Progress in Nonlinear Dynamics and Chaos
Vol. 6, No. 1, 2018, 49-64
ISSN: 2321 – 9238 *(online)*
Published on 19 April 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/pindac.v6n1a6

Progress in **Nonlinear Dynamics and Chaos**

Transmuted Weibull-Inverse Exponential Distribution with Applications to Medical Science and Engineering

Kawsar Fatima[1], Uzma Jan **and** ***S.P.Ahmad***

Department of Statistics, University of Kashmir, Srinagar, India
[1]Corresponding authors. email: kawsarfatima@gmail.com ;
peeruzma1@gmail.com

Received 2 March 2018; accepted 16 April 2018

Abstract. The main aim of the manuscript is to study a generalization of Weibull Inverse Exponential model using the transmutation technique and analyze the influence of the additional parameter to the base model. Several statistical properties of the proposed model have been derived. The parameters of the new model have been obtained using the maximum likelihood estimation. Moreover, the three real life data sets have been used to compare the base model and the generalized model for the lifetime analysis.

Keywords: Weibull Inverse Model, Transmutation Technique, Reliability analysis, Maximum Likelihood Estimation, Renyi Entropy, Order Statistics, AIC and BIC.

AMS Mathematics Subject Classification (2010): 60K10

1. Introduction

The one parameter exponential distribution is continuous analogue of the geometric distribution. This distribution gives us the description of the time between the events in a Poisson process. This model is widely applicable in life testing and is well-known for its memory less property. Due to its constant failure rate, this probability model is inappropriate for the analysis of the data with bathtub failure rates and inverted bathtub failure rates. In order to overcome such shortcomings and improve the flexibility and competence of the model, the one parameter inverted exponential distribution was studied by Keller and Kamath [1]. Because of its inverted bathtub failure rate, it is widely competent model for the exponential distribution.

The probability density function (pdf) and the cumulative density function (cdf) of inverse exponential distribution are respectively given as:

$$g(x)=\frac{\lambda}{x^2}e^{\frac{-\lambda}{x}} \quad ;x>0,\lambda>0. \tag{1.1}$$

$$G(x)=e^{\frac{-\lambda}{x}} \quad ;x>0,\lambda>0. \tag{1.2}$$

The cumulative distribution function of the Weibull inverse exponential distribution is given by:

$$F(x)=1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}, \tag{1.3}$$

where λ is a scale parameter, α and β are shape parameters. The corresponding probability density function is given by

$$f(x)=\frac{\alpha\beta\lambda}{x^2}e^{\frac{\lambda}{x}}\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta-1}e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}} \quad ;x>0. \tag{1.4}$$

2. Transmuted Weibull inverse exponential distribution

There arise certain situations in the real life problems where the classical distributions fail to model the lifetime data. As such, many distributions in the statistical distributional theory have been generalized to model lifetime datasets and have found widespread applications in reliability theory. These probability models have become much more applicable to analyze different failure rates including unimodal, monotonic pattern and bathtub shaped behavior. Various techniques in the past have been devised to improve the flexibility and the competence of the models by adding an extra parameter to the existing distributions. The different generalization methods in the statistical literature include Marshall Olkin [2], Kumaraswamy G [3], Mc Donald G [4], Lomax G [5] and Weibull G [6] and transmutation technique proposed by Shaw and Buckley [7].

According to the Quadratic Rank Transmutation Map, (QRTM), approach a random variable X is said to have transmuted distribution if its cumulative distribution function cdf is given by

$$F_T(x) = (1+\theta)F(x) - \theta\left[F(x)\right]^2, \ |\theta| \le 1, \tag{2.1}$$

where $F(x)$ is the cdf of the base distribution. It must be noted that when $\theta = 0$, the proposed model reduces to base distribution.

Differentiating equation (2.1) with respect to x gives the pdf of the transmuted model as

$$f_T(x) = f(x)\left[(1+\theta) - 2\theta F(x)\right]. \tag{2.2}$$

Here $f(x)$ is the probability density function of the base model.

Faton Merovci [8] obtained the transmuted Rayleigh distribution and discussed its important properties. Further, Afaq et. al. [9] studied the transmuted inverse Rayleigh distribution and derived its different characteristic properties. Uzma et al. [10] compared the transmuted Exponentiated inverse Weibull distribution with its different sub models.

A random variable X is said to have a Transmuted Weibull Inverse Exponential distribution with parameters α,β,λ and θ if the cumulative density function is given by:

$$F_T(x) = \left\{1 - e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\left\{1 + \theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}. \tag{2.3}$$

The corresponding pdf of the proposed distribution is given by:

$$f_T(x) = \frac{\alpha\beta\lambda}{x^2} e^{\frac{\lambda}{x}}\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta-1} e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\left\{1-\theta+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}. \tag{2.4}$$

for $x > 0, \alpha, \beta, \lambda > 0$ and $|\theta| \le 1$, where λ, α and β are the scale and shape parameters representing the different patterns of the transmuted Weibull inverse exponential distribution and θ is the transmuted parameter.

