$.*
c) *If $x \in Y$ is adjacent to u_i and u_j then no vertex $x' \in Y$ is adjacent to both u_{i+1} and u_{j+2}.*
d) *Suppose $\alpha(G) = m - 1$ where $m \leq (n+3)/2$ and $\{x_1, x_2, x_3, ..., x_{m-1}\} \subseteq Y$ is an $(m-1)$ − element independent set. Then no member of this set is adjacent to $m-2$ or more vertices on the cycle.*

Lemma 6. *AC_5-free graph of order* 16 *with no independent set of 5 vertices contains an isomorphic copy of* 4 K_4.

Proof. Let G be a C_5-free graph of order 16 with no independence set of 5 vertices. First note that $\delta(G) \geq 3$ by lemma 3 as $r(C_5, K_4) = 13$.

Remark 1. Suppose that G contains a K_4, then any vertex outside of K_4 can be adjacent to at most one vertex of the K_4. Furthermore, any two adjacent vertices together can be adjacent to at most one vertex of a K_4.

Case 1: The minimum degree is 3.
Say w is a vertex of minimum degree. Then $G[V(G)\backslash\Gamma^*(w)]$ will satisfy the conditions of lemma 3. Thus, using all possible cases and using remark 1, any two non-adjacent vertices of $G[\Gamma(w)]$ will give rise to an independent set of size five containing them. Therefore, $G[\Gamma(w)]\cong K_3$. This will result in the required $4K_4$.

Case 2: The minimum degree greater than or equal to 4.
Say w is a vertex of minimum degree. Then $G[V(G) \backslash \Gamma^*(w)]$ will satisfy all conditions of lemma 2. Thus, using remark 1 one sees that there are no possible extensions.

Case 3: The minimum degree greater than or equal to 5. Say w is a vertex of minimum degree. Then $G[\Gamma(w)]$ will contain a $C_3 \cup K_2$, $P_3 \cup K_2$, $K_{1,4}$, $K_{1,3} \cup K_1$, $2K_2$ or at least two isolated vertices. It is worth noting that, we will get C_5 or else an independent set of size 5 directly in all cases, other than when $G[\Gamma(w)] = C_3 \cup K_2$. In this case, let $H = G[V(G)\backslash \Gamma^*(w)]$. Then, H has 10 vertices. In order to avoid a C_5 each these three vertices of the C_3 in $G[\Gamma(w)]$ will each have to be adjacent to two vertices of H and these neighborhoods will have to be non-overlapping. Select v_1, v_2 and v_3 from the three neighborhoods. Moreover, in order to avoid a C_5 the two vertices of the K_2 in $G[\Gamma(w)]$ will share a common neighbor in H, say v, as illustrated in the following figure.

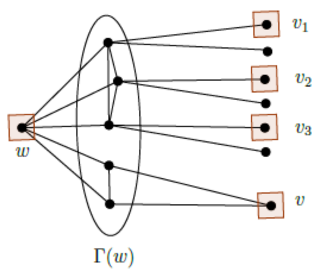

Figure 1: The only option left in case 3

Then, in order to avoid a C_5, $\{w,v_1,v_2,v_3,v\}$ will be an independent set, contrary to the assumption.

Lemma 7. *A* C_6*-free graph of order* 20 *with no independent set of 5 vertices contains an isomorphic copy of* $4K_5$.
Proof. Let G be a C_5-free graph of order 20 with no independence set of 5 vertices. First note that, by lemma 1, $\delta(G) \geq 4$ as $r(C_6, K_4) = 16$.

Remark 2. Any vertex can be adjacent to one vertex of a disjoint K_5. Further any two adjacent vertices together can be adjacent to at most one vertex of a K_5.
Case 1: The minimum degree is 4.
Say w is a vertex of minimum degree. Then $G[V(G)\backslash\Gamma^*(w)]$ will satisfy the conditions of lemma 4. Thus, using all possible cases and using remark 2, it follows that any two non-adjacent vertices of $G[\Gamma(w)]$ will give rise to an independent set of size five containing them. Therefore, $G[\Gamma(w)]\cong K_4$. This will result in the required $4K_5$.

Case 2: The minimum degree greater than or equal to 5.
Suppose that G is a C_6 -free graph on 20 with no independent set of 5 vertices. Then as $r(C_5, K_5) = 17$ (see [3, 10]) there exists a cycle $U = (u_1, u_2, ..., u_5, u_1)$ of length 5. Let $X = \{u_1, u_2, ..., u_5\}$. Define $H=G[X^c]$ as the induced subgraph of G not containing the vertices of the cycle and $H_1=G[X]$. Then, $|V(H)|=15$ and $|V(H_1)| = 5$.

