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Progress in **Nonlinear Dynamics and Chaos**

Blow-up Phenomena for a Class of Degenerate Parabolic Problems with Multiple Nonlinearities

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Abstract. In this paper, we study the blow-up solution to a nonlinear degenerate parabolic equation

$$u_t = u\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)+\alpha_0|\nabla u|^p-\alpha_1|\nabla u|^{2q}+\alpha_2\int_O u^s\mathrm{d}x+\alpha_3 u^{p(x)}-\alpha_4 u^{q_2}$$

under nonlinear boundary condition. By constructing some appropriate auxiliary functions and using first-order differential inequality technique, an explicit formula of lower bound for blow-up time is derived.

Key words: Multiple nonlinearities; nonlinear degenerate parabolic equations; blow-up; nonlinear boundary condition.

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1. Introduction

Lower bounds for blow-up time of solution to degenerate parabolic problem have been extensively studied in the last 10 years [1-6]. Payne and Song in [1] considered an initial-boundary value problem for parabolic equations of the form

$$\frac{\partial u}{\partial t}=\Delta u+u^p-|\nabla u|^{q_1} \quad \text{in} \quad O\times(0,T^*) \tag{1}$$

where

$$u=0 \quad \text{on} \quad \partial O\times(0,T^*), \quad u(x,0)=u_0(x)\geq 0 \quad \text{in} \quad O.$$

Here O is a bounded domain in $\mathbb{R}^3$, Δ is the Laplace operator, ∇ is the gradient operator, ∂O is the boundary of O, and T^* is the possible blow-up time. A lower bound for the blow-up time T^* was determined under the condition $p \le q_1$, and the relative result in [7] was extended to the case with nonlinear boundary condition.

In [8] the authors studied the question of blow-up for the solution to the problem

$$u_t = \Delta u + \int_O u^s \mathrm{d}x - \alpha u^{q_2} \quad \text{in} \quad O \times (0, T^*), \tag{2}$$

$$u(x,0) = u_0(x) \ge 0 \quad \text{in} \quad O.$$

with both homogeneous Dirichlet boundary condition and homogeneous Neumann boundary condition. They obtained the lower bounds for the blow-up time under the above two boundary conditions. Later, others generalized this result to the case of nonlinear boundary condition [9] or Robin condition [2].

In this paper, we consider the following nonlinear parabolic equation generalized from (1) and (2)

$$u_t = u\,\mathrm{div}\left(|\nabla u|^{p-2} \nabla u\right) + \alpha_0(x)|\nabla u|^p - \alpha_1(x)|\nabla u|^{q_1} + \alpha_2(x)\int_O u^s \mathrm{d}x + \alpha_3(x) u^{p(x)} - \alpha_4(x) u^{q_2} \tag{3}$$

with the following nonlinear boundary condition

$$\frac{\partial u}{\partial \vec{n}} = g(u) \quad \text{in} \quad O \times (0, T^*) \tag{4}$$

and the initial condition

$$u(x,0) = u_0(x) \ge 0 \quad \text{in} \quad O. \tag{5}$$

Here $p > 2$, $\vec{n}$ is the unit outer normal vector of ∂O, and $\frac{\partial u}{\partial \vec{n}}$ is outward normal derivative of u on the boundary ∂O which is assumed to be sufficiently smooth. Moreover, we assume that

$$1 < q^- := \inf_{x\in O} q(x) \le q(x) \le q^+ := \sup_{x \in O} q(x) < +\infty,\ 0 < \alpha_0(x) \le c_0 := \sup_{x\in O} \alpha_0(x) < +\infty,$$

$$0 < c_1 := \inf_{x\in O} \alpha_1(x) \le \alpha_1(x) < +\infty,\ 0 < \alpha_2(x) \le c_2 := \sup_{x\in O} \alpha_2(x) < +\infty,$$

$$0 < \alpha_3(x) \le c_3 := \sup_{x\in O} \alpha_3(x) < +\infty,\ 0 < c_4 := \inf_{x\in O} \alpha_4(x) \le \alpha_4(x) < +\infty.$$

As indicated in [1,7,8,9], we also need $q_1 > 1$, $q_2 > 1$. Reference [7] assumed that $s > q_2 > 1$, here we release this restriction by $s > 0$.

