Progress in Nonlinear Dynamics and Chaos
Vol. 6, No. 2, 2018, 97-104
ISSN: 2321 – 9238 (online)
Published on 27 December 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/pindac.v6n2a34

Blow-up Phenomena for a Class of Degenerate Parabolic Problems under Robin Boundary Condition

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Received 6 December 2018; accepted 26 December 2018

Abstract. This concerns with the blow-up solution to a general quasilinear degenerate parabolic equation $u_t = u^{\sigma}\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + \alpha_0(x)u^{\sigma-1}|\nabla u|^p + \alpha_1(x) + \alpha_2(x)u^{\gamma} + \alpha_3(x)u^{p(x)}$ under Robin boundary condition. By constructing some appropriate auxiliary functions and using first order differential inequalities technique, we derive conditions which guarantee the blow-up of solution. Moreover, Lower bound and upper bound for blow-up time are determined if the solution blows up.

Keywords: General quasilinear degenerate parabolic; Blow-up; upper bound; Lower bound; Robin boundary condition.

AMS Mathematics Subject Classification (2010): 35B40, 35K35

1. Introduction

The phenomena of blow-up for parabolic problem received much attention in the last few decades (see, for instance, [1-6]). In the above-mentioned works, many different approaches have been developed in dealing with various nonlinear parabolic problems, such as the existence of global solution, blow-up solution, upper bound on blow-up time, blow-up rate and asymptotic behavior of solutions. For example, Pinasco in [7] considered an initial-boundary value problem for parabolic equation of the form

$$\begin{cases} u_t = \Delta u + u^{p(x)}, & x \in O, t > 0 \\ u(x,t) = 0, & x \in \partial O, t > 0 \\ u(x,0) = u_0(x) \geq 0, & x \in O \end{cases} \tag{1}$$

where $O \subset \mathbb{R}^d\,(d>3)$ is a bounded domain with appropriately smooth boundary ∂O, the function $O \to (1,+\infty)$ satisfies

$$1 < p^- := \inf_{x\in O} p(x) \le p(x) \le p^+ := \sup_{x\in O} p(x)\,. \tag{2}$$

He proved that solutions to problem (1) blows up when $p^- > 1$, and later the relative theory was extended in [8] in which the authors concluded that blow-up phenomenon occurs in finite time if and only if $p^+ > 1$. Moreover, they showed that there are functions $p(x)$ and domains O such that all solutions to problem (1) blow up in finite time. The authors in [9] obtained that the solution to problem (1) blows up in finite time when the initial energy is positive. Finally, Mohammad, Ghaemi and Hesaaraki in [10] obtained the lower bound of blow-up time if the solution blows up.

In this paper, we are concerned with the more complicated case:

$$u_t = u^\sigma \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \alpha_0(x)u^{\sigma-1}|\nabla u|^p + \alpha_1(x) + \alpha_2(x)u^\gamma + \alpha_3(x)u^{p(x)} \tag{3}$$

In $O_\infty = O\times(0,\infty)$ with the Robin boundary condition

$$\frac{\partial u}{\partial n} + ku = 0 \quad \text{on} \quad \partial O\times(0,\infty) \tag{4}$$

and the initial condition

$$u(x,0) = u_0(x) \ge 0 \quad \text{in} \quad O \tag{5}$$

where $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on the boundary ∂O and $p(x)$ still satisfies (2). As indicated in [11], we make the following assumptions:

(a) The parameters of problem (3)-(5) satisfy $\sigma \in [1,2)$, $p > 2$ and $k \ge 0$.

(b) $\alpha_i s$ are integrable function satisfying $0 \le \alpha_i \le c_i$, $i = 1,2,3$.

In this paper, by constructing some appropriate auxiliary functions and using first order differential inequalities technique we investigate the more general problem (3)-(5). In section 2, we develop a sufficient condition on the initial data, which guarantees that blow-up of solution dose occur. Moreover, an upper bound of blow-up time is derived. In section 3, a lower bound of blow-up time is obtained when blow-up occurs.

Unfortunately, although we extend the conclusion in [7,8,10] to the more complicated model, we must replace $d \ge 3$ by $d > p$ $(p > 2)$. For this we do not know how sharp the condition is.