Suppose that there exists an independent set Y in H of size 4 consisting of the four vertices y_1, y_2,y_3 and y_4.In order to avoid an independent set of size 5, each vertex of X must be adjacent to at least one vertex of Y. Clearly, no vertex of Y can be adjacent to three vertices of X as two of these three adjacent vertices will have to be consecutive vertices of the C_5 (which will result in a C_6 containing all the vertices of X). Therefore, each vertex of Y is adjacent to at most two vertices of X and if they are adjacent to two vertices of X they must be length 2 apart. As $|X| = 5$, we get that without loss of generality y_1 will have exactly two red neighbors in X length 2 apart (say $u_1$and u_3). Then without loss of generality u_2 is adjacent to y_2 and in order to avoid a red C_6, y_2 cannot be adjacent to any other vertex of X. Without loss of generality $\{y_3, y_4\}$ will have either two vertices or one vertex or no vertices that are adjacent to two vertices of X. Furthermore, any vertex of $\{y_3, y_4\}$ adjacent to 2 vertices of X must be adjacent to either u_3 and u_5 or u_1 and u_4 or u_1 and u_3. Moreover, if y_3 is adjacent to u_3, u_5 and y_4 is adjacent to u_4, u_1 then $(u_1, u_5, y_3, u_3, u_4, y_4, u_1)$ will be a cycle of length 6, contrary to our assumption. If, y_3 is adjacent to u_1, $u_3$wewill get that y_4 will be forced to be adjacent to u_4, u_5 and then $(u_1, u_2, u_3, u_4, y_4, u_5, u_1)$ will be a cycle of length 6, contrary to our assumption. Therefore, without loss of generality, by symmetry, we are left with the two possibilities where y_3 is adjacent to u_5 and y_4 is adjacent to u_4 or else y_3 is adjacent to u_3, u_5 and y_4 is adjacent to u_4. In the first option, in order to avoid an independent set of size 5, u_1 must be adjacent to u_3. As y_2 cannot be adjacent to any other vertex of X it must be adjacent to 4 vertices (as minimum degree of G is greater than 5) outside of $X \cup Y$(say w_1,w_2,$w_3$and w_4), as illustrated in the Figure 2.

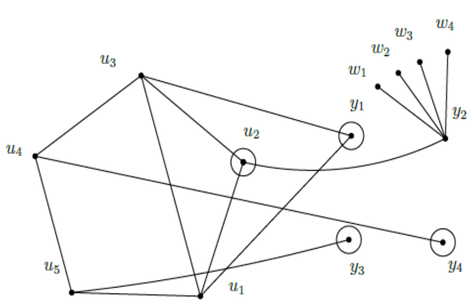

Figure 2: The first possibility

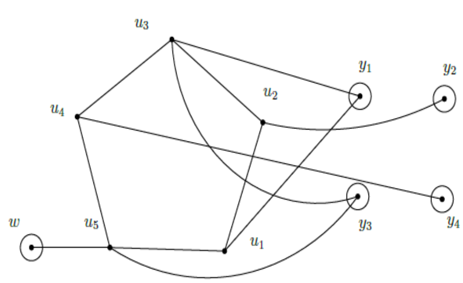

Figure 3: The second possibility

In order to avoid an independent set of size 5 consisting of $\{w_1,u_2,y_1,y_3,y_4\}$ we get w_1 must be adjacent to u_2. Similarly, we can conclude that all other vertices of $W = \{w_1, w_2, w_3, w_4\}$ too have to be adjacent to u_2. But then $G[W] = K_4$, as otherwise any two non-adjacent vertices of W together with $\{y_1, y_3, y_4\}$ will form an independent set of 5 vertices, contrary to the assumption. This will give us a C_6 containing all vertices W, contrary to our assumption.

In the second option, as u_5 cannot be adjacent to u_2 (may be adjacent to u_3), it will have to be adjacent to some other vertex (since the minimum degree of G is greater than 5) outside of $X \cup Y$ (say w), as illustrated in the Figure 3. But then, w cannot be adjacent to any vertex of Y as any such occurrence will lead to some C_6. Therefore, $Y \cup \{w\}$ will be an independent set of size 5, contrary to our assumption.

Thus, the initial assumption is false. That is, there is no independent set of order 4 in H. By lemma 4, we can conclude that H is equal to one of the five graphs $R_{15,1}, R_{15,2}, R_{15,3}, R_{15,4}$ or $R_{15,5}$. Now consider any two vertices of U, say u and v, and suppose that $(u, v) \notin E(G)$. Since there is no C_6 in G, each of the vertices u and v must be adjacent to at most one vertex of each copy of K_5 in H. Therefore, we can select vertex x_1 in the first K_5, vertex x_2 in the second K_5 and vertex x_3 in the third K_5 such that x_1, x_2 and x_3 are independent and not adjacent to u or v. This gives us that $\{u, v, x_1, x_2, x_3\}$ is an independent set of order 5, a contradiction. Therefore, $(u, v) \in E(G)$. Since u, v are arbitrary vertices in U, we can conclude that X induces a K_5asrequired.