Since the initial data $u_0(x)$ in (5) is nonnegative, we have by the parabolic maximum principles [10,11] that u is nonnegative in $O \times (0, T^*)$. In the next section, we will find a lower bound for the blow-up time when blow-up occurs.

2. A lower bound for the blow-up time

In this section we seek the lower bound for the blow-up time T^*. To this end, we define an auxiliary function of the form

$$E(t)=\int_O u^{psn+1}\mathrm{d}x \quad \text{with} \quad n>1, \quad \rho_1=\min_{\partial O}|x|, \quad \rho_0=\min_{\partial O}|x\cdot\vec{n}| \tag{6}$$

and make an assumption on $g(z)$

$$g(z)\le kz^{\sigma}, \tag{7}$$

where k is a positive constant. Our assumption is weaker than the one in [10], it requires $0\le g(z)\le kz^{\sigma}$, and σ depends on the choice of $E(t)$. Here we allow σ to be any positive constant. Furthermore, reference [12] indicated that if $c_0-1-nps\le 0$, the solution will not blow-up in finite time. So we consider the case $c_0-1-nps>0$.

The main result of this article is formulated in the following theorem:

Theorem 1 Let $u(x,t)$ be the nonnegative classical solution to problem (3)-(5), and g satisfies (7). Then for any

$$\frac{1}{(p-2)s}<n<\frac{c_0-1}{ps},$$

the blow-up time T^* is bounded from below by

$$\int_{E(0)}^{\infty}\frac{\mathrm{d}\tau}{A_0+A_{11}\tau^{\frac{pns+1}{pns+(\sigma p/2)}}+(A_1+A_6)\tau^{\frac{3}{2}}+(A_2+A_7)\tau^3+(A_5+A_{10})\tau},$$

where A_0, A_1, A_2, A_5, A_6, A_7, A_{10} and A_{11} are positive constants to be determined later.

Proof: First we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t)&=(psn+1)\int_O u^{psn}u_t\mathrm{d}x\\
&=(psn+1)\int_O u^{psn}\left(u\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)+\alpha_0|\nabla u|^p\right)\mathrm{d}x\\
&\quad-(psn+1)\int_O\alpha_1(x)u^{psn}|\nabla u|^{q_1}\mathrm{d}x+(psn+1)\int_O\alpha_2(x)u^{psn}\mathrm{d}x\int_O u^s\mathrm{d}x\\
&\quad+(psn+1)\int_O\alpha_3(x)u^{psn+p(x)}\mathrm{d}x-(psn+1)\int_O\alpha_4(x)u^{psn+q_2}\mathrm{d}x\\
&\le-\frac{(c_0-1-nps)(psn+1)}{(sn+1)^p}\int_O\left|\nabla u^{1+ns}\right|^p\mathrm{d}x+(psn+1)\int_{\partial O}u^{psn}u|\nabla u|^{p-2}\frac{\partial u}{\partial n}\mathrm{d}x\\
&\quad-(psn+1)\int_O\alpha_1(x)u^{psn}|\nabla u|^{q_1}\mathrm{d}x+(psn+1)|O|\int_O\alpha_2(x)u^{psn+s}\mathrm{d}x\\
&\quad+(psn+1)\int_O\alpha_3(x)u^{psn+p(x)}\mathrm{d}x-(psn+1)\int_O\alpha_4(x)u^{psn+q_2}\mathrm{d}x.
\end{aligned}\tag{8}$$

As long as $q_2>1$, we apply the holder inequality to get

$$(psn+1)\int_O \alpha_4(x)u^{psn+q_2}\,\mathrm{d}x \ge c_4(psn+1)\int_O u^{psn+q_2}\,\mathrm{d}x \ge c_4(psn+1)|O|^{\frac{1-q_2}{psn+1}} E(t)^{\frac{psn+q_2}{psn+1}}. \tag{9}$$

For convenience, let $v = u^{\frac{psn+q_1}{q_1}}$. It follows that

$$(psn+1)\int_O \alpha_1(x)u^{psn}|\nabla u|^{q_1}\,\mathrm{d}x = (psn+1)\left(\frac{psn+q_1}{q_1}\right)^{-q_1}\int_O \alpha_1(x)|\nabla v|^{q_1}\,\mathrm{d}x. \tag{10}$$