By selecting different values for parameters α, β, λ and θ, the various possible shapes for the pdf and cdf of the TWIE distribution are given in Figure 1 and Figure 2 as below:

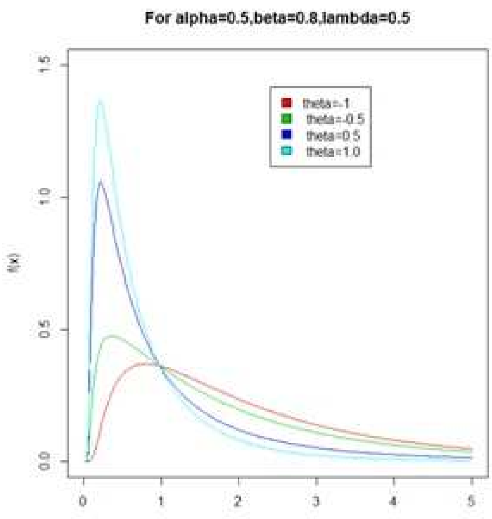

Figure 1: graph of density function

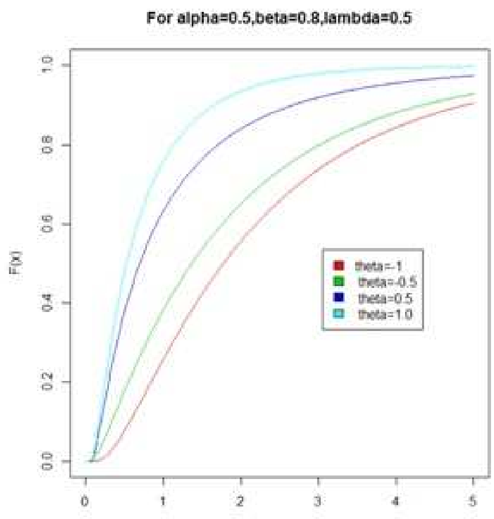

Figure 2: Graph of distribution function

3.1. Reliability analysis

The reliability (survival) function of TWIE distribution is given by

$$R(x)=1-F(x)=1-\left[\left\{1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\right];x>0;\ \alpha,\beta,\lambda,\theta>0. \quad (3.1)$$

The hazard function (failure rate) is given by

$$h(x)=\frac{f(x)}{R(x)}=\frac{\alpha\beta\lambda}{x^2}\frac{e^{\frac{\lambda}{x}}\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta-1}e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\left\{1-\theta+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}}{1-\left[\left\{1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\right]}. \quad (3.2)$$

The Reverse Hazard function of the TWIE distribution is obtained by

$$\phi(x)=\frac{f(x)}{F(x)}=\frac{\alpha\beta\lambda}{x^2}\frac{e^{\frac{\lambda}{x}}\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta-1}e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\left\{1-\theta+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}}{\left[\left\{1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\right]}. \quad (3.3)$$

The graphical representation of the reliability function and hazard function for the TWIE distribution is shown in figure 3 and 4 respectively.

4. Mixture representation

The TWIE density function given in (2.4) can be rewritten as

$$f_T(x)=(1-\theta)\alpha\beta g(x)\frac{[G(x)]^{\beta-1}}{[1-G(x)]^{\beta+1}}e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)^{\beta}}+\alpha\beta\theta g(x)\frac{[G(x)]^{\beta-1}}{[1-G(x)]^{\beta+1}}e^{-2\alpha\left(\frac{G(x)}{1-G(x)}\right)^{\beta}} \quad (4.1)$$

By using the power series for the exponential function, we obtain

$$e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)^{\beta}}=\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left\{\frac{G(x)}{1-G(x)}\right\}^{i\beta}\ \&\ e^{-2\alpha\left(\frac{G(x)}{1-G(x)}\right)^{\beta}}=\sum_{j=0}^{\infty}\frac{(-1)^j(2\alpha)^j}{j!}\left\{\frac{G(x)}{1-G(x)}\right\}^{j\beta}$$

$$f_T(x) = (1-\theta)\alpha\beta g(x)\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\frac{[G(x)]^{\beta(i+1)-1}}{[1-G(x)]^{\beta(i+1)+1}} + \alpha\beta\theta g(x)\sum_{j=0}^{\infty}\frac{(-1)^j(2\alpha)^j}{j!}\frac{[G(x)]^{\beta(j+1)-1}}{[1-G(x)]^{\beta(j+1)+1}} \quad (4.2)$$

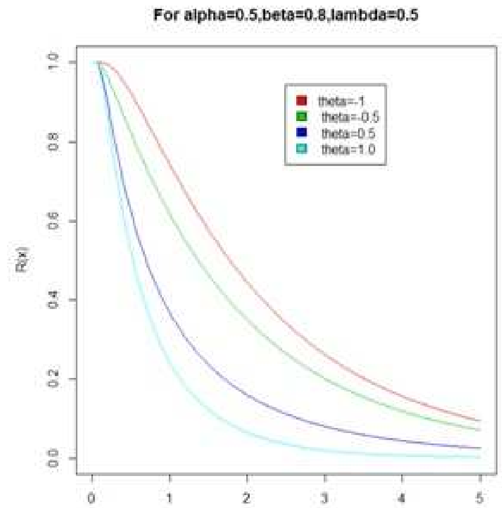

Figure 3: Graph of reliability function

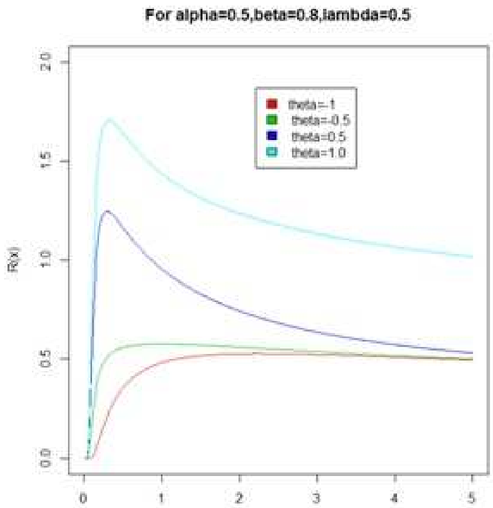

Figure 4: Graph of hazard function

Also from generalized binomial theorem, we have

$$(1-G(x))^{-\{\beta(i+1)+1\}} = \sum_{k=0}^{\infty}\frac{\Gamma(\beta(i+1)+k+1)}{k!\Gamma(\beta(i+1)+1)}(G(x))^k. \quad (4.3)$$