Lemma 8. *A C_n-free graph (where $n \geq 7$) of order $4(n-1)$ with no independent set of 5 vertices contains an isomorphic copy of $4K_{n-1}$.*

Proof: Suppose that G is a C_n-free graph on $4(n-1)$ with no independent set of 5vertices. Then as $r(C_{n-1}, K_5) = 4n-7$ (see[3,10]) there exists a cycle $(u_l, u_2, \ldots, u_{n-1}, u_1)$ of length $n-1$. Let $X = \{u_l, u_2, \ldots, u_{n-1}\}$. Define $H = G[X^c]$ as the induced subgraph of G not containing the vertices of the cycle and $H_1 = G[X]$. Then, $|V(H)| = 3(n-1)$ and $|V(H_1)| = n-1$.

Suppose that there exists an independent set Y in H of size 4 consisting of the four vertices y_1, y_2, y_3 and y_4. That is, $\alpha(G) = 4$. From lemma 5(d) (as $5 \leq (n+3)/2$), it follows that every vertex y is adjacent to at most two vertex of the cycle C_{n-1}.

Case 1: $n \geq 10$

Then as $n-1 > 8$, we will get a independent set of size 5, containing Y; a contradiction.

Case 2: $n = 9$

In order to avoid an independent set of size 5, each vertex of X must be adjacent to at least one vertex of Y. Therefore as $|X| = 8$, we get that each y_i $(1 \leq i \leq 4)$ will have exactly two disjoint neighbors in X and without loss of generality then we have the following subcases. In the first subcase, we assume that without loss of generality Y has a vertex which is adjacent to two vertices of X which are length 2 apart in the C_8. In the second subcase we assume that without loss of generality Y has a vertex which is adjacent to two vertices of X which are length 3 apart in the C_8. In the third subcase we assume that without loss of generality all vertices of Y are adjacent to two vertices of X which are length 4 apart in theC_8.

Subcase 2.1: Y has a vertex which is adjacent to two vertices of X which arelength 2 apart in theC_8.

Without loss of generality assume that y_1 is adjacent in to u_1 and u_3 and y_2 is adjacent in to

u_2. In order to avoid an independent set of size 5 consisting of $\{u_1, u_3, y_2, y_3, y_4\}$, we get that (u_1, u_3) is an edge. Without loss of generality, then y_2 is adjacent to one vertex of $\{u_8, u_7, u_6\}$. In the first possibility, as (u_2, u_8) must be an edge and therefore we will get a cycle $(u_3, ..., u_8, y_2, u_2, u_1, u_3)$ of length 9, contrary to the assumption. In the next possibility, as (u_2, u_7) must be an edge and therefore we will get a cycle $(u_3, u_4, u_5, u_6, u_7, y_2, u_2, u_1, y_1, u_3)$ of length 9, contrary to the assumption. In the last possibility without loss of generality gives rise to two scenarios as illustrated in Figure 4 and Figure 5.

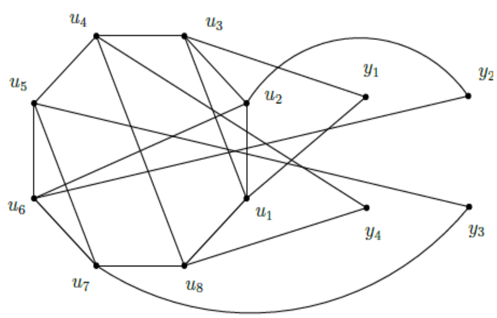

Figure 4: The first scenario

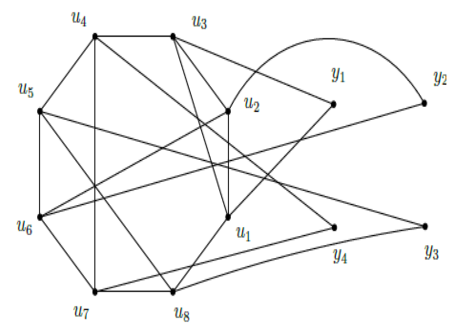

Figure 5: The second scenario

In the first scenario, without loss of generality, y_3 is adjacent to u_5 and u_7 and y_4 is adjacent to u_4 and u_8. Moreover, (u_2, u_6), (u_4, u_8) and (u_5, u_7) must be edges. Thus, we get a cycle $(u_2, u_1, y_1, u_3, u_4, u_8, u_7, u_5, u_6, u_2)$ of length 9, contrary to the assumption.