Using the Sobolev inequality derived in [14] (see 2.10) or [15] (see (4.10)), we have

$$\chi_1\int_O |\nabla v|^{q_1}\,\mathrm{d}x \ge \int_O |v|^{q_1}\,\mathrm{d}x, \tag{11}$$

where χ_1 is a positive constant to be determined later. Therefore, combining (11) and Holder inequality we get

$$(psn+1)\int_O \alpha_1(x)u^{psn}|\nabla u|^{q_1}\,\mathrm{d}x \ge \chi_1 c_1(psn+1)\left(\frac{psn+q_1}{q_1}\right)^{-q_1}|O|^{\frac{1-q_1}{psn+1}} E(t)^{\frac{psn+q_1}{psn+1}}. \tag{12}$$

Further, using (9) and (12), we replace (8) by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le & -\frac{(\alpha_0-1-nps)(psn+1)}{(sn+1)^p}\int_O |\nabla u^{1+ns}|^p\,\mathrm{d}x \\ & - c_1\chi_1(psn+1)\left(\frac{psn+q_1}{q_1}\right)^{-q_1}|O|^{\frac{1-q_1}{psn+1}} E(t)^{\frac{psn+q_1}{psn+1}} \\ & + (psn+1)|O|\int_O \alpha_2(x)u^{psn+s}\mathrm{d}x + (psn+1)\int_O \alpha_3(x)u^{psn+p(x)}\mathrm{d}x \\ & - c_4(psn+1)|O|^{\frac{1-q_2}{psn+1}} E(t)^{\frac{psn+q_2}{psn+1}} + (psn+1)\int_{\partial O} u^{psn}u|\nabla u|^{p-2}\frac{\partial u}{\partial n}\mathrm{d}x.\end{aligned} \tag{13}$$

Now, we focus on the term $(psn+1)|O|\int_O \alpha_2(x)u^{psn+s}\mathrm{d}x$ in (13). Using Holder and Young inequalities twice, we have

$$\begin{aligned} & \int_O u^{psn+s}\mathrm{d}x \le |O|^{\frac{1}{psn+s+1}}\left(\int_O u^{psn+s+1}\mathrm{d}x\right)^{\frac{psn+s}{psn+s+1}} \\ & \le \frac{1}{psn+s+1}|O| + \frac{psn+s}{psn+s+1}\int_O u^{psn+s+1}\mathrm{d}x \\ & \le \frac{1}{psn+s+1}|O| + \frac{psn+s}{psn+s+1}\left(\int_O u^{\frac{3}{2}(psn+1)}\mathrm{d}x\right)^{\frac{2s}{psn+1}}\left(\int_O u^{psn+1}\mathrm{d}x\right)^{\frac{psn+1-2s}{psn+1}} \\ & \le \frac{1}{psn+s+1}|O| + \frac{psn+s}{psn+s+1}\frac{2s}{psn+1}\int_O u^{\frac{3}{2}(psn+1)}\mathrm{d}x + \frac{psn+s}{psn+s+1}\frac{psn+1-2s}{psn+1}\int_O u^{psn+1}\mathrm{d}x \end{aligned} \tag{14}$$

and

$$\int_O \left|\nabla u^{\frac{1}{2}(psn+1)}\right|^2 \mathrm{d}x \le \frac{(psn+1)^2}{4(ns+1)^2}\left(\int_O |\nabla v|^p \,\mathrm{d}x\right)^{\frac{2}{p}} \left(\int_O v^{\frac{psn+1}{(ns+1)}-2}\mathrm{d}x\right)^{\frac{p-2}{p}}$$

$$\le \frac{(psn+1)^2}{2p(ns+1)^2}\int_O |\nabla v|^p \,\mathrm{d}x + \frac{p-2}{p}\frac{(psn+1)^2}{4(ns+1)^2}\int_O v^{\frac{psn+1}{(ns+1)}-2}\mathrm{d}x$$

$$\le \frac{(psn+1)^2}{2p(ns+1)^2}\int_O \left|\nabla u^{1+ns}\right|^p \mathrm{d}x + \frac{p-2}{p}|O|^{\frac{2(ns+1)}{psn+1}}\frac{(psn+1)^2}{4(ns+1)^2}E(t)^{\frac{psn+1-2(ns+1)}{psn+1}} \tag{15}$$

where $v = u^{1+ns}$. Then, we connect (14) and (16) by using the integral inequality derived in [6] (see (2.16)), namely