2. The blow-up solution

In this section, we mainly seek the sufficient conditions for the blow-up. To this end, our investigate will make use of the following auxiliary function

$$E(t)=\int_O u\mathrm{d}x \tag{6}$$

where u is the solution to problem (3)-(5).

Theorem 2.1. If $\alpha_0>\sigma$, the solution to the problem (3)-(5) will blow-up at finite time T^* and

$$T^*\le C_0^{-1}E(0)^{1-p\nu}$$

where $\nu=\dfrac{\sigma-1}{p}+1$, $\mu=\dfrac{p\alpha_0}{2(p-1)}-\dfrac{\sigma}{2}+1$, C_0 is a positive constant to be determined later.

Proof: Assume that the problem (3)-(5) does not blow-up at finite time, that is to say, problem (3)-(5) admits a weak solution in O_T with $T>0$. It follows from (b) in Definition 1 that for any $0\le\varphi\in L^\infty\left(O_T\times(t_1,t_2)\right)\cap L^p\left(t_1,t_2;W_0^{1,p}(O)\right)$ with

$$\frac{\partial}{\partial t}\varphi\in L^2\left(O_T\times(t_1,t_2)\right)\quad(T\ge t_2>t_1>0)$$

there holds

$$\begin{aligned}&\int_O u(t_2)\varphi(t_2)-u(t_1)\varphi(t_1)\mathrm{d}x-\int_{t_1}^{t_2}\int_O u\frac{\partial\varphi}{\partial t}\mathrm{d}x\mathrm{d}t\\&+\int_{t_1}^{t_2}\int_O\frac{1}{\nu^{p-1}}u^\upsilon\left|\nabla u^\upsilon\right|^{p-2}\nabla u^\upsilon\cdot\nabla\varphi+\frac{\sigma-\alpha_0}{\nu^p}\left|\nabla u^\upsilon\right|^p\varphi\mathrm{d}x\mathrm{d}t\\&-\int_{t_1}^{t_2}\int_O\left(\alpha_1+\alpha_2u^\gamma+\alpha_2u^{p(x)}\right)\varphi\mathrm{d}x\mathrm{d}t\\&-\int_{t_1}^{t_2}\int_O\left(\alpha_1+\alpha_2u^\gamma+\alpha_2u^{p(x)}\right)\varphi\mathrm{d}x\mathrm{d}t-k^{p-1}\int_{\partial O}u^{p+\sigma-1}\varphi=0.\end{aligned}\tag{7}$$

Choosing $\varphi=1$ as test function in (7), we have

$$\begin{aligned}&\int_O u(t_2)\mathrm{d}x-\int_O u(t_1)\mathrm{d}x\\&=\frac{\sigma-\alpha_0}{\nu^p}\int_{t_1}^{t_2}\int_O\left|\nabla u^\upsilon\right|^p\mathrm{d}x\mathrm{d}t+k^{p-1}\int_{\partial O}u^{p+\sigma-1}\varphi+\int_{t_1}^{t_2}\int_O\left(\alpha_1+\alpha_2u^\gamma+\alpha_2u^{p(x)}\right)\mathrm{d}x\mathrm{d}t\end{aligned}\tag{8}$$

Further, by letting $t_1\to0^+$, (6) and (8) lead to

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}\ge\frac{\alpha_0-\sigma}{\nu^p}\int_O\left|\nabla u^\upsilon\right|^p\mathrm{d}x \tag{9}$$

where we have used the assumption (b).

Next, we pay our attention to the term $\int_O\left|\nabla u^\upsilon\right|^p\mathrm{d}x$. Using the Sobolev inequality with $d > p$ derived in [12], we have

$$\left(\int_O (u^{pv})^{\frac{d}{d-p}} \mathrm{d}x\right)^{\frac{d-p}{d}} = \left\|u^v\right\|^p_{L^{\frac{dp}{d-p}}(O)} \le (C_s(d,p))^p \left\|\nabla u^v\right\|^p_{L^p(O)} = (C_s(d,p))^p \int_O \left|\nabla u^v\right|^p \mathrm{d}x \qquad (10)$$

where $C_s(d,p)$ is the best constant of Sobolev inequality. Further, using Holder inequality, we obtain

$$\int_O u \mathrm{d}x \le |O|^{\frac{dpv-d+p}{dpv}} \left(\int_O u^{\frac{dpv}{d-p}} \mathrm{d}x\right)^{\frac{d-p}{dpv}}. \qquad (11)$$

