Blow-up Phenomena for a Class of Degenerate Parabolic Problems with Multiple Nonlinearities

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Abstract. In this paper, we study the blow-up solution to a nonlinear degenerate parabolic equation

\[ u_t = \text{div}(\nabla |u|^{p-2} \nabla u) + \alpha_1 |\nabla u|^q - \alpha_2 |\nabla u|^r + \alpha_3 \int_{\partial O} u^s \, d\nu + \alpha_4 u^{\nu(x)} - \alpha_5 u^n \]

under nonlinear boundary condition. By constructing some appropriate auxiliary functions and using first-order differential inequality technique, an explicit formula of lower bound for blow-up time is derived.

Key words: Multiple nonlinearities; nonlinear degenerate parabolic equations; blow-up; nonlinear boundary condition.

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1. Introduction

Lower bounds for blow-up time of solution to degenerate parabolic problem have been extensively studied in the last 10 years [1-6]. Payne and Song in [1] considered an initial-boundary value problem for parabolic equations of the form

\[ \frac{\partial u}{\partial t} = \Delta u + u^s - |\nabla u|^k \text{ in } O \times (0,T^*) \]

where

\[ u = 0 \text{ on } \partial O \times (0,T^*), \ u(x,0) = u_0(x) \geq 0 \text{ in } O. \]

Here \( O \) is a bounded domain in \( \mathbb{R}^3 \), \( \Delta \) is the Laplace operator, \( \nabla \) is the gradient operator, \( \partial O \) is the boundary of \( O \), and \( T^* \) is the possible blow-up time. A lower
bound for the blow-up time $T'$ was determined under the condition $p \leq q_1$, and the relative result in [7] was extended to the case with nonlinear boundary condition.

In [8] the authors studied the question of blow-up for the solution to the problem

$$u_t = \Delta u + \int_\Omega u' \, dx - \alpha u^q; \quad \text{in } O \times (0, T'),$$

with both homogeneous Dirichlet boundary condition and homogeneous Neumann boundary condition. They obtained the lower bounds for the blow-up time under the above two boundary conditions. Later, others generalized this result to the case of nonlinear boundary condition [9] or Robin condition [2].

In this paper, we consider the following nonlinear parabolic equation generalized from (1) and (2)

$$u_t = \text{div} \left( |\nabla u|^{p-2} \nabla u \right) + \alpha(x) - \alpha'(x) |\nabla u|^q + \alpha_1(x) \int_\Omega u' \, dx + \alpha_2(x) \alpha_3(x) - \alpha_4(x) u^q; \quad \text{in } O \times (0, T'),$$

with the following nonlinear boundary condition

$$\frac{\partial u}{\partial n} = g(u); \quad \text{in } O \times (0, T').$$

and the initial condition

$$u(x, 0) = u_0(x) \geq 0; \quad \text{in } O.$$

Here $p > 2$, $\bar{n}$ is the unit outer normal vector of $\partial O$, and $\frac{\partial u}{\partial n}$ is outward normal derivative of $u$ on the boundary $\partial O$ which is assumed to be sufficiently smooth. Moreover, we assume that

$$1 < q^- := \inf_{x \in \partial O} q(x) \leq q(x) \leq q^+ := \sup_{x \in \partial O} q(x) < +\infty, \quad 0 < \alpha_i(x) \leq \alpha(x) \leq \alpha_i(x) < +\infty,$$

$$0 < c_i := \inf_{x \in \partial O} \alpha(x) \leq \alpha_i(x) < +\infty, \quad 0 < \alpha_i(x) \leq \alpha_(x) \leq c_i := \sup_{x \in \partial O} \alpha_i(x) < +\infty,$$

$$0 < \alpha_i(x) \leq c_i := \sup_{x \in \partial O} \alpha_i(x) < +\infty, \quad 0 < c_i := \inf_{x \in \partial O} \alpha_i(x) \leq \alpha_i(x) < +\infty.$$

As indicated in [1,7,8,9], we also need $q_i > 1, \quad q_2 > 1$. Reference [7] assumed that $s > q_2 > 1$, here we release this restriction by $s > 0$.