In the second scenario, without loss of generality, y_3 is adjacent to u_5 and u_8 and y_4 is adjacent to u_4 and u_7. Moreover, (u_2, u_6), (u_5, u_8) and (u_4, u_7) must be edges. Thus, we get a cycle $(u_1, y_1, u_3, u_4, u_7, u_8, u_5, u_6, u_2, u_1)$ of length 9, contrary to the assumption.

Subcase 2.2: *Y* has a vertex which is adjacent to two vertices of *X* which are length 3 apart in the C_8

Without loss of generality assume that y_1 is adjacent to u_1 and u_4 and y_2 is adjacent to u_3. In order to avoid an independent set of size 5 consisting of $\{u_1, u_4, y_2, y_3, y_4\}$, we get that (u_1,u_4) is an edge. Without loss of generality, then y_2 must be adjacent to one vertex of $\{u_2,u_7,u_8,u_6\}$.

The first possibility, is impossible as we get a cycle $(u_1,u_2,y_2,u_3,...,u_8,u_1)$ of length 9, contrary to the assumption. In the second possibility we get a cycle $(u_7, y_2, u_3, u_2, u_1, y_1, u_4, u_5, u_6, u_7)$ of length 9, contrary to the assumption. In the third possibility we get a cycle$(u_5,u_4,u_1,u_2,u_3,y_2,u_8,u_7,u_6,u_5)$of length 9,contrary to the assumption. In the fourth possibility is without loss of generality y_3 is adjacent to u_2. But then the only options for y_3 to be adjacent to is to two vertices of $\{u_5, u_8, u_7\}$. However, if y_3 is adjacent to u_2 and u_5 it leads to $(u_1,u_4,u_3,u_2,y_3,u_5,u_6,u_7,u_8,u_1)$,a cycle of length 9 and if y_3 is adjacent in red to u_2 and u_8 it leads to $(u_8,y_3,u_2,u_1,u_4,u_3,y_2,u_6,u_7,u_8)$ a cycle of length 9, contrary to the assumption. Therefore, we are left with the scenario when y_3 is adjacent to u_2 and u_7. But then in order to avoid an independent set of size 5, we get that (u_2, u_7) and (u_3, u_6) are edges. This is illustrated in Figure 6.

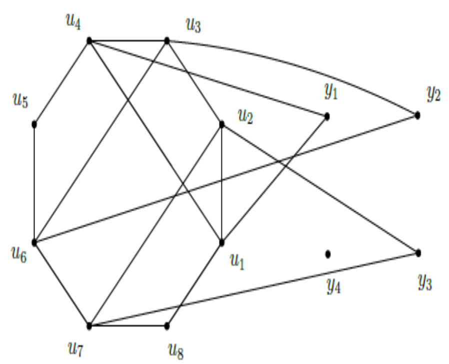

Figure 6: The only remaining scenario

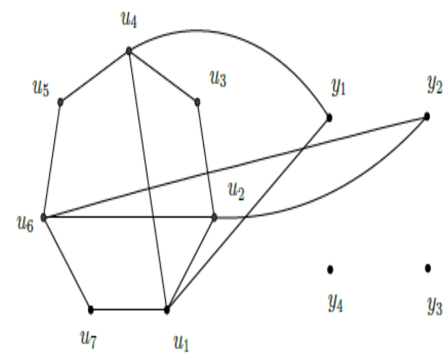

Figure 7: The only remaining scenario of subcase 3.2

However, even in this scenario too we get a cycle (u_1, u_8, u_7, u_2, u_3, u_6, u_5, u_4, y_1, u_1) of length 9, contrary to the assumption.

Subcase 2.3: All vertices of Y are adjacent to two vertices of X which are length 4 apart in the C_8

Without loss of generality assume that y_1 is adjacent in to u_1 and u_5 and y_2 is adjacent to u_2 and u_6 and y_3 is adjacent to u_4 and u_8. In order to avoid an independent set of size 5, we get that (u_1, u_5), (u_2, u_6) and (u_4, u_8) are edges. But then we get cycle (u_3, u_4, u_8, u_7, u_6, u_5, y_1, u_1, u_2, u_3) of length 9, contrary to the assumption.

Case 3: $n = 8$

In order to avoid an independent set of size 5, each vertex of X must be adjacent to at least one vertex of Y. Therefore as $|X| = 7$, we get that without loss of generality each y_i $(1 \leq i \leq 2)$ will have exactly two disjoint neighbors in X such that $\Gamma_R(x) \cap \Gamma_R(x) \cap X = \phi$ for all $x \in \{y_1, y_2\}$and $y \in \{y_3, y_4\}$. This will give rise to the following subcases. In the first subcase, we assume that without loss of generality $\{y_1, y_2\}$has a vertex which is adjacent to two vertices of X which are length 2 apart in theC_7.In the second subcase we assume that without loss of generality both $\{y_1, y_2\}$ are adjacent to two vertices of X which are length 3 apart in the C_7.