Therefore, $\int_O \left|\nabla u^v\right|^p \mathrm{d}x$ can be estimated by

$$\int_O \left|\nabla u^v\right|^p \mathrm{d}x \ge |O|^{\frac{d-p-dpv}{d}} (C_s(d,p))^{-p} E_\varepsilon(t)^{pv}. \qquad (12)$$

Next, using (12) and the assumption that $\alpha_0 > \sigma$ to (9), we arrive at

$$\frac{E'(t)}{E(t)^{pv}} \ge \frac{\alpha_0 - \sigma}{v^p} |O|^{\frac{d-p-dpv}{d}} (C_s(d,p))^{-p}. \qquad (13)$$

Integration of (13) from 0 to t leads to

$$E(t)^{1-pv} \le E(0)^{1-pv} - C_0 t \qquad (14)$$

were

$$C_0 = \frac{\alpha_0 - \sigma}{v^p} |O|^{\frac{d-p-dpv}{d}} (C_s(d,p))^{-p} (pv-1).$$

Since inequality (14) does not hold if $E(0)^{1-pv} - C_0 t \le 0$, that is, for $t \ge C_0^{-1} E(0)^{1-pv}$, we thus conclude that the solution u blows up at some finite time T^* and T^* is bounded above by

$$T^* \le C_0^{-1} E(0)^{1-pv}.$$

The proof is complete.

Remark 2.1. From [11] and the Theorem 2.1 we see that if $0 < \alpha_0 < 1$ and $\alpha_i \le 0$ $(i = 1,2,3)$, the problem (3)-(5) admits a weak solution, and that if $\alpha_0 > \sigma$ and $\alpha_i \ge 0$ $(i = 1,2,3)$, the problem (3)-(5) has no weak solutions. However, our proof does not work if $\alpha_0 \in [1,\sigma]$ and the claim in [11] can not hold when $\alpha_i > 0$ $(i = 1,2,3)$, we can not obtain the longtime behavior of solution to problem (3)-(5) in the cases $\alpha_0 \in [1,\sigma]$ or $\alpha_i > 0$ $(i = 1,2,3)$.

3. Lower bound for the blow-up time

In this section we seek the lower bound for the blow-up time T^* by some appropriate measures. To this end, we define another auxiliary function of the form

$$M(t)=\int_O u(t)^{1+\mu m}\mathrm{d}x \quad \text{with} \quad m\geq \frac{p^+d-2p-dpv}{(\mu+1)p}. \tag{15}$$

Theorem 3.1. Let $u(x,t)$ be the nonnegative classical solution to problem (3)-(5). Then, if $0<\gamma<1$, T^* is bounded from below by

$$\int_{M(0)}^{\infty}\frac{\mathrm{d}\xi}{k_1+k_2\xi^{\frac{\mu m}{\mu m+1}}+k_3\xi^{\frac{\mu m+\gamma}{\mu m+1}}+k_4\xi^2},$$

where k_1, k_2, k_3 and k_4 are positive constants to be determined later.

Proof: It follows from (3) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}M(t)=(1+\mu m)\int_O u^{\mu m}u_t\mathrm{d}x\\
&=(1+\mu m)\int_O u^{\mu m}\left(u^{\sigma}div\left(|\nabla u|^{p-2}\nabla u\right)+\alpha_0u^{\sigma-1}|\nabla u|^{p}\right)\mathrm{d}x\\
&\quad+(1+\mu m)\int_O u^{\mu m}\left(\alpha_1+\alpha_2u^{\gamma}+\alpha_3u^{p(x)}\right)\mathrm{d}x\\
&\leq-\frac{(1+\mu m)(\alpha_0-\sigma-\mu m)p^p}{(pv+\mu m)^pv^p}\int_O\left|\nabla u^{v+\frac{\mu m}{p}}\right|^p\mathrm{d}x-k(1+\mu m)\int_{\partial O}u^{\mu m}u^{\sigma}|\nabla u|^{p-2}u\mathrm{d}x\\
&\quad+(1+\mu m)\int_O u^{\mu m}(\alpha_1(x)+\alpha_2(x)u^{\gamma})\mathrm{d}x+(1+\mu m)\int_O\alpha_3(x)u^{\mu m+p(x)}\mathrm{d}x.
\end{aligned} \tag{16}$$