Substituting Equation (4.3) in (4.2), we find that we have

$$f_T(x) = (1-\theta)\sum_{i,k=0}^{\infty}\frac{(-1)^i\alpha^{i+1}\beta\Gamma(\beta(i+1)+k)}{i!k!\Gamma(\beta(i+1)+1)}(\beta(i+1)+k)g(x)(G(x))^{\beta(i+1)+k-1} + \alpha\theta \times \sum_{j,k=0}^{\infty}\frac{(-1)^j(2\alpha)^j\beta\Gamma(\beta(j+1)+k)}{j!k!\Gamma(\beta(j+1)+1)}(\beta(j+1)+k)g(x)(G(x))^{\beta(j+1)+k-1}$$

$$f_T(x) = (1-\theta)\sum_{i,k=0}^{\infty}\frac{(-1)^i\alpha^{i+1}\beta\Gamma(\beta(i+1)+k)}{i!k!\Gamma(\beta(i+1)+1)}\frac{\lambda(\beta(i+1)+k)}{x^2}e^{\frac{-\lambda(\beta(i+1)+k)}{x}} + \alpha\theta \times \sum_{j,k=0}^{\infty}\frac{(-1)^j(2\alpha)^j\beta\Gamma(\beta(j+1)+k)}{j!k!\Gamma(\beta(j+1)+1)}\frac{\lambda(\beta(j+1)+k)}{x^2}e^{\frac{-\lambda(\beta(j+1)+k)}{x}}$$

$$= (1-\theta)\sum_{i,k=0}^{\infty}\frac{(-1)^i\alpha^{i+1}\beta\Gamma(\beta(i+1)+k)}{i!k!\Gamma(\beta(i+1)+1)}g(x;\lambda(\beta(i+1)+k)) + \alpha\theta\times\sum_{j,k=0}^{\infty}\frac{(-1)^j(2\alpha)^j\beta\Gamma(\beta(j+1)+k)}{j!k!\Gamma(\beta(j+1)+1)}g(x;\lambda(\beta(j+1)+k)) \quad (4.4)$$

5. Statistical properties of the TWIE distribution

This section provides some basic statistical properties of the transmuted Weibull Inverse exponential distribution.

5.1. Moments of the TWIE distribution

Theorem 5.1. If $X \sim TWIE\ (\alpha,\beta,\lambda,\theta)$, then r[th] moment of a continuous random variable X is given as follow:

$$\mu_r = (1-\theta)\sum_{i,k=0}^{\infty}\omega_{i,k}\beta\{\lambda(\beta(i+1)+k)\}^r\Gamma(1-r) + \alpha\theta\sum_{j,k=0}^{\infty}\omega_{j,k}\beta\{\lambda(\beta(j+1)+k)\}^r\Gamma(1-r),\ \forall r<1.$$

Proof: Let X is an absolutely continuous non-negative random variable with PDF $f_T(x)$, then the r^{th} moment of X can be obtained by:

$$\mu_r = E(x^r) = \int_0^{\infty} x^r f_T(x)\,dx.$$

$$= (1-\theta)\sum_{i,k=0}^{\infty}\frac{(-1)^i\alpha^{i+1}\beta\Gamma(\beta(i+1)+k)}{i!k!\Gamma(\beta(i+1)+1)}\int_0^{\infty}x^r g(x;\lambda(\beta(i+1)+k))dx + \alpha\theta$$

$$\times\sum_{j,k=0}^{\infty}\frac{(-1)^j(2\alpha)^j\beta\Gamma(\beta(j+1)+k)}{j!k!\Gamma(\beta(j+1)+1)}\int_0^{\infty}x^r g(x;\lambda(\beta(j+1)+k))dx$$

$$\mu_r = I_1 + I_2\ . \tag{5.1}$$

where, $I_1 = (1-\theta)\sum_{i,k=0}^{\infty}\frac{(-1)^i\alpha^{i+1}\beta\Gamma(\beta(i+1)+k)}{i!k!\Gamma(\beta(i+1)+1)}\int_0^{\infty}x^r\,\frac{\lambda(\beta(i+1)+k)}{x^2}e^{\frac{-\lambda(\beta(i+1)+k)}{x}}dx$

Suppose $t = \frac{\lambda(\beta(i+1)+k)}{x}$ then after some simplifications, we have

$$I_1 = (1-\theta)\sum_{i,k=0}^{\infty}\frac{(-1)^i\alpha^{i+1}\beta\Gamma(\beta(i+1)+k)}{i!k!\Gamma(\beta(i+1)+1)}\{\lambda(\beta(i+1)+k)\}^r\Gamma(1-r). \tag{5.2}$$