Since the initial data $u_0(x)$ in (5) is nonnegative, we have by the parabolic maximum principles [10,11] that $u$ is nonnegative in $O \times (0, T')$. In the next section, we will find a lower bound for the blow-up time when blow-up occurs.
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2. A lower bound for the blow-up time
In this section we seek the lower bound for the blow-up time $T$. To this end, we define an auxiliary function of the form

$$E(t) = \int_0^t u^{p_n+s} \, dx \quad \text{with} \quad n > 1, \quad \rho_1 = \min_{\partial_0} |x|, \quad \rho_2 = \min_{\partial_0} |x \cdot n|$$

and make an assumption on $g(z)$

$$g(z) \leq k z^\sigma,$$  \hspace{1cm} (7)

where $k$ is a positive constant. Our assumption is weaker than the one in [10], it requires $0 \leq g(z) \leq k z^\sigma$, and $\sigma$ depends on the choice of $E(t)$. Here we allow $\sigma$ to be any positive constant. Furthermore, reference [12] indicated that if $c_0 - 1 - n p s \leq 0$, the solution will not blow-up in finite time. So we consider the case $c_0 - 1 - n p s > 0$.

The main result of this article is formulated in the following theorem:

**Theorem 1** Let $u(x,t)$ be the nonnegative classical solution to problem (3)-(5), and $g$ satisfies (7). Then for any

$$\frac{1}{(p-2)s} < n < \frac{c_0 - 1}{p s},$$

the blow-up time $T$ is bounded from below by

$$\int_{E(0)}^{\infty} \frac{d\tau}{A_0 + A_1 \tau^{p_n+s(p+2)/2} + (A_1 + A_0) \tau^2 + (A_1 + A_2) \tau^3 + (A_1 + A_0) \tau},$$

where $A_0, A_1, A_2, A_3, A_4, A_5$ and $A_6$ are positive constants to be determined later.

**Proof:** First we compute

$$\frac{d}{dt} E(t) = (p s n + 1) \int_0^t u^{p_n+s} u' \, dx$$

$$= (p s n + 1) \int_0^t u^{p_n+s} \left( |\nabla u|^{p-2} \nabla u + \alpha_0 |\nabla u|^p \right) \, dx$$

$$- (p s n + 1) \int_0^t \alpha_1(x) u^{p_n+s} |\nabla u|^p \, dx + (p s n + 1) \int_0^t \alpha_2(x) u^{p_n+s} \, dx \int_0^t u' \, dx$$

$$+ (p s n + 1) \int_0^t \alpha_3(x) u^{p_n+s} \, dx - (p s n + 1) \int_0^t \alpha_4(x) u^{p_n+s} \, dx$$

$$\leq \frac{(c_0 - 1 - n p s)(p s n + 1)}{(p s n + 1)} \int_0^t |\nabla u|^{p_n+s} \, dx$$

$$- (p s n + 1) \int_0^t \alpha_1(x) u^{p_n+s} |\nabla u|^p \, dx + (p s n + 1) \int_0^t \alpha_2(x) u^{p_n+s} \, dx$$

$$+ (p s n + 1) \int_0^t \alpha_3(x) u^{p_n+s} \, dx - (p s n + 1) \int_0^t \alpha_4(x) u^{p_n+s} \, dx.$$
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\[(psn + 1) \int_0^1 \alpha_s(x) u^{psn+q} dx \geq c_1 (psn + 1) \int_0^1 u^{psn+q} dx \geq c_1 (psn + 1) \| \frac{1}{sn+1} E(t) \|^{psn+q}_{sn+1}. \tag{9}\]

For convenience, let \( v = u^\frac{1}{psn+q} \). It follows that

\[(psn + 1) \int_0^1 \alpha_s(x) u^{psn+q} \| \nabla u \|_p dx = (psn + 1) \left( \frac{psn + q}{q_t} \right)^{\frac{1}{p}} \int_0^1 \alpha_s(x) \| \nabla u \|_p dx. \tag{10}\]