Subcase 3.1: $\{y_1, y_2\}$ has a vertex which is adjacent to two vertices of X which are length 2 apart in the C_7.

Without loss of generality assume that y_1 is adjacent to u_1 and u_3. Moreover, y_2 is adjacent to u_2 and u_4 or y_2 is adjacent to u_2 and u_5 or else y_2 is adjacent to u_4 and u_6. In order to avoid an independent set of size 5 consisting of $\{u_1, u_3, y_2, y_3, y_4\}$, we get that ($u_1$,$u_3$)is an edge. In the first possibility we will get a cycle (u_1,u_3,u_2,y_2,u_4,u_5,u_6,u_7, u_1) of length 8, contrary to the assumption. In the second possibility, we will get a cycle (u_1, y_1, u_3, u_2, y_2, u_5, u_6, u_7, u_1) of length 8, contrary to the assumption. Therefore, we may assume the only remaining option that y_2 is adjacent to u_4 and u_6. In order to avoid an independent set of size 5 consisting of $\{u_1, u_3, y_2, y_3, y_4\}$, we get that ($u_4$, u_6) is a edge. Since without loss of generality, y_3 is adjacent to two vertices distinct from $\{u_1, u_3, u_4, u_6\}$, we get three possibilities given by y_3 is adjacent to u_2 and u_5 or y_3 is adjacent to u_2 and u_7 or else y_3 is adjacent to u_5 and u_7. This will give a cycles (u_1,u_3,u_2,y_3,u_5,u_4,u_6,u_7,u_1),

$(u_4,u_3,u_1,u_2,y_3,u_7,u_6,u_5,u_4)$, $(u_1,u_2,u_3,u_4,u_6,u_5,y_3,u_7,u_1)$ respectively in the three cases, contrary to the assumption.

Subcase 3.2: Both $\{y_1, y_2\}$ are adjacent to two vertices of X which are length 3 apart in the C_7.
Without loss of generality assume that y_1 is adjacent to u_1 and u_4. By symmetry, we may assume that without loss of generality y_2 is adjacent to u_2 and u_5 or u_2 and u_6. In order to avoid an independent set of size 5 consisting of $\{u_1, u_4, y_2, y_3, y_4\}$, we get that (u_1,u_4) is an edge. In the first possibility, y_2 is adjacent to u_2 and u_5. However, in order to avoid an independent set of size 5, (u_2, u_5) is an edge and thus we will get a cycle $(u_1, u_4, u_3, u_2, y_2, u_5, u_6, u_7, u_1)$ of length 8, contrary to the assumption. In the second possibility, y_2 is adjacent to u_2 and u_6. In order to avoid an independent set of size 5 consisting of $\{u_2, u_6, y_1, y_3, y_4\}$, we get that (u_2, u_6) is an edge. This scenario isillustratedinFigure7.Thus, we will get a cycle $(u_1, y_1, u_4, u_3, u_2, y_2, u_6, u_7, u_1)$ of length 8, contrary to the assumption.

Case 4: $n = 7$
In order to avoid an independent set of size 5, each vertex of X must be adjacent to at least one vertex of Y. Note that in order to avoid a C_7, no vertex of Y can be adjacent to two vertices of X which are length 1 apart in the C_6. Thus, we assume that without loss of generality there are two main subcases generated by when Y has a vertex which is adjacent to two vertices of X which are length 2 or 3 apart in the C_6 and no other vertex of Y is adjacent to these two neighbors (subcase 4.1) and when such a situation doesn't exist (subcase4.2).

Subcase 4.1: Y has a vertex which is adjacent to two vertices of X which are length 2 or 3 apart in the C_6 and no other vertex of Y is adjacent to these two neighbors