Similarly, $I_2 = \alpha\theta\sum_{j,k=0}^{\infty}\frac{(-1)^j(2\alpha)^j\beta\Gamma(\beta(j+1)+k)}{j!k!\Gamma(\beta(j+1)+1)}\int_0^{\infty}x^r\,\frac{\lambda(\beta(j+1)+k)}{x^2}e^{\frac{-\lambda(\beta(j+1)+k)}{x}}dx$

$$I_2 = \alpha\theta\sum_{j,k=0}^{\infty}\frac{(-1)^j(2\alpha)^j\beta\Gamma(\beta(j+1)+k)}{j!k!\Gamma(\beta(j+1)+1)}\{\lambda(\beta(j+1)+k)\}^r\Gamma(1-r) \tag{5.3}$$

Substituting Equation (5.3) and Equation (5.2) in Equation (5.1), then the r[th] moment of TWIE distribution is given by

$$\mu_r = (1-\theta)\sum_{i,k=0}^{\infty}\omega_{i,k}\beta\{\lambda(\beta(i+1)+k)\}^r\Gamma(1-r) + \alpha\theta\sum_{j,k=0}^{\infty}\omega_{j,k}\beta\{\lambda(\beta(j+1)+k)\}^r\Gamma(1-r),\ \forall r<1. \tag{5.4}$$

where $\omega_{i,k} = \frac{(-1)^i\alpha^{i+1}\Gamma(\beta(i+1)+k)}{i!k!\Gamma(\beta(i+1)+1)}$ & $\omega_{j,k}\ \frac{(-1)^j(2\alpha)^j\Gamma(\beta(j+1)+k)}{j!k!\Gamma(\beta(j+1)+1)}$.

Note that the above series does not exist for $r>1$. Therefore, the r[th] moment of TWIE distribution does not exist since the expression in Equation (5.4) only exist for $r<1$.

5.2. Harmonic mean

The harmonic mean (H) is given by:

$$\frac{1}{H} = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_T(x)dx.$$

Using (4.4) we obtain

$$\frac{1}{H} = (1-\theta)\sum_{i,k=0}^{\infty} \frac{(-1)^i \alpha^{i+1} \beta\Gamma(\beta(i+1)+k)}{i!k!\Gamma(\beta(i+1)+1)} \int_0^\infty \frac{1}{x} g(x;\lambda(\beta(i+1)+k))dx + \alpha\theta$$

$$\times \sum_{j,k=0}^{\infty} \frac{(-1)^j (2\alpha)^j \beta\Gamma(\beta(j+1)+k)}{j!k!\Gamma(\beta(j+1)+1)} \int_0^\infty \frac{1}{x} g(x;\lambda(\beta(j+1)+k))dx$$

Suppose $t = \frac{\lambda(\beta(i+1)+k)}{x}$ then after some simplifications, we have

$$\frac{1}{H} = (1-\theta)\sum_{i,k=0}^{\infty} \omega_{i,k} \frac{\beta}{\{\lambda(\beta(i+1)+k)\}} + \alpha\theta \sum_{j,k=0}^{\infty} \omega_{j,k} \frac{\beta}{\{\lambda(\beta(j+1)+k)\}}, \quad (5.5)$$

where $\omega_{i,k} = \frac{(-1)^i \alpha^{i+1}\Gamma(\beta(i+1)+k)}{i!k!\Gamma(\beta(i+1)+1)}$ & $\omega_{j,k} \frac{(-1)^j (2\alpha)^j \Gamma(\beta(j+1)+k)}{j!k!\Gamma(\beta(j+1)+1)}$.

5.3. Moment generating function (MGF)

In this sub section, we derive the moment generating function of TWIE distribution.

Theorem 5.2. Let X have a TWIE distribution. Then moment generating function of X denoted by $M_X(t)$ is given by:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left\{ \begin{aligned} &(1-\theta)\sum_{i,k=0}^{\infty} \omega_{i,k} \beta\{\lambda(\beta(i+1)+k)\}^r \Gamma(1-r) \\ &\qquad + \alpha\theta \sum_{j,k=0}^{\infty} \omega_{j,k} \beta\{\lambda(\beta(j+1)+k)\}^r \Gamma(1-r) \end{aligned} \right\} \quad (5.6)$$

Proof: By definition

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_T(x)dx.$$

Using Taylor series

$$M_X(t) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f_T(x)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty x^r f_T(x)dx. = \sum_{i=0}^{\infty} \frac{t^r}{r!} E(X^r)$$

$$\therefore M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left\{ \begin{aligned} &(1-\theta)\sum_{i,k=0}^{\infty} \omega_{i,k} \beta\{\lambda(\beta(i+1)+k)\}^r \Gamma(1-r) + \\ &\qquad \alpha\theta \sum_{j,k=0}^{\infty} \omega_{j,k} \beta\{\lambda(\beta(j+1)+k)\}^r \Gamma(1-r) \end{aligned} \right\}$$

This completes the proof.

