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Progress in Nonlinear Dynamics and Chaos

Ulam Stability for System of Nonlinear Implicit Fractional Differential Equations

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Abstract. In the present paper, we study the Ulam-type stability of solutions for system of nonlinear implicit fractional differential equations. The main techniques are based on method of successive approximations. An illustrative example is also given.

Keywords: Fraction integral and derivative, system of fractional differential equations, initial value problem, successive approximations, existence and stability of solutions.

AMS Mathematics Subject Classification (2010): 26A33, 34A08, 34A34, 34D20

1. Introduction

In 1940, Ulam [22] proposed a general Ulam stability problem in the talk before the Mathematics Club of University of Wisconsin in which he discussed a number of important unsolved problems. In the following year, Hyers [12] affirmatively answered partially to the Ulams' question. Further in 1978, Rassias [20-21] generalized the results of Hyers' and since then the stability of functional equations have been investigated by many researchers as an emerging field of mathematical analysis [1,2,5,13,14,16,18] and the books [15,17,20-23].

Existence and uniqueness of solutions of various class of fractional differential equations are recently studied by the authors in [3,4,6-11] by using variety of techniques.

In this paper, we will study four Ulam-type stabilities of solution of nonlinear initial value problem (IVP)

$$\begin{cases} \mathfrak{D}_1^\alpha x(t) = f(t, x(t), \mathfrak{D}_1^\alpha x(t)), & t \in J = [1, T], T > 1, \\ x^{(k)}(1) = x_k, \quad x_k \in \mathbb{R}^n, & k = 0, 1, \cdots, m-1, \end{cases} \tag{1}$$

for system of implicit fractional differential equations for some $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, where $f: J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear continuous function, $x: J \to \mathbb{R}^n$ and $\mathfrak{D}_1^\alpha$ is the Caputo-Hadamard derivative of order α.

The rest of the paper is organized as follows: in Section 2, we give the definitions and preliminary results. In Section 3, we prove the four Ulam-type stabilities. An illustrative example is given in last section.

2. Preliminaries

Let $\mathbb{B} = C^m(J, \mathbb{R}^n)$ be a Banach space of continuous functions from J into $\mathbb{R}^n$ having

m^{th} order derivatives with supremum norm $||\cdot||_{\mathbb{B}}$. The well-known function frequently used in the solution of fractional differential equations is the Mittag-Leffler function

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad z\in\mathbb{R}, Re(\alpha)>0, \tag{1}$$

where $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt, x>0,$ is the Gamma function.

Definition 1. *[17]The Hadamard fractional integral of order $\alpha>0$ for a continuous function $g(t):[1,+\infty)\to\mathbb{R}$ is defined as*

$$\mathfrak{J}_1^\alpha g(t)=\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}g(s)\frac{ds}{s},\quad \alpha>0. \tag{2}$$

Definition 2. *[17] The Caputo-Hadamard fractional derivative of order α for a continuous function $g(t):[1,+\infty)\to\mathbb{R}$ is defined as*

$$\mathfrak{D}_1^\alpha g(t)=\frac{1}{\Gamma(n-\alpha)}\int_1^t(\log\frac{t}{s})^{n-\alpha-1}\delta^n(g)(s)\frac{ds}{s},\quad n-1<\alpha<n, \tag{3}$$

where $\delta^n=(t\frac{d}{dt})^n, n\in\mathbb{N}$.

Lemma 1. *[17] If $m-1<\alpha\le m, m\in\mathbb{N}$ and $g\in C^m[1,T]$, then*

$$\mathfrak{J}_1^\alpha[\mathfrak{D}_1^\alpha g(t)]=g(t)-\sum_{k=0}^{m-1}\frac{g^{(k)}(1)}{\Gamma(k+1)}(\log t)^k.$$

Lemma 2. *[17] For all $\mu>0$ and $\nu>-1$,*

$$\frac{1}{\Gamma(\mu)}\int_1^t(\log\frac{t}{s})^{\mu-1}(\log s)^\nu\frac{ds}{s}=\frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log t)^{\mu+\nu}.$$

Lemma 3. *[17] Let $g(t)=t^\mu$, where $\mu\ge 0$ and if $m-1<\alpha\le m, m\in\mathbb{N}$, then*

$$\mathfrak{D}_1^\alpha(\log t)^\mu=\begin{cases}0, & if\quad \mu\in\{0,1,\cdots,m-1\},\\ \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(\log t)^{\mu+\alpha}, & if\quad \mu\in\mathbb{N},\mu\ge m\quad or\quad \mu\notin\mathbb{N},\mu>m-1.\end{cases}$$