Subcase 4.1.1: Y has a vertex which is adjacent to two vertices of X which are length 2 apart in the C_6 and no other vertex of Y is adjacent to these two neighbors.
As Y has a vertex which is adjacent to two vertices of X which are length 2 apart in the C_6, without loss of generality, assume that y_1 is adjacent to u_1 and u_3 and no other vertex of Y is adjacent to u_1 and u_3. Then, either y_2 is adjacent to u_2 and u_4 or y_2 is adjacent to u_2 and u_5 or else y_2 is adjacent to u_4 and u_6. In order to avoid an independent set of size 5 consisting of $\{u_1, u_3, y_2, y_3, y_4\}$, we get that (u_1, u_3) is an edge.
In the first possibility, we will get a cycle $(u_1,u_3,u_2,y_2,u_4,u_5,u_6,u_1)$ of length 7, contrary to the assumption. In the second possibility we get a cycle $(u_1,y_1,u_3,u_2,y_2,u_5,u_6,u_1)$ of length7, contrary to the assumption. In the last option, we get that y_2 is adjacent to u_4 and u_6. Next note that, y_3 cannot be adjacent to both u_2 and u_5 as then we will get a cycle $(u_1,y_1,u_3,u_2,y_2,u_5,u_6,u_1)$ of length7, contrary to the assumption. Similarly, y_4 cannot be adjacent to both u_2 and u_5. Therefore, without loss of generality (excluding the possibilities we have already discussed in case 4.1.1), we get that y_3 is adjacent to u_5 and y_4 is adjacent to u_2. Moreover, one can observe that in order to avoid a C_7 and the cases already discussed in case 4.1.1 both y_3 and y_4 cannot be adjacent to no more vertices of X. In order to avoid an independent set of size5, we get that (u_4,u_6) is an edge. Also, in the induced subgraph of X, in order to avoid a C_7, the only other possible edge is either (u_1,u_4) or (u_3,u_6), but not both. Therefore, in the graph of $G[X\cup Y]$

illustrated in the following figure 8. Without loss of generality (u_3,u_6) is marked as a broken edge to indicate that it is a possible edge (see figure 2) and its worth noting that $\{y_1, y_2, u_2, u_5\}$ forms an independent set of size 4.

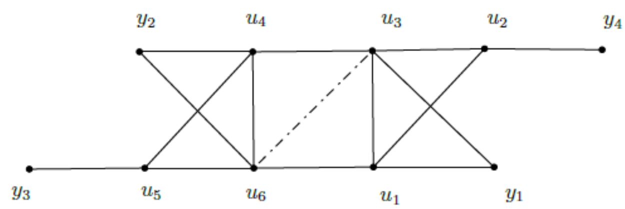

Figure 8: The structure of the graph induced by $G[X \cup Y]$

The graph G is an order 24 graph with no C_7 and no independent set of size 5. Therefore, by applying lemma 1, we get that the degree of each of the vertices of G is greater than or equal to 5. Therefore, u_4 has to be adjacent to some vertex outside $X \cup Y$ (say w). In order to avoid an independent set of size 5 consisting of $\{w,y_1,y_2,u_2,u_5\}$, we get that w must be adjacent to one of the vertices of $\{y_1,y_2,u_2,u_5\}$.This will give a C_7 containing w, contrary to the assumption.

Subcase 4.1.2: Y has a vertex which is adjacent to two vertices of X which are length 3 apart in the C_6 and no other vertex of Y is adjacent to these two neighbors

Without loss of generality, assume that y_1 is adjacent to u_1 and u_4 and no other vertex of Y is adjacent u_1 and u_4. In order to avoid an independent set of size 5, we get that (u_1, u_4) is an edge. Hence, one of the vertices in $\{y_2,y_3,y_4\}$ (say y_2), must be adjacent to either u_2 and u_5 or u_2 and u_6 or u_3 and u_5 or else u_3 and u_6. However, if y_2 is adjacent to u_2 and u_5 (see Figure 9), we will get a cycle $(u_1,u_4,u_3,u_2,y_2,u_5,u_6,u_1)$of length 7, contrary to the assumption. Similarly, if y_2 is adjacent to u_2 and u_6, we will get a cycle $(u_4,y_1,u_1,u_6,y_2,u_2,u_3,u_4)$ and if y_2 is adjacent to u_3 and u_5, we will get a cycle $(u_1,y_1,u_4,u_3,y_2,u_5,u_6,u_1)$ and if y_2 is adjacent to u_3 and u_6, we will get a cycle $(u_1,u_4,u_5,u_6,y_2,u_3,u_2,u_1)$, contrary to the assumption.

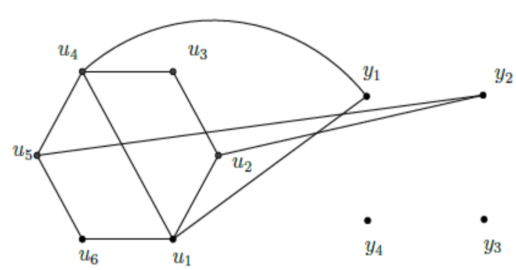

Figure 9: In subcase 4.2 when y_2 is adjacent to u_2 and u_5

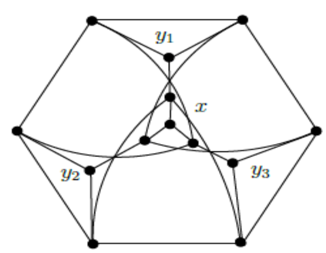

Figure 10: The unique $r(C_4, K_5)$ - critical graph R_{13}

Subcase 4.2: Y has a vertex which is adjacent to two vertices of X which are length 2 or 3 apart in the C_6 and some other vertex of Y is adjacent in to at least one of these two neighbors