Using the Sobolev inequality derived in [14] (see 2.10) or [15] (see (4.10)), we have

\[\chi \int_0^1 \| \nabla u \|_p dx \geq \int_0^1 \| v \|_p dx, \tag{11}\]

where \( \chi \) is a positive constant to be determined later. Therefore, combining (11) and Holder inequality we get

\[(psn + 1) \int_0^1 \alpha_s(x) u^{psn+q} \| \nabla u \|_p dx \geq \chi c_1 (psn + 1) \left( \frac{psn + q}{q_t} \right)^{\frac{1}{p}} \| \frac{1}{sn+1} E(t) \|^{psn+q}_{sn+1}. \tag{12}\]

Further, using (9) and (12), we replace (8) by

\[\frac{d}{dt} E(t) \leq - \frac{(\alpha_s - 1 - nps)(psn + 1)}{(sn + 1)^r} \int_0^1 \| \nabla u \|_p dx \]

\[\quad - c_1 \chi (psn + 1) \left( \frac{psn + q}{q_t} \right)^{\frac{1}{p}} \| \frac{1}{sn+1} E(t) \|^{psn+q}_{sn+1}\]

\[\quad + (psn+1) \| \int_0^1 \alpha_s(x) u^{psn+q} dx + (psn + 1) \| \int_0^1 \alpha_s(x) u^{psn+q} dx \]

\[\quad \leq c_1 (psn + 1) \| \frac{1}{sn+1} E(t) \|^{psn+q}_{sn+1} + (psn + 1) \int_0^1 u^{psn+q} \| \nabla u \|_p ^2 \frac{du}{dn} dx. \tag{13}\]

Now, we focus on the term \( (psn + 1) \| \int_0^1 \alpha_s(x) u^{psn+q} dx \) in (13). Using Holder and Young inequalities twice, we have

\[\int_0^1 u^{psn+q} dx \leq \| \int_0^1 u^{psn+q} dx \|^{psn+q}_{psn+1} \]

\[\leq \frac{1}{psn+s+1} \| + \frac{psn+s}{psn+s+1} \int_0^1 u^{psn+q} dx \]

\[\leq \frac{1}{psn+s+1} \| + \frac{psn+s}{psn+s+1} \left( \int_0^1 u^{\frac{1}{psn+1}} dx \right)^{\frac{2s}{psn+1} + \frac{psn+1-2s}{psn+1}} \]

\[\leq \frac{1}{psn+s+1} \| + \frac{psn+s}{psn+s+1} \left( \int_0^1 u^{\frac{1}{psn+1}} dx \right)^{\frac{2s}{psn+1} + \frac{psn+1-2s}{psn+1}} \]

and
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\[
\int_0^1 \left| \nabla u^{\frac{1}{2}} (x) \right|^2 \, dx \leq \frac{(psn+1)^2}{4(ns+1)^2} \left( \int_0^1 \left| \nabla v^{\frac{1}{2}} \right|^2 \, dx \right)^{\frac{p+1}{p}} \left( \int_0^1 \left| v^{\frac{1}{2}} (psn+1)^{2-2} \right|^2 \, dx \right)^{\frac{p-2}{p}} \leq \frac{(psn+1)^2}{2p(ns+1)^2} \int_0^1 \left| \nabla v^{\frac{1}{2}} \right|^2 \, dx + \frac{p-2}{p} \frac{(psn+1)^2}{4(ns+1)^2} \int_0^1 v^{\frac{1}{2}} \, dx \leq \frac{(psn+1)^2}{2p(ns+1)^2} \int_0^1 \left| \nabla u^{1^{\frac{1}{2}}} \right|^2 \, dx + \frac{p-2}{p} \frac{(psn+1)^2}{4(ns+1)^2} E(t) \frac{(psn+2-2m+1)}{psn+1} \tag{15}
\]

where \( v = u^{1^{\frac{1}{2}}} \). Then, we connect (14) and (16) by using the integral inequality derived in [6] (see (2.16)), namely