$$\int_O u^{\frac{3}{2}(psn+1)}\mathrm{d}x \le \frac{3^{\frac{3}{4}}}{2\rho_0^{\frac{3}{2}}}E(t)^{\frac{3}{2}} + \frac{\sqrt{2}}{3^{\frac{3}{4}}}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}}\left[\frac{E(t)^3}{4\chi_2^3} + \frac{3}{4}\chi_2\int_O \left|\nabla u^{\frac{1}{2}(psn+1)}\right|^2 \mathrm{d}x\right] \tag{16}$$

to obtain

$$(psn+1)|O|\int_O \alpha_2(x)u^{psn+s}\mathrm{d}x \le \frac{c_2(psn+1)}{psn+s+1}|O|^2 + A_1E(t)^{\frac{3}{2}} + A_2E(t)^3 + A_5E(t)$$
$$+ A_3\chi_2\int_O \left|\nabla u^{1+ns}\right|^p \mathrm{d}x + A_4\chi_2 E(t)^{\frac{psn+1-2(ns+1)}{psn+1}} \tag{17}$$

where χ_2 is another positive constant to be determined later,

$$A_1 = 3^{\frac{3}{4}}c_2\rho_0^{-\frac{3}{2}}(psn+1)|O|\frac{psn+s}{psn+s+1}\frac{s}{psn+1},\ A_2 = \frac{\sqrt{2}}{2}3^{-\frac{3}{4}}sc_2\chi_2^{-3}|O|\frac{psn+s}{psn+s+1}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}},$$

$$A_3 = \frac{1}{2\sqrt{2}}3^{\frac{1}{4}}c_2|O|\frac{(psn+s)s}{psn+s+1}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}}\frac{(psn+1)^2}{p(ns+1)^2},$$

$$A_4 = \frac{1}{4\sqrt{2}}3^{\frac{1}{4}}c_2\frac{p-2}{p}|O|^{\frac{2(ns+1)}{psn+1}+1}\frac{(psn+1)^2}{(ns+1)^2}\frac{(psn+s)s}{psn+s+1}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}},\ A_5 = c_2|O|\frac{(psn+s)(psn+1-2s)}{psn+s+1}.$$

Next we give a bound for the term $(psn+1)\int_O \alpha_3(x)u^{psn+p(x)}\mathrm{d}x$ in (13). For each $t>0$, we divide O into two sets,

$$O_0 = \{x\in O \mid u(x,t)<1\},\quad O_1 = \{x\in O \mid u(x,t)\ge 1\}.$$

It follows that

$$
\begin{aligned}
&(psn+1)\int_O \alpha_3(x)u^{psn+p(x)}\mathrm{d}x \\
&\le c_3(psn+1)\int_{O_0} u^{psn+p^-}\mathrm{d}x + c_3(psn+1)\int_{O_1} u^{psn+p^+}\mathrm{d}x \\
&\le \left(\frac{c_3(psn+1)}{psn+p^-+1}+\frac{c_3(psn+1)}{psn+p^++1}\right)|O| + c_3(psn+1)\frac{psn+p^-}{psn+p^-+1}\int_O u^{psn+p^-+1}\mathrm{d}x \\
&\quad + c_3(psn+1)\frac{psn+p^+}{psn+p^++1}\int_O u^{psn+p^++1}\mathrm{d}x \\
&\le c_3(psn+1)\left[\frac{psn+p^-}{psn+p^-+1}\frac{2p^-}{psn+1}+\frac{psn+p^+}{psn+p^++1}\frac{2p^+}{psn+1}\right]\int_O u^{\frac{3}{2}(psn+1)}\mathrm{d}x \\
&\quad + c_3(psn+1)\left[\frac{psn+p^-}{psn+p^-+1}\frac{psn+1-2p^-}{psn+1}+\frac{psn+p^+}{psn+p^++1}\frac{psn+1-2p^+}{psn+1}\right]E(t).
\end{aligned}
\tag{18}
$$