5.4. Characteristic function
In this sub section, we derive the Characteristic function of TWIE distribution.

Theorem 5.3. Let *X* have a TWIE distribution. Then characteristic function of *X* denoted by $\phi_X(t)$ is given by:

$$\phi_X(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left\{\begin{array}{l}(1-\theta)\sum_{i,k=0}^{\infty}\omega_{i,k}\beta\{\lambda(\beta(i+1)+k)\}^r\Gamma(1-r)\ + \\ \qquad\qquad \alpha\theta\sum_{j,k=0}^{\infty}\omega_{j,k}\beta\{\lambda(\beta(j+1)+k)\}^r\Gamma(1-r)\end{array}\right\} \tag{5.7}$$

Proof: By definition

$$\phi_X(t)=E\left(e^{itx}\right)=\int_0^{\infty}e^{itx}f_T(x)dx.$$

Using Taylor series

$$\phi_X(t)=\int_0^{\infty}\left(1+itx+\frac{(itx)^2}{2!}+\cdots\right)f_T(x)dx.$$

$$=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^{\infty}x^r f_T(x)dx=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(X^r)$$

$$\therefore\phi_X(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left\{\begin{array}{l}(1-\theta)\sum_{i,k=0}^{\infty}\omega_{i,k}\beta\{\lambda(\beta(i+1)+k)\}^r\Gamma(1-r)\ + \\ \qquad\qquad \alpha\theta\sum_{j,k=0}^{\infty}\omega_{j,k}\beta\{\lambda(\beta(j+1)+k)\}^r\Gamma(1-r)\end{array}\right\}$$

This completes the proof.

6. Quantile function, median and random number generation
This section deals with obtaining the quantile function, median and generating random numbers of TWIE distribution.

6.1. Quantile function and median
Theorem 5.5. Let the random variable X follow TWIE distribution. Then, the q^{th} quantile $Q(u)$ of the TWIE distribution is given by:

$$Q(u)=\frac{\lambda}{\left[\log\left[1+\left\{\frac{-1}{\alpha}\log\left(\frac{(\theta-1)+\sqrt{(1+\theta)^2-4\theta u}}{2\theta}\right)\right\}^{\frac{-1}{\beta}}\right]\right]}.$$

Proof: The Quantile function is denoted by $Q(u)$ and can be mathematically calculated as follows:

$$Q(u) = F_T^{-1}(u), \qquad 0 < u < 1. \tag{6.1}$$

$\therefore$ The corresponding quantile function for the proposed model is given by:

$$Q(u) = \frac{\lambda}{\log\left[1 + \left\{\frac{-1}{\alpha}\log\left(\frac{(\theta-1) + \sqrt{(1+\theta)^2 - 4\theta u}}{2\theta}\right)\right\}^{\frac{-1}{\beta}}\right]}. \tag{6.2}$$

where U has the uniform U (0,1) distribution. We obtain the median of the TWIE distribution by substituting u=0.5 in equation (6.2). Hence, the median of the proposed model is calculated as:

$$\text{Median} = F_T^{-1}(0.5) = \frac{\lambda}{\log\left[1 + \left\{\frac{-1}{\alpha}\log\left(\frac{(\theta-1) + \sqrt{1+\theta^2}}{2\theta}\right)\right\}^{\frac{-1}{\beta}}\right]}. \tag{6.3}$$

6.2. Random number generation

In order to generate the random numbers from the transmuted Weibull inverted exponential distribution, the method of inversion is used as follows:

$$u = \left\{1 - e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\left\{1 + \theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\},$$

where u ~ U (0, 1). After simplification this yields

$$x = \frac{\lambda}{\log\left[1 + \left\{\frac{-1}{\alpha}\log\left(\frac{(\theta-1) + \sqrt{(1+\theta)^2 - 4\theta u}}{2\theta}\right)\right\}^{\frac{-1}{\beta}}\right]}. \tag{6.4}$$

One can use equation (6.4) to generate random numbers when the parameters are known.

7. Order statistics

Order statistics finds many applications in statistical theory and modeling. It can be applied in studying the reliability of a system and life testing. If $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics obtained from the random sample $X_1, X_2, \ldots, X_n$ drawn from TWIE distribution $(\alpha, \beta, \lambda, \tau)$ with cumulative density function and probability density function given in the equations (2.3) and (2.4) respectively, then the probability density function of the order statistics is given as below:

$$f_{Tr}(x)=\frac{n!}{(r-1)!(n-r)!}\left[F_T(x)\right]^{r-1}\left[1-F_T(x)\right]^{n-r} f_T(x) \qquad for\ 1\le r\le n\,. \quad (7.1)$$

Using the equations (2.3) and (2.4), the pdf of the first order statistic $X_{(1)}=\min(X_1, X_2,\dots X_n)$ is given by

$$f_{T_1}(x)=\frac{n\alpha\beta\lambda}{x^2}\left[1-\left(\left\{1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\right)\right]^{n-1} e^{\frac{\lambda}{x}}\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta-1} e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\left\{1-\theta+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\} \quad . (7.2)$$

Similarly, the pdf of the nth order statistic $X_{(n)}=\max(X_1, X_2,\dots X_n)$ is given as follows:

$$f_{Tn}(x)=\frac{n\alpha\beta\lambda}{x^2}\left[\left\{1-e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\}\right]^{n-1} e^{\frac{\lambda}{x}}\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta-1} e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\left\{1-\theta+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}}\right\} \quad . (7.3)$$

8. Joint distribution function of i^th and j^th order statistics

The joint density function of (x_i, x_j) for $1\le i\le j\le n$ is given by

$$f_{i:j:n}(x_i, x_j)=C\left[F(x_i)\right]^{i-1}\left[F(x_j)-F(x_i)\right]^{j-i-1}\left[1-F(x_j)\right]^{n-j} f(x_i)\, f(x_j), \qquad (8.1)$$

where $C=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}$.