\[
\int_0^1 u^{\frac{1}{2}} \, dx \leq \frac{3}{2} \left( E(t)^2 + \frac{\sqrt{2}}{3} \left( \frac{\rho}{\rho_0} + 1 \right) \right)^{\frac{1}{2}} \frac{E(t)^{\frac{3}{2}}}{4X_2} \left( \int_0^1 \left| \nabla u^{\frac{1}{2}} \right|^2 \, dx \right) \tag{16}
\]

to obtain

\[
(psn+1)\frac{\partial}{\partial t}(x)u^{psn+1} \leq \frac{c_2}{psn+s+1} \frac{(psn+1)}{\rho} \left( \frac{\rho}{\rho_0} + 1 \right)^{\frac{1}{2}} \frac{E(t)^{\frac{3}{2}}}{4X_2} \left( \int_0^1 \left| \nabla u^{\frac{1}{2}} \right|^2 \, dx \right) + A_1 \frac{1}{\rho^2} \left( p + 1 \right) \frac{(psn+1)}{psn+s+1} \tag{17}
\]

where \( X_2 \) is another positive constant to be determined later,

\[
A_1 = 3^3 c_2 \rho_0 \left( psn+1 \right) \frac{s}{psn+s+1}, \quad A_2 = \frac{\sqrt{2}}{3} \left( \frac{3}{3} \frac{s}{psn+s+1} \right)^{\frac{1}{2}} \left( \frac{\rho}{\rho_0} + 1 \right)^{\frac{1}{2}},
\]

\[
A_3 = \frac{1}{2\sqrt{2}} \left( \frac{1}{psn+s+1} \left( \frac{\rho}{\rho_0} + 1 \right)^{\frac{1}{2}} \right) \frac{(psn+1)^{2}}{\rho ns+1}.
\]

Next we give a bound for the term \((psn+1)\frac{\partial}{\partial t}(x)u^{psn+1}\)dx in (13). For each \( t > 0 \), we divide \( O \) into two sets,

\[
O_h = \{ x \in O | u(x,t) < 1 \}, \quad O_c = \{ x \in O | u(x,t) \geq 1 \}.
\]

It follows that
Here we have used the Holder and Young inequalities. Furthermore, using (15) and (16) to $\int_0^{\frac{2}{\alpha}} u_t^{(m+\rho_1)} \, dx$, then $\int_0^{\frac{2}{\alpha}} u_t^{(m+\rho_1)} \, dx$ can be estimated by

$$\int_0^{\frac{2}{\alpha}} u_t^{(m+\rho_1)} \, dx \leq c_1 \left( \frac{u_{m+\rho_1}}{psn+p^*+1} \right)^\frac{1}{\alpha} \left[ \frac{2p^-}{psn+p^*+1} + \frac{psn+p^*}{psn+p^*+1} \int_0^{\frac{2}{\alpha}} u_t^{(m+\rho_1)} \, dx \right] \left[ \frac{2p^*}{psn+p^*+1} \right] \left[ \frac{2p^*}{psn+p^*+1} \right] E(t).$$

(18)

where

$$A_0 = \frac{1}{2} c_{\rho_0}^{\frac{3}{2}} \frac{p-2}{p} \left( \frac{\rho_1}{\rho_0} \right)^{\frac{3}{2}} \left( \frac{1}{psn+p^*+1} \right)^{\frac{2p^-}{psn+p^*+1} + \frac{psn+p^*}{psn+p^*+1} \int_0^{\frac{2}{\alpha}} u_t^{(m+\rho_1)} \, dx \right] \left[ \frac{2p^*}{psn+p^*+1} \right] \left[ \frac{2p^*}{psn+p^*+1} \right] E(t).$$

(19)
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Next, we pay our attention to the term \((psn+1)\int_0 u^{psn}u|\nabla u|^{p-2} \frac{\partial u}{\partial n} \, dx\) in (13). Making use of the nonlinear boundary condition, it follows from (4) that

\[
(psн+1)\int_0 u^{psн}u|\nabla u|^{p-2} \frac{\partial u}{\partial n} \, dx \\
\leq k(psн+1)\left(\int_0 u^{2psн} \, dx\right)^{p-2}
\leq k(psн+1)\int_0 u^{2psн} \, dx \\
\leq k(psн+1)\int_0 u^{2psн} \, dx \\
\leq k(psн+1)\int_0 u^{2psн} \, dx
\]