Here we have used the Holder and Young inequalities. Furthermore, using (15) and (16) to $\int_O u^{\frac{3}{2}(psn+1)}\mathrm{d}x$, then $(psn+1)\int_O \alpha_3(x)u^{psn+p(x)}\mathrm{d}x$ can be estimated by

$$
\begin{aligned}
(psn+1)\int_O \alpha_3(x)u^{psn+p(x)}\mathrm{d}x \le{}& \frac{c_3(psn+1)}{psn+p^-+1}|O| + \frac{c_3(psn+1)}{psn+p^++1}|O| + A_6E(t)^{\frac{3}{2}} + A_7E(t)^3 \\
&+ A_8\chi_2\int_O \left|\nabla u^{1+ns}\right|^p \mathrm{d}x + A_9\chi_2 E(t)^{\frac{psn+1-2(ns+1)}{psn+1}} + A_{10}E(t)
\end{aligned}
\tag{19}
$$

where

$$
A_6 = \frac{1}{2}3^{\frac{3}{4}}c_3\rho_0^{-\frac{3}{2}}(psn+1)\left[\frac{psn+p^-}{psn+p^-+1}\frac{2p^-}{psn+1}+\frac{psn+p^+}{psn+p^++1}\frac{2p^+}{psn+1}\right],
$$

$$
A_7 = \frac{\sqrt{2}}{4}c_3\chi_2^{-3}3^{-\frac{3}{4}}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}}(psn+1)\left[\frac{psn+p^-}{psn+p^-+1}\frac{psn+1-2p^-}{psn+1}+\frac{psn+p^+}{psn+p^++1}\frac{psn+1-2p^+}{psn+1}\right],
$$

$$
A_8 = \frac{3}{8\sqrt{2}}c_3\rho_0^{-\frac{3}{2}}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}}(psn+1)\frac{(psn+1)^2}{p(ns+1)^2}\left[\frac{psn+p^-}{psn+p^-+1}\frac{2p^-}{psn+1}+\frac{psn+p^+}{psn+p^++1}\frac{2p^+}{psn+1}\right],
$$

$$
\begin{aligned}
A_9 ={}& \frac{1}{2\sqrt{2}}3^{\frac{1}{4}}c_3\frac{p-2}{p}(psn+1)|O|^{\frac{2(ns+1)}{psn+1}}\frac{(psn+1)^2}{4(ns+1)^2}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}} \\
&\cdot\left[\frac{psn+p^-}{psn+p^-+1}\frac{2p^-}{psn+1}+\frac{psn+p^+}{psn+p^++1}\frac{2p^+}{psn+1}\right],
\end{aligned}
$$

$$
A_{10} = c_3(psn+1)\left[\frac{psn+p^-}{psn+p^-+1}\frac{2p^-}{psn+1}+\frac{psn+p^+}{psn+p^++1}\frac{2p^+}{psn+1}\right].
$$

Next, we pay our attention to the term $(psn+1)\int_{\partial O} u^{psn} u |\nabla u|^{p-2} \frac{\partial u}{\partial n} \mathrm{d}x$ in (13). Making use of the nonlinear boundary condition, it follows from (4) that

$$
\begin{aligned}
&(psn+1)\int_{\partial O} u^{psn} u |\nabla u|^{p-2} \frac{\partial u}{\partial n} \mathrm{d}x \\
&\le k \frac{(psn+1)}{(ns+1)^{p-2}} \int_{\partial O} u^{2ns+\sigma} \left|\nabla u^{ns+1}\right|^{p-2} \mathrm{d}x \le k \frac{(psn+1)}{(ns+1)^{p-2}} \int_{O} u^{2ns+\sigma} \left|\nabla u^{ns+1}\right|^{p-2} \mathrm{d}x.
\end{aligned}
$$

Since $p \ge 2$, we can apply Holder and Young inequalities to get

$$
\begin{aligned}
&(psn+1)\int_{\partial O} u^{psn} u |\nabla u|^{p-2} \frac{\partial u}{\partial n} \mathrm{d}x \\
&\le k \frac{(psn+1)}{(ns+1)^{p-2}} \left(\int_O \left|\nabla u^{ns+1}\right|^p \mathrm{d}x\right)^{\frac{p-2}{p}} \left(\int_O u^{pns+\frac{\sigma p}{2}} \mathrm{d}x\right)^{\frac{2}{p}} \\
&\le k \frac{p-2}{p} \frac{(psn+1)}{(ns+1)^{p-2}} \int_O \left|\nabla u^{ns+1}\right|^p \mathrm{d}x + k \frac{2}{p} \frac{(psn+1)}{(ns+1)^{p-2}} \int_O u^{pns+\frac{\sigma p}{2}} \mathrm{d}x \\
&\le k \frac{p-2}{p} \frac{(psn+1)}{(ns+1)^{p-2}} \int_O \left|\nabla u^{ns+1}\right|^p \mathrm{d}x + k \frac{2}{p} |O|^{\frac{(\sigma p/2)-1}{pns+(\sigma p/2)}} \frac{(psn+1)}{(ns+1)^{p-2}} E(t)^{\frac{pns+1}{pns+(\sigma p/2)}}.
\end{aligned}
\tag{20}
$$