Then the joint distribution function of the ith and jth order statistics of TWIE distribution is as follows:

$$f_{T\,i:j:n}(x)=C\left(\left\{1-e^{-\alpha\left(h_{(i)}\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(h_{(i)}\right)^{-\beta}}\right\}\right)^{i-1}\left(\left\{1-e^{-\alpha\left(h_{(j)}\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(h_{(j)}\right)^{-\beta}}\right\}-\left(\left\{1-e^{-\alpha\left(h_{(i)}\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(h_{(i)}\right)^{-\beta}}\right\}\right)\right)^{j-i-1}$$

$$\left(1-\left\{1-e^{-\alpha\left(h_{(j)}\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(h_{(j)}\right)^{-\beta}}\right\}\right)^{n-j}\frac{\alpha\beta\lambda}{x_i^{\,2}}e^{\frac{\lambda}{x_i}}\left(h_{(i)}\right)^{-\beta-1}e^{-\alpha\left(h_{(i)}\right)^{-\beta}}\left\{1-\theta+2\theta e^{-\alpha\left(h_{(i)}\right)^{-\beta}}\right\}$$

$$\frac{\alpha\beta\lambda}{x_j^{\,2}}e^{\frac{\lambda}{x_j}}\left(h_{(j)}\right)^{-\beta-1}e^{-\alpha\left(h_{(j)}\right)^{-\beta}}\left\{1-\theta+2\theta e^{-\alpha\left(h_{(j)}\right)^{-\beta}}\right\}$$

where $h_{(k)}=e^{\frac{\lambda}{x_k}}-1$ for $k=i,\,j$. (8.2)

For the special case $i=1$ and $j=n$, we get the joint distribution of minimum and maximum order statistics as follows:

$$f_{1n}(x)=n(n-1)\left[F(x_n)-F(x_1)\right]^{n-2}f(x_1)f(x_n)$$

$$f_{1n}(x)=n(n-1)\left(\left\{1-e^{-\alpha\left(h_{(n)}\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(h_{(n)}\right)^{-\beta}}\right\}-\left(\left\{1-e^{-\alpha\left(h_{(1)}\right)^{-\beta}}\right\}\left\{1+\theta e^{-\alpha\left(h_{(1)}\right)^{-\beta}}\right\}\right)\right)^{n-2}$$

$$\frac{\alpha\beta\lambda}{x_1^{\,2}}e^{\frac{\lambda}{x_1}}\left(h_{(1)}\right)^{-\beta-1}e^{-\alpha\left(h_{(1)}\right)^{-\beta}}\left\{1-\theta+2\theta e^{-\alpha\left(h_{(1)}\right)^{-\beta}}\right\} \qquad (8.3)$$

$$\frac{\alpha\beta\lambda}{x_n^{\,2}}e^{\frac{\lambda}{x_j}}\left(h_{(n)}\right)^{-\beta-1}e^{-\alpha\left(h_{(n)}\right)^{-\beta}}\left\{1-\theta+2\theta e^{-\alpha\left(h_{(n)}\right)^{-\beta}}\right\}$$

where $h_{(n)}=e^{\frac{\lambda}{x_n}}-1$ and $h_{(1)}=e^{\frac{\lambda}{x_1}}-1$.

9. Parameter estimation

In this section maximum likelihood estimators and inference for TWIE distribution are discussed. In order to estimate the unknown parameters of the TWIE distribution we use the technique of maximum likelihood estimation. The maximum likelihood estimates (MLE's) of the parameters that are inherent within the Transmuted Weibull Inverse Exponential distribution function are obtained as follows:

Let $x_1, x_2, ...x_n$ be a random sample of size n from TWIE distribution. Then the likelihood function is given by

$$L=\prod_{i=1}^{n}\left[\frac{\alpha\beta\lambda}{x_i^{\,2}}e^{\frac{\lambda}{x_i}}\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta-1}e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}\left\{1-\theta+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}\right\}\right]. \qquad (9.1)$$

Taking log on both sides of (9.1), we get

$$\log L = n\log\alpha + n\log\beta + n\log\lambda - 2\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\frac{\lambda}{x_i} -$$

$$\alpha\sum_{i=1}^{n}\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta} - (\beta+1)\sum_{i=1}^{n}\log\left(e^{\frac{\lambda}{x_i}}-1\right) + \sum_{i=1}^{n}\log\left((1-\theta)+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}\right).$$

On differentiating log likelihood function with respect to α, β, λ and θ and equating them to zero, we get the system of nonlinear equations as

$$\frac{\partial}{\partial\alpha}\log L = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta} - 2\theta\sum_{i=1}^{n}\frac{\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}\times e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}}{\left(1-\theta+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}\right)} = 0$$

$$\frac{\partial}{\partial\beta}\log L = \frac{n}{\beta} + \alpha\sum_{i=1}^{n}\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}\log\left(e^{\frac{\lambda}{x_i}}-1\right) - \sum_{i=1}^{n}\log\left(e^{\frac{\lambda}{x_i}}-1\right) + 2\theta\sum_{i=1}^{n}\frac{\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta} e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}}{\left(1-\theta+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}\right)} = 0$$

$$\frac{\partial}{\partial\lambda}\log L = \frac{n}{\lambda} + \sum_{i=i}^{n}\frac{1}{x_i} + \alpha\beta\sum_{i=1}^{n}\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta-1}\frac{e^{\frac{\lambda}{x_i}}}{x_i} -$$

$$(\beta+1)\sum_{i=1}^{n}\frac{e^{\frac{\lambda}{x_i}}}{\left(e^{\frac{\lambda}{x_i}}-1\right)}\frac{1}{x_i} + 2\theta\sum_{i=1}^{n}\frac{\frac{1}{x_i}e^{\frac{\lambda}{x_i}}e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}e^{\alpha\beta\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta-1}}}{\left(1-\theta+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}\right)} = 0$$

$$\frac{\partial}{\partial\theta}\log L = 2\sum_{i=1}^{n}\frac{e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}-1}{\left(1-\theta+2\theta e^{-\alpha\left(e^{\frac{\lambda}{x_i}}-1\right)^{-\beta}}\right)} = 0 .$$

It can be clearly seen that the above equations are not in explicit form as such the estimates of the unknown parameters are obtained by solving the normal equations simultaneously using the Newton Raphson algorithm.