Without loss of generality, assume that y_1 is adjacent to u_1 and w length 2 or 3 apart. If one

other vertex in Y is adjacent to at least one vertex $\{u_1, w\}$ and not adjacent to any other vertex in X outside $\{u_1, w\}$it would lead to one of the previous two subcases. Therefore, without loss of generality, we may assume that, y_1 is adjacent to u_1 and u_3 and y_2 is adjacent to u_1 and u_4 or else y_1 is adjacent to u_1 and u_3 and y_2 is adjacent to u_1 and u_5. In the first possibility, in order to avoid the previous subcases, without loss of generality the only option left is for y_3 to be adjacent to u_2 and u_5 or y_3 is adjacent to u_2 and u_6. If y_3 to be adjacent to u_2 and u_5, we get a cycle $(u_1, u_6, u_5, y_3, u_2, u_3, y_1, u_1)$ of length 7 and if y_3 to be adjacent to u_2 and u_6, we get a cycle $(u_4, y_2, u_1, u_6, y_3, u_2, u_3, u_4)$ of length 7, contrary to the assumption. In the second possibility, in order to avoid a C_7, we get $\{u_2, u_4, u_6\}$ will induce an independent set of size 3. But then $\{y_1, y_2, u_2, u_4, u_6\}$ will induce an independent set of size 5 contrary to the assumption.
Now continuing with the proof of the lemma for all four cases we may assume that H is a C_n-free graph of order $3(n-1)$ with no independent set of size 4. By [7], we can deduce that H is equal to $3K_{3n-3,1}$, $3K_{3n-3,2}$, ...,$2K_{3n-3,6}$, as $n \geq 5$. In any case H contains a copy of a $3K_{n-1}$.Now consider any two vertices of $V(C_{n-1})$ say u and v. Since there is no C_n in G, each of the vertices u and v must be adjacent to at most one vertex each of the three copies of K_4 in H. Also, there can be at most two edges connecting a given K_{n-1} in the copy of the $3K_{n-1}$ to the remaining $2K_{n-1}$ subgraphs. Therefore, as $n > 5$, we can select three vertices x_1, x_2 and x_3 in first K_{n-1}, second K_{n-1} and third K_{n-1} respectively, such that x_1, x_2 and x_3 and are not adjacent to u or v. As $\{u, v, x_1, x_2, x_3\}$ cannot be an independent set of size 5, this will force $(u,v)\in E(G)$. Therefore, $V(C_n)$will induce a K_{n-1}, since u,v are arbitrary elements of $V(C_n)$. Hence the lemma.

4. Main result

Theorem 1. *If* $n \geq 4$, *then*

$$r_*(C_n, K_5) = \begin{cases} 13 & \text{if} \quad n = 4 \\ 3n-1 & \text{if} \quad n \geq 5 \end{cases}$$

Proof: We break up the proof into two cases.

Case $n = 4$
Let R_{13} (see Figure 13) represent the unique Ramsey critical (C_4, K_5) graph (see [8]). Let R^*_{13} represent the graph of order 14, obtained from R_{13} by adding a vertex y and connecting it to the vertices y_1, y_2, y_3. Color the edges of $K_{13}\sqcup K_{1,12} \cong K_{14} - \text{e}$, using red and blue such that the red graph is isomorphic to R^*_{13}. Then the blue graph will be is isomorphic to the complement of the graph R^*_{13}in $K_{14} - \text{e}$ where $e = (x, y)$.

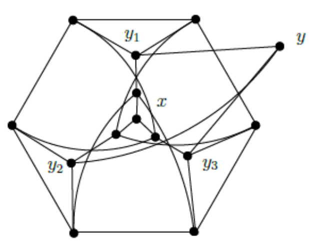

Figure 11: A graph isomorphic to the red coloring of $K_{14}-e$ which contains no C_4 and which contains no K_5 in the complement with respect to $K_{14}-e$. Notice the missing edge e, is refers to the edge in $(R^*_{13})^c$ connecting the nodes labeled y and x

Case $n \geq 5$
Color the graph $K_{4(n-1)+1} \setminus K_{n-2}$ using red and blue colors, such that the red graph consists of a $3K_{n-1} \cup (K_{n-1} \sqcup K_{1,1})$ as illustrated in the following figure.