Noting that $0<\gamma<1$, and applying the Holder inequalities, we have

$$\int_O\alpha_1(x)u^{\mu m}\mathrm{d}x\leq c_1|O|^{\frac{1}{\mu m+1}}M(t)^{\frac{\mu m}{\mu m+1}} \tag{17}$$

and

$$\int_O\alpha_2(x)u^{\gamma+\mu m}\mathrm{d}x\leq c_2|O|^{\frac{1-\gamma}{\mu m+1}}M(t)^{\frac{\mu m+\gamma}{\mu m+1}}. \tag{18}$$

Next, we pay attention to the term $\int_O\alpha_3(x)u^{\mu m+p(x)}\mathrm{d}x$ in (16). For each $t>0$, we divide O into two sets,

$$O_0=\{x\in O\,|\,u(x,t)<1\},\quad O_1=\{x\in O\,|\,u(x,t)\geq 1\}.$$

It follows that

$$
\begin{aligned}
&\int_O \alpha_3(x) u^{\mu m+p(x)}\mathrm{d}x \\
&= c_3\int_{O_0} u^{\mu m+p(x)}\mathrm{d}x + c_3\int_{O_1} u^{\mu m+p(x)}\mathrm{d}x \le c_3\int_{O_0} u^{\mu m+p^-}\mathrm{d}x + c_3\int_{O_1} u^{\mu m+p^+}\mathrm{d}x \\
&\le c_3\int_O u^{\mu m+p^-}\mathrm{d}x + c_3\int_O u^{\mu m+p^+}\mathrm{d}x.
\end{aligned} \tag{19}
$$

By the Holder and Young inequalities to the terms on the right of (19), we have

$$
\begin{aligned}
\int_O u^{\mu m+p^-}\mathrm{d}x \le &\left(1-\frac{(\mu m+p^-)(d+p)}{d(pv+\mu m)+2p(1+\mu m)}\right)|O| \\
&+\frac{(\mu m+p^-)(d+p)}{d(pv+\mu m)+2p(1+\mu m)}\int_O u^{\frac{d}{d+p}(pv+\mu m)+(1+\mu m)\frac{2p}{d+p}}\mathrm{d}x
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
\int_O u^{\mu m+p^+}\mathrm{d}x \le &\left(1-\frac{(\mu m+p^+)(d+p)}{d(pv+\mu m)+2p(1+\mu m)}\right)|O| \\
&+\frac{(\mu m+p^+)(d+p)}{d(pv+\mu m)+2p(1+\mu m)}\int_O u^{\frac{d}{d+p}(pv+\mu m)+(1+\mu m)\frac{2p}{d+p}}\mathrm{d}x
\end{aligned} \tag{21}
$$

where the condition $m \ge \dfrac{p^+d-2p-dpv}{(\mu+1)p}$ in (15) has been used. Again by Holder inequality, we have

$$
\int_O u^{\frac{d}{d+p}(pv+\mu m)+(1+\mu m)\frac{2p}{d+p}}\mathrm{d}x \le \left(\int_O u^{1+\mu m}\mathrm{d}x\right)^{\frac{2p}{d+p}}\left(\int_O \left(u^{pv+\mu m}\right)^{\frac{d}{d-p}}\mathrm{d}x\right)^{\frac{d-p}{d+p}} \tag{22}
$$

and

$$
\begin{aligned}
&\int_O u^{\frac{d}{d+p}(pv+\mu m)+(1+\mu m)\frac{2p}{d+p}}\mathrm{d}x \\
&\le \left(\int_O u^{1+\mu m}\mathrm{d}x\right)^{\frac{2p}{d+p}} (C_s(d,p))^{\frac{pd}{d+p}}\left[\chi^{-1}\chi\int_O \left|\nabla u^{v+\frac{\mu m}{p}}\right|^p \mathrm{d}x\right]^{\frac{d}{d+p}} \\
&\le \left(\chi^{-\frac{d}{2p}}[C_s(d,p)]^{\frac{d}{2}}\int_O u^{1+\mu m}\mathrm{d}x\right)^{\frac{2p}{d+p}}\left[\chi\int_O \left|\nabla u^{v+\frac{\mu m}{p}}\right|^p \mathrm{d}x\right]^{\frac{d}{d+p}}
\end{aligned} \tag{23}
$$

where χ is a positive constant to be determined later. Then, we connect (21) and (22) by using the Sobolev inequality with $d > p$ derived in [12], namely