10. Fisher information matrix

For the three parameters of TWIE $(x;\alpha,\beta,\lambda,\theta)$ all the second order derivatives of the log-likelihood function exist. Thus, the inverse dispersion matrix is given by:

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\lambda}\\ \hat{\theta}\end{pmatrix} \sim N\left(\begin{pmatrix}\alpha\\ \beta\\ \lambda\\ \theta\end{pmatrix}, \begin{pmatrix}\hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} & \hat{V}_{\alpha\lambda} & \hat{V}_{\alpha\theta}\\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} & \hat{V}_{\beta\lambda} & \hat{V}_{\beta\theta}\\ \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\beta} & \hat{V}_{\lambda\lambda} & \hat{V}_{\lambda\theta}\\ \hat{V}_{\theta\alpha} & \hat{V}_{\theta\beta} & \hat{V}_{\theta\lambda} & \hat{V}_{\theta\theta}\end{pmatrix}\right) \tag{10.1}$$

$$V^{-1} = -E\begin{pmatrix}\hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} & \hat{V}_{\alpha\lambda} & \hat{V}_{\alpha\theta}\\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} & \hat{V}_{\beta\lambda} & \hat{V}_{\beta\theta}\\ \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\beta} & \hat{V}_{\lambda\lambda} & \hat{V}_{\lambda\theta}\\ \hat{V}_{\theta\alpha} & \hat{V}_{\theta\beta} & \hat{V}_{\theta\lambda} & \hat{V}_{\theta\theta}\end{pmatrix} \tag{10.2}$$

where $V_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha\partial\alpha}$, $\hat{V}_{\beta\beta} = \frac{\partial^2 L}{\partial\beta\partial\beta}$, $\hat{V}_{\lambda\lambda} = \frac{\partial^2 L}{\partial\lambda\partial\lambda}$, $\hat{V}_{\theta\theta} = \frac{\partial^2 L}{\partial\theta\partial\theta}$, $V_{\alpha\lambda} = \frac{\partial^2 L}{\partial\alpha\partial\lambda} = V_{\lambda\alpha}$,

$V_{\alpha\theta} = \frac{\partial^2 L}{\partial\alpha\partial\theta} = V_{\theta\alpha},\ \hat{V}_{\beta\alpha} = \frac{\partial^2 L}{\partial\beta\partial\alpha} = \hat{V}_{\alpha\beta}$, $\hat{V}_{\beta\lambda} = \frac{\partial^2 L}{\partial\beta\partial\lambda} = \hat{V}_{\lambda\beta}$, $\hat{V}_{\beta\theta} = \frac{\partial^2 L}{\partial\beta\partial\theta} = \hat{V}_{\theta\beta}$,

$\hat{V}_{\lambda\theta} = \frac{\partial^2 L}{\partial\lambda\partial\theta} = \hat{V}_{\theta\lambda}$ and so on.

By deriving the inverse dispersion matrix, the asymptotic variances and covariances of the ML estimators for α, β, λ and θ are obtained.

11. Data analysis

In this section three real data sets are analyzed for the purpose of illustration.

Data set I: Consider a data set corresponding to remission times (in months) of a random sample of 124 bladder cancer patients given in Lee and Wang [11]. The data set is given as follows : 0.08, 2.09, 2.73, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.22, 3.52, 4.98, 6.99, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 15.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.93, 8.65, 12.63 and 22.69. The summary of the data is given in Table 1.

Data set II: The second data set is the failure times of 84 Aircraft Windshield. The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the nonstructural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in replacement of the windshield. These data on failure times are reported in the book “Weibull Models” by Murthy et al. [12] .The failure times of 84 Aircraft Windshield is: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663. The summary of the data is given in Table 2.

Data set III: The third data set which is discussed in Smith and Naylor [13].The data are about the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. The observed data are as follows: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56,57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75,76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131,143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341and 376. The summary of the data is given in Table 3.

Table 1: Data summary for the data set first

Minimum	1st Qu.	Median	Mean	3rd Qu.	Max.	Variance	Skewness	Kurtosis
0.080	3.295	6.050	9.311	11.680	79.050	112.178	3.3184	18.548

Table 2: Data summary for the data set second

Minimum	1st Qu.	Median	Mean	3rd Qu.	Max.	Variance	Skewness	Kurtosis
0.040	1.866	2.385	2.563	3.376	4.663	1.239	0.0865	2.365

Table 3: Data summary for the data set third

Minimum	1st Qu.	Median	Mean	3rd Qu.	Max.	Variance	Skewness	Kurtosis
12.00	54.75	70.00	99.82	112.80	376.00	6580.122	1.796	5.614

Table 4: MLEs of the model parameters using real data sets, the resulting SEs parentheses and criteria for comparison