Since \(p \geq 2\), we can apply Holder and Young inequalities to get

\[
(psн+1)\int_0 u^{psн}u|\nabla u|^{p-2} \frac{\partial u}{\partial n} \, dx \\
\leq k(psн+1)(\int_0 u^{psн+1} \, dx)^{p-2}
\leq k(p-2)(psн+1)\int_0 u^{psн+1} \, dx
\]

By taking (17), (19) and (20) into (13), we have

\[
\frac{d}{dt}E(t) \leq c(psn+1)\frac{\|u\|^{psn+p+1}}{psn+p+1} + \frac{c(psn+1)}{psn+p+1}||\partial u||^{psn+p+1} + \frac{c(psn+1)}{psн+1}||\nabla u||^{psн+1}
\]

\[
+ \frac{2}{p}(\int_0 u^{psн+1} \, dx)^{p-2} + \frac{c(psn+1)}{(sn+1)^p} \int_0 \nabla u^{psн+1} \, dx
\]

(21)

Here we have used the conditions that \(n > \frac{1}{(p-2)s}\) and \(p > 2\). In order to remove the terms which contain the unknown constants \(\chi_1\) and \(\chi_2\) and the negative terms, we present the following three inequalities obtained by Young inequality

\[
E(t) \leq \frac{psn+1}{psn+1-2(ns+1)} E(t) + \frac{psn+1}{2(ns+1)}
\]
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\[ E(t) \leq \frac{psn + 1}{psn + q_1} \left( \frac{psn + q_1}{psn + q_1} \right)^{\frac{psn + q_1}{psn + q_1}} + \frac{q_1 + 1}{psn + q_1}, \quad E(t) \leq \frac{psn + 1}{psn + q_2} \left( \frac{psn + q_2}{psn + q_2} \right)^{\frac{psn + q_2}{psn + q_2}} + \frac{q_2 - 1}{psn + q_2} \]

and insert them into (21), we have

\[
\frac{d}{dt} E(t) \leq A_1 \int_0^t \left[ \nabla u^{1+q} \right] \, dx + A_3 E(t) + A_{10} + A_1 \left( \frac{psn + q_1}{psn + q_1} \right)^{\frac{1}{q_1}} E(t)^{\frac{1}{q_1}} + (A_1 + A_3) E(t)^3 + (A_4 + A_5) E(t),
\]

where

\[
A_0 = \frac{c_1 (psn + 1)}{psn + p^{-1} + 1} |O| + \frac{c_1 (psn + 1)}{psn + p^{-1} + 1} |O| + \frac{c_2 (psn + 1)}{psn + s + 1} |O| + \frac{c_2 (psn + 1)}{psn + s + 1} |O|^{\frac{s}{psn + s + 1}}
\]

\[
+ (A_1 + A_3) X_2 \frac{psn + 1}{2(n+1)} + c_3 (psn + 1) \frac{q_1 + 1}{psn + 1} \left| O \right|^{\frac{1-a}{psn + 1}} \frac{psn + q_1}{q_1} X_1.
\]

\[
A_1 = \frac{2k}{p} \left| O \right|^{\frac{a p n / 2 + 1}{p s n + 1}} \left( \frac{psn + 1}{psn + p^{-1} + 1} \right)^{\frac{1}{q_1}}, \quad A_2 = k \frac{p - 2}{p} \left( \frac{psn + 1}{psn + p^{-1} + 1} \right)^{\frac{1}{q_1}} + (A_1 + A_3) X_2 - \frac{(c_0 - 1 - nps)(psn + 1)}{(sn + 1)^{\frac{1}{q_1}}},
\]

\[
A_3 = (A_1 + A_3) X_3 \frac{psn + 1}{psn + 1 - 2(n+1)} - c_1 (psn + 1) \left| O \right|^{\frac{1-a}{psn + 1}} \frac{psn + q_1}{psn + 1}
\]

\[
- c_3 \left| O \right|^{\frac{1-a}{psn + 1}} (psn + 1) \frac{psn + q_1}{q_1} \left( \frac{psn + q_1}{q_1} \right)^{\frac{1}{q_1}} X_1.
\]