$$
\left(\int_O (u^{pv+\mu m})^{\frac{d-p}{d}}\mathrm{d}x\right)^{\frac{d-p}{d}} = \left\|u^{v+\frac{\mu m}{p}}\right\|^p_{L^{\frac{dp}{d-p}}(O)} \le (C_s(d,p))^p\left\|\nabla u^{v+\frac{\mu m}{p}}\right\|^p_{L^p(O)} = (C_s(d,p))^p\int_O\left|\nabla u^{v+\frac{\mu m}{p}}\right|^p\mathrm{d}x \tag{24}
$$

to obtain

$$\begin{aligned}&\int_O \alpha_3(x)u^{\mu m+p(x)}\mathrm{d}x\\&\le\left(1-\frac{(\mu m+p^-+p^+)(d+p)}{d(pv+\mu m)+2p(1+\mu m)}\right)|O|\\&+\frac{(\mu m+p^-+p^+)(d+p)}{d(pv+\mu m)+2p(1+\mu m)}\frac{p}{d+p}\left(\chi^{-\frac{d}{2p}}[C_s(d,p)]^{\frac{d}{2p}}\int_O u^{1+\mu m}\mathrm{d}x\right)^2\\&+\frac{\chi d}{d+p}\frac{(\mu m+p^-+p^+)(d+p)}{d(pv+\mu m)+2p(1+\mu m)}\int_O\left|\nabla u^{v+\frac{\mu m}{p}}\right|^p\mathrm{d}x.\end{aligned}\tag{25}$$

Choosing

$$\chi=\frac{(c_0-\sigma-\mu)p^p[d(mpv+\mu m)+2p(1+\mu m)]}{\alpha_3 mdv^p(pv+\mu)^p(\mu m+p^-+p^+)}\tag{26}$$

and inserting (17), (18) and (25) into (16), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t)\le k_1+k_2M(t)^{\frac{\mu m}{\mu m+1}}+k_3M(t)^{\frac{\mu m+\gamma}{\mu m+1}}+k_4M(t)^2\tag{27}$$

where

$$k_1=2c_3(1+\mu m)\left(2-\frac{(\mu m+p^-+p^+)(d+p)}{d(pv+\mu m)+2p(1+\mu m)}\right)|O|,\quad v=\frac{\sigma-1}{p}+1,$$

$$k_2=c_1(1+\mu m)|O|^{\frac{1}{1+\mu m}},\quad k_3=c_2(1+\mu m)|O|^{\frac{1}{1+\mu m}},$$

$$k_4=c_3(1+\mu m)\frac{p(\mu m+p^-+p^+)}{d(pv+\mu m)+2p(1+\mu m)}\chi^{-\frac{d}{p}}[C_s(d,p)]^{\frac{d}{p}}.$$

Finally, an integration of the differential inequality (26) from 0 to t leads to

$$\int_{M(0)}^{\infty}\frac{\mathrm{d}\xi}{k_1+k_2\xi^{\frac{\mu m}{\mu m+1}}+k_3\xi^{\frac{\mu m+\gamma}{\mu m+1}}+k_4\xi^2}\le t$$

from which we derive a lower bound for T^*

$$T^*\ge\int_{M(0)}^{\infty}\frac{\mathrm{d}\xi}{k_1+k_2\xi^{\frac{\mu m}{\mu m+1}}+k_3\xi^{\frac{\mu m+\gamma}{\mu m+1}}+k_4\xi^2}.$$

Thus, the proof is complete.

Remark 3.1. Theorem 3.1 remains valid if the condition $0<\gamma<1$ is replaced by $\gamma>0$. In fact, if $\gamma\ge1$, $\int_O\alpha_2(x)u^{\gamma+\mu m}\mathrm{d}x$ can be bounded from above in terms of $M(t)$,

$\int_O \left|\nabla u^{v+\frac{\mu m}{p}}\right|^p \mathrm{d}x$ and the undetermined constant χ. Further, we end the proof by choosing a suitable χ instead of the one in (27).

Acknowledgement. This work was supported by National Nature Science Foundation of China (Grant No. 71171164; 11426176) and the Science and Technology Foundation of Guizhou province (Grant No.[2015]2076). The authors are sincerely grateful to the reference and the Associate Editor handling the paper for their valuable comments.

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