By taking (17), (19) and (20) into (13), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E(t) \le\ & \frac{c_3(psn+1)}{psn+p^-+1}|O| + \frac{c_3(psn+1)}{psn+p^++1}|O| + \frac{c_2(psn+1)}{psn+s+1}|O| \\
&+ \left[k \frac{p-2}{p} \frac{(psn+1)}{(ns+1)^{p-2}} + A_3\chi_2 + A_8\chi_2 - \frac{(c_0-1-nps)(psn+1)}{(sn+1)^p} \right] \int_O \left|\nabla u^{1+ns}\right|^p \mathrm{d}x \\
&+ k \frac{2}{p} |O|^{\frac{(\sigma p/2)-1}{pns+(\sigma p/2)}} \frac{(psn+1)}{(ns+1)^{p-2}} E(t)^{\frac{pns+1}{pns+(\sigma p/2)}} \\
&- c_1\chi_1(psn+1)\left(\frac{psn+q_1}{q_1}\right)^{-q_1} |O|^{\frac{1-q_1}{psn+1}} E(t)^{\frac{psn+q_1}{psn+1}} - c_4(psn+1)|O|^{\frac{1-q_2}{psn+1}} E(t)^{\frac{psn+q_2}{psn+1}} \\
&+ (A_1+A_6)E(t)^{\frac{3}{2}} + (A_2+A_7)E(t)^3 + (A_5+A_{10})E(t) + (A_4+A_9)\chi_2 E(t)^{\frac{psn+1-2(ns+1)}{psn+1}}.
\end{aligned}
\tag{21}
$$

Here we have used the conditions that $n > \frac{1}{(p-2)s}$ and $p > 2$. In order to remove the terms which contain the unknown constants χ_1 and χ_2 and the negative terms, we present the following three inequalities obtained by Young inequality

$$
E(t)^{\frac{psn+1-2(ns+1)}{psn+1}} \le \frac{psn+1}{psn+1-2(ns+1)} E(t) + \frac{psn+1}{2(ns+1)},
$$

$$E(t)\le \frac{psn+1}{psn+q_1}E(t)^{\frac{psn+q_1}{psn+1}}+\frac{q_1+1}{psn+q_1},\ E(t)\le \frac{psn+1}{psn+q_2}E(t)^{\frac{psn+q_2}{psn+1}}+\frac{q_2-1}{psn+q_2}$$

and insert them into (21), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t)&\le A_{12}\int_O\left|\nabla u^{1+ns}\right|^p \mathrm{d}x+A_{13}E(t)+A_{10}+A_{11}E(t)^{\frac{pns+1}{pns+(\sigma p/2)}}\\&+(A_1+A_6)E(t)^{\frac{3}{2}}+(A_2+A_7)E(t)^3+(A_5+A_{10})E(t)\end{aligned}\tag{22}$$

where

$$\begin{aligned}A_{10}&=\frac{c_3(psn+1)}{psn+p^-+1}|O|+\frac{c_3(psn+1)}{psn+p^++1}|O|+\frac{c_2(psn+1)}{psn+s+1}|O|+c_4(psn+1)\frac{q_2-1}{psn+1}|O|^{\frac{1-q_2}{psn+1}}\\&+(A_4+A_9)\chi_2\frac{psn+1}{2(ns+1)}+c_1(psn+1)\frac{q_1+1}{psn+1}|O|^{\frac{1-q_1}{psn+1}}\left(\frac{psn+q_1}{q_1}\right)^{-q_1}\chi_1,\end{aligned}$$

$$A_{11}=\frac{2k_1}{p}|O|^{\frac{(\sigma p/2)-1}{pns+(\sigma p/2)}}\frac{(psn+1)}{(ns+1)^{p-2}},\ A_{12}=k_1\frac{p-2}{p}\frac{(psn+1)}{(ns+1)^{p-2}}+(A_3+A_8)\chi_2-\frac{(c_0-1-nps)(psn+1)}{(sn+1)^p},$$