Data sets	Distributions	β	λ	θ	α	Log-likelihood	AIC	BIC
Data set 1	TWIED	0.76412 (0.06947)	0.10394 (0.21553)	-0.94792 (0.15604)	0.05189 (0.07750)	-397.3775	**802.755**	**814.0361**
	WIED	1.01279 (0.06689)	0.20481 (0.15639)	–	0.02108 (0.01859)	-399.9863	805.9726	814.4335
Data set 2	TWIED	1.68453 (0.14287)	0.10771 (0.03387)	-0.81003 (0.13481)	0.00627 (0.00236)	-132.5814	**273.1629**	**282.9335**
	WIED	1.97620 (0.14045)	0.15803 (0.03524)	–	0.00369 (0.00097)	-136.2736	278.5471	285.8751
Data set 3	TWIED	0.97406 (0.22977)	83.75871 (42.90951)	0.50686 (0.39088)	0.95409 (0.79457)	-390.7368	**789.4736**	**798.5802**
	WIED	1.31289 (0.10322)	1.61969 (0.65571)	–	0.00413 (0.00134)	-397.0588	800.1176	806.9476

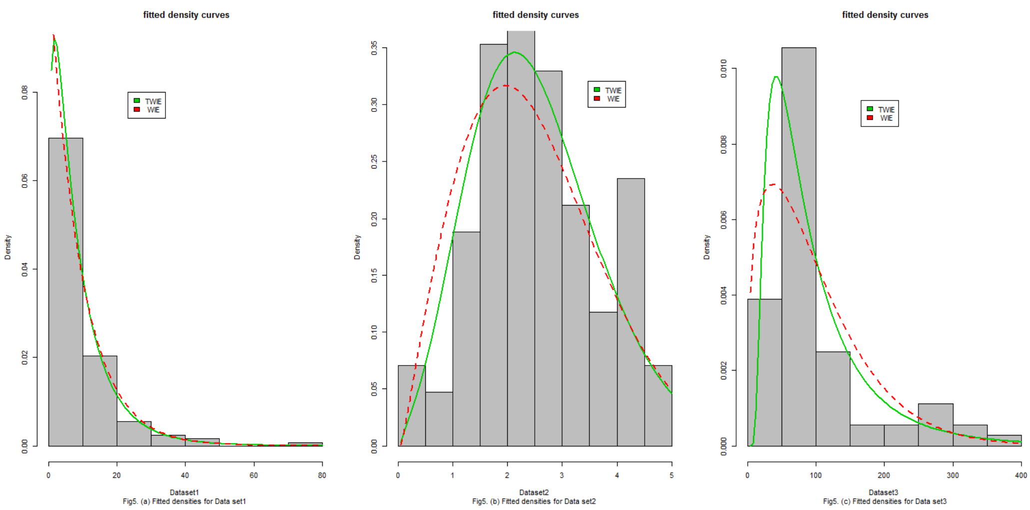

Figure 5: Graphs of the fitted Transmuted Weibull-Inverse Exponential (TWIE) and Weibull-Inverse Exponential (WIE) distributions for data sets 1, 2 and 3.

12. Conclusion

In this paper, several structural properties of the transmuted Weibull Inverse Exponential model have been studied which is obtained by adding a transmuted parameter to the base model. The parameters have been estimated by maximum likelihood estimation. Three real data sets have been considered for the comparison of the base model with the proposed model. Further, the lesser values of AIC and BIC conclude that the newly proposed distribution fits better to the real life data sets and can receive wider applications in life testing.

Acknowledgement. The authors are very grateful for the reviewers and the editors for the worthy comments and suggestions which helped us to improve our research article.

REFERENCES

2 A.Z.Keller, A.R.R.Kamath and U.D.Perera, Reliability analysis of CNC machine tools, *Reliability Engineering,* 3 (1982) 449–473.

3 A.M.Marshall and I.Olkin, A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families, *Biometrika,* 84 (1997) 641–652.

4 G.M.Cordeiro and M.de Castro, A new family of generalized distributions, *Journal of Statistical Computation and Simulation*, 81(2011) 883–893.

5 C.Alexander, G.M.Cordeiro, E.M.M.Ortega and J.M.Sarabia, Generalized beta generated distributions, *Computational Statistics and Data Analysis,* 56 (2012) 1880–1897.

6 G.M.Cordeiro, E.M.M.Ortega, B.V.Popovi´c and R.R.Pescim, The Lomax generator of distributions: Properties, minification process and regression model, *Applied Mathematics and Computation,* 247 (2014) 465–486.

7 M.Bourguignon, R.B.Silva, G.M.Cordeiro, The Weibull-G family of probability distributions, *Journal of Data Science,* 12 (2014) 53–68.

8 W.Shaw and I.Buckley, The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, *Research Report.* 2009

9 F.Merovci, Transmuted Rayleigh distribution, *Austrian Journal of Statistics,* 42 (2013) 21-31.

10 I.Ahmad, S.P.Ahmad and A.Ahmed, Transmuted inverse Rayleigh distribution: A generalization of the inverse Rayleigh distribution. *Mathematical Theory and Modeling*, 4 (2014) 90-98.

11 U.Jan, K.Fatima, S.P.Ahmad, Transmuted exponentiated inverse weibull distribution with applications in medical sciences, *International Journal of Mathematics Trends and Technology,* 50 (2017) 160-167.

12 E.T.Lee and J.W.Wang, Statistical methods for survival data analysis, 3rd edition, *John Wiley and Sons, New York, USA.* 2003

13 DNP.Murthy, M.Xi, and R.Jiangs, *Weibull models, Wiley, Hoboken*, 2004, 297.

14 R.L.Smith and J.C.Naylor, A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, *Applied Statistics*, 36 (1987) 358–369.