Lemma 4. *[19] For any $t\in[1,T]$,*

$$u(t)\le a(t)+b(t)\int_1^t(\log\frac{t}{s})^{\alpha-1}u(s)\frac{ds}{s},$$

where all the functions are not negative and continuous. The constant $\alpha>0, b$ is a bounded and monotonic increasing function on $[1,T)$, then,

$$u(t)\le a(t)+\int_1^t\left[\sum_{n=1}^{\infty}\frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\log\frac{t}{s})^{n\alpha-1}a(s)\right]\frac{ds}{s},\quad t\in[1,T).$$

Remark 1. *Under the hypothesis of Lemma 4, if $a(t)$ be a nondecreasing function on $[1,T)$. Then*

$$u(t)\le a(t)E_\alpha(b(t)\Gamma(\alpha)\log t^\alpha).$$

Definition 3. *A function $x\in\mathbb{B}$ is said to be a solution of problem (1) if x satisfies nonlinear implicit fractional differential system of equations*
$\mathfrak{D}_1^\alpha x(t)=f(t,x(t),\mathfrak{D}_1^\alpha x(t))$ *on J together with initial conditions* $x^{(k)}(1)=x_k, k=0,1,\cdots,m-1,\ x_k\in\mathbb{R}^n,\ m-1<\alpha\le m,\ m\in\mathbb{N}$.

3. Ulam-type stability

In this section, we present our main results concerning the stability of solutions for IVP (1)

The following lemma is proved in [11] which is equivalence of IVP (1) with integral equation

$$x(t) = \sum_{k=0}^{m-1} \frac{x_k}{\Gamma(k+1)} (\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} p(s) \frac{ds}{s}, \quad t \in J, \tag{4}$$

where $p \in \mathbb{B}$ satisfies the functional equation

$$p(t) = f(t, \sum_{k=0}^{m-1} \frac{x_k}{\Gamma(k+1)} (\log t)^k + \mathfrak{J}_1^\alpha p(t), p(t)), \quad t \in J. \tag{5}$$

Lemma 5. *[11] Suppose that $f: J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Then system (1) is equivalent to the fractional integral equation (5).*

Next, we make the following assumptions:

$f: J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous function and satisfies the Lipschitz-type condition: for $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^n$ there exist constants $M > 0$ and $0 < N < 1$ such that

$$||f(t, x, y) - f(t, \tilde{x}, \tilde{y})|| \leq M||x - \tilde{x}|| + N||y - \tilde{y}||, \quad t \in J.$$

(H2) Let $\Phi \in C(J, \mathbb{R}_+)$ be a nondecreasing function. There exists a constant $K > 0$ satisfying $0 < K\theta < 1$ and

$$\| \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} \Phi(s) \frac{ds}{s} \| \leq K\Phi(t), \quad t \in J,$$

where $\theta = \frac{M}{1-N} > 0.$

Let $\varepsilon > 0$ and $\Phi: J \to \mathbb{R}_+$ be a continuous function. We consider the following inequations:

$$||\mathfrak{D}_1^\alpha y(t) - f(t, y(t), \mathfrak{D}_1^\alpha y(t))|| \leq \varepsilon, \qquad t \in J, \tag{6}$$

$$||\mathfrak{D}_1^\alpha y(t) - f(t, y(t), \mathfrak{D}_1^\alpha y(t))|| \leq \Phi(t), \quad t \in J, \tag{7}$$

$$||\mathfrak{D}_1^\alpha y(t) - f(t, y(t), \mathfrak{D}_1^\alpha y(t))|| \leq \varepsilon\Phi(t), \quad t \in J. \tag{8}$$

Definition 4. *Problem (1) is Ulam-Hyers stable if there exists a real number $K_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y: J \to \mathbb{R}^n$ in $\mathbb{B}$ of inequality (7), there exists a solution $x: J \to \mathbb{R}^n$ of Problem (1) in $\mathbb{B}$ with*

$$||y(t) - x(t)|| \leq \varepsilon K_f; \qquad t \in J.$$

Definition 5. *Problem (1) is generalized Ulam-Hyers stable if there exists $\psi \in C(\mathbb{R}_+, \mathbb{R}_+), \psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y: J \to \mathbb{R}^n$ in $\mathbb{B}$ of inequality (7), there exists a solution $x: J \to \mathbb{R}^n$ of Problem (1) in $\mathbb{B}$ with*

$$