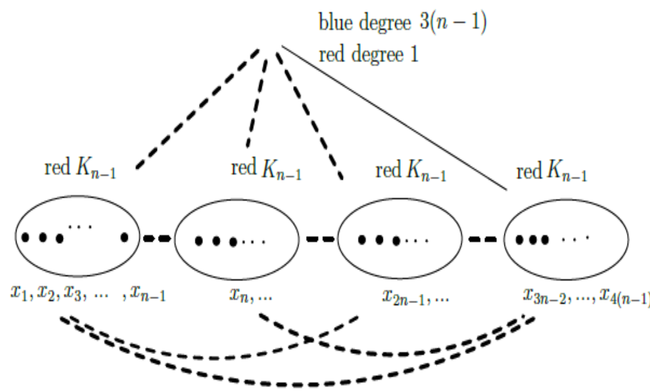

Figure 12: A coloring of $K_{4(n-1)+1} - K_{n-2}$ which contains no red $C_n$and no blue K_5

Therefore, $r_*(C_n, K_5) \geq 3n - 1$. Next to show that, $r_*(C_n, K_5) \leq 3n - 1$, consider any red/blue coloring of a graph $G = K_{4(n-1)+1} \setminus K_{n-2}$ contains no red C_n and no blue K_5.Let H be the graph obtained by deleting the vertex of degree $3n-1$ (say v) from G.
Then H is a graph on $4(n - 1)$ vertices such that it contains no red C_n or a blue K_4. Therefore, by lemma 6, lemma 7 and lemma 8 we get that H with a red $4K_{n-1}$. Let us denote the sets of vertices of the four connected components by V_1, V_2, V_3 and V_4 respectively. Since there is no red C_n in the coloring, v has at most one red neighbor in each of the four sets V_1, V_2, $V_3$and V_4.

Case 1: v is adjacent to at exactly three vertices of some $V_i(1 \leq i \leq 4)$.
Without loss of generality, we may assume that v is adjacent to all the vertices of V_1, V_2 and V_3. That is, v is adjacent to at least 5 vertices of each $V_i(1 \leq i \leq 3)$. Select $v_1 \in V_1$, $v_2 \in V_2$ and $v_3 \in V_3$ such that v_1 has no red neighbors in $G[V_2 \cup V_3 \cup V_4 \cup \{v\}]$, v_2 has no red neighbors in $G[V_1 \cup V_3 \cup V_4 \cup \{v\}]$ and v_3 has no red neighbors in $G[V_1 \cup V_2 \cup V_4 \cup \{v\}]$(this is possible because each V_i can have at most 4 vertices with red neighbors outside V_i). Because there are no red C_n's in the coloring, we can find a $v_4 \in V_4$ such that $\{v, v_4\}$ is colored blue. Then, $\{v, v_1, v_2, v_3, v_4\}$will induce a blue K_5, a contradiction.

Case 2: v is adjacent to exactly two vertices of some V_i (say V_4).
Without loss of generality, we may assume that v is adjacent to all the vertices of V_1,V_2 and all but one vertex of V_3. That is, v is adjacent to at least 5 vertices of each V_i $(1 \leq i \leq 2)$ and at least 4 vertices of V_3. Since there are at least two vertices of V_4 adjacent to v in blue, select $v_4 \in V_4$ such that v_3 has no red neighbors in $G[V_3 \cup \{v\}]$. Next select any vertex $v_3 \in V_3$ such that (v_3, v) is colored blue. Then, $\{v, v_3, v_4\}$ will induce a blue K_3. Finally select $v_1 \in V_1$ and $v_2 \in V_2$ such that v_1 has no red neighbors in $G[V_2 \cup V_3 \cup V_4 \cup \{v\}]$ and v_2 has no red neighbors in $G[V_1 \cup V_3 \cup V_4 \cup \{v\}]$ (this is possible because each V_i can have at most 4 vertices with red neighbors outside V_i). Then, $\{v, v_1, v_2, v_3, v_4\}$ will induce a blue K_5, a contradiction.

Case 3: Both case 1 and 2, do not hold.

Without loss of generality, we may assume that v is adjacent to at least 5 vertices of each V_1 and at least 4 vertices of ($2 \leq i \leq 4$). Since there are at least three vertices of V_4 adjacent to v in blue, select $v_4 \in V_4$ such that v_4 has no red neighbors in $G[V_2 \cup V_3 \cup \{v\}]$. Next select $v_2 \in V_2$ and $v_3 \in V_3$ such that (v_2,v), (v_3,v), (v_2,v_3) are colored blue. Then, $\{v, v_2, v_3, v_4\}$ will induce a blue K_4. Finally select $v_1 \in V_1$ such that v_1 has no red neighbors in $G[V_2 \cup V_3 \cup V_4 \cup \{v\}]$ (this is possible because each V_i can have at most 4 vertices with red neighbors outside V_i). Then, $\{v, v_1, v_2, v_3, v_4\}$ will induce a blue K_5, a contradiction. Hence $r_*(C_n, K_4) \leq 3n - 1$. Therefore, $r_*(C_n, K_4) = 3n - 1$.

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