$$\begin{aligned}A_{13}&=(A_4+A_9)\chi_2\frac{psn+1}{psn+1-2(ns+1)}-c_4(psn+1)|O|^{\frac{1-q_2}{psn+1}}\frac{psn+q_2}{psn+1}\\&-c_1|O|^{\frac{1-q_1}{psn+1}}(psn+1)\frac{psn+q_1}{psn+1}\left(\frac{psn+q_1}{q_1}\right)^{-q_1}\chi_1.\end{aligned}$$

Now we show the proof that from (22) we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)\le A_0+A_{11}E(t)^{\frac{pns+1}{pns+(\sigma p/2)}}+(A_1+A_6)E(t)^{\frac{3}{2}}+(A_2+A_7)E(t)^3+(A_5+A_{10})E(t).\tag{23}$$

Indeed, when

$$\frac{(c_0-1-nps)(psn+1)}{(sn+1)^p}\le k_1\frac{p-2}{p}\frac{(psn+1)}{(ns+1)^{p-2}},\tag{24}$$

we choose $\chi_2>0$ such that $A_{12}\le 0$ and $A_{13}\le 0$. Then a direct calculation tells us that (23) holds by removing all the negative terms. When

$$\frac{(c_0-1-nps)(psn+1)}{(sn+1)^p}>k_1\frac{p-2}{p}\frac{(psn+1)}{(ns+1)^{p-2}},$$

we can fix $\chi_2>0$ to make $A_{12}=0$. For this case, if

$$(A_4+A_9)\chi_2\le c_4(psn+1)|O|^{\frac{1-q_2}{psn+1}}\frac{psn+q_2}{psn+1},\tag{25}$$

then we choose $\chi_1=0$ such that $A_{13}\le 0$. We can remove the negative terms $A_{13}E(t)$ to obtain (23); If not, we choose a suitable $\chi_1>0$ to make $A_{13}=0$.This indicates that (23) always holds whether (24) or (25) holds or not.

From (22), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le A_0 + A_{11}E(t)^{\frac{pns+1}{pns+(\sigma p/2)}} + (A_1 + A_6)E(t)^{\frac{3}{2}} + (A_2 + A_7)E(t)^3 + (A_5 + A_{10})E(t).$$

An integration leads to

$$T^* \ge \int_{E(0)}^{\infty} \frac{\mathrm{d}\tau}{A_0 + A_{11}\tau^{\frac{pns+1}{pns+(\sigma p/2)}} + (A_1 + A_6)\tau^{\frac{3}{2}} + (A_2 + A_7)\tau^3 + (A_5 + A_{10})\tau}.$$

The proof of Theorem 1 is achieved. □

3. Discussion

This work can be extended to the more general case, that is

$$u_t = u\mathrm{div}\left(\left|\nabla u\right|^{p-2}\nabla u\right) + \alpha_0(x)\left|\nabla u\right|^p + \alpha_2(x)\int_O u^s \mathrm{d}x + \alpha_3(x)u^{p(x)} - f(x,u,\nabla u) \qquad (26)$$

with the following nonlinear boundary condition(4) and the initial condition (5). Here f is a positive function belonging to $L(O\times\mathbb{R}\times\mathbb{R}_+)$. Indeed, using (9), and removing the negative term generated from $f(x,u,\nabla u)$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le& -\frac{(\alpha_0 - 1 - nps)(psn+1)}{(sn+1)^p}\int_O \left|\nabla u^{1+ns}\right|^p \mathrm{d}x + (psn+1)\int_{\partial O} u^{psn} u \left|\nabla u\right|^{p-2}\frac{\partial u}{\partial n}\mathrm{d}x \\ &+ (psn+1)\left|O\right|\int_O \alpha_2(x)u^{psn+s}\mathrm{d}x + (psn+1)\int_O \alpha_3(x)u^{psn+p(x)}\mathrm{d}x.\end{aligned} \qquad (27)$$

For this, we can derive a lower bound of blow-up time T^* for problem (26) by inserting (17), (19) and (20) into (27) and choosing a suitable χ_2. But the lower bound of blow-up time T^* obtained here is smaller than the one in Theorem 1.

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