Now we show the proof that from (22) we can get

\[
\frac{d}{dt} E(t) \leq A_1 + A_3 E(t)^{\frac{psn + q_1}{psn + q_1}} + (A_1 + A_3) E(t)^3 + (A_4 + A_5) E(t).
\]

Indeed, when

\[
\frac{(c_0 - 1 - nps)(psn + 1)}{(sn + 1)^{\frac{1}{q_1}}} \leq k \frac{p - 2}{p} \left( \frac{psn + 1}{psn + p^{-1} + 1} \right)^{\frac{1}{q_1}},
\]

we choose \( \chi_2 > 0 \) such that \( A_2 \leq 0 \) and \( A_3 \leq 0 \). Then a direct calculation tells us that (23) holds by removing all the negative terms. When

\[
\frac{(c_0 - 1 - nps)(psn + 1)}{(sn + 1)^{\frac{1}{q_1}}} > k \frac{p - 2}{p} \left( \frac{psn + 1}{psn + p^{-1} + 1} \right)^{\frac{1}{q_1}},
\]

we can fix \( \chi_2 > 0 \) to make \( A_2 = 0 \). For this case, if

\[
(A_1 + A_3) X_2 \leq c_3 (psn + 1) \left| O \right|^{\frac{1-a}{psn + 1}} \frac{psn + q_1}{psn + 1},
\]

then we choose \( \chi_1 = 0 \) such that \( A_3 \leq 0 \). We can remove the negative terms \( A_3 E(t) \) to obtain (23); If not, we choose a suitable \( \chi_1 > 0 \) to make \( A_3 = 0 \). This indicates that (23) always holds whether (24) or (25) holds or not.

From (22), we obtain
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\[ \frac{d}{dt} E(t) \leq A_i + A_j E(t)^{\frac{m+1}{m+\delta}} + (A_k + A_l) E(t)^{\delta} + (A_m + A_n) E(t) \cdot \]

An integration leads to

\[ T^* \geq \int_{E(0)}^{\infty} \frac{d\tau}{A_i + A_j \tau^{\frac{m+1}{m+\delta}} + (A_k + A_l) \tau^{\delta} + (A_m + A_n) \tau} \cdot \]

The proof of Theorem 1 is achieved. □

3. Discussion

This work can be extended to the more general case, that is

\[ u_t = \text{div} \left( \left| \nabla u \right|^{p-2} \nabla u \right) + \alpha_t(x) \left| \nabla u \right|^p + \alpha(x) \int_0^t u^r \, dx + \alpha(x) u^{p(t)} - f(x,u,\nabla u) \quad (26) \]

with the following nonlinear boundary condition (4) and the initial condition (5). Here \( f \) is a positive function belonging to \( L(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+). \) Indeed, using (9), and removing the negative term generated from \( f(x,u,\nabla u), \) we have

\[ \frac{d}{dt} E(t) \leq - \left( \alpha_t - 1 - nps \right)(psn + 1) \int_0^\alpha \left| \nabla u \right|^{p+1} \, dx + (psn + 1) \int_0^{psn+1} u^{pm} \left| \nabla u \right|^{p+1} \frac{\partial u}{\partial n} \, dx \]

\[ + (psn + 1) \int_0^{psn+1} \alpha_t(x) u^{p(t)} \, dx + (psn + 1) \int_0^{psn+1} \alpha_t(x) u^{p(t)} \cdot (27) \]

For this, we can derive a lower bound of blow-up time \( T^* \) for problem (26) by inserting (17), (19) and (20) into (27) and choosing a suitable \( \chi^* \). But the lower bound of blow-up time \( T^* \) obtained here is smaller than the one in Theorem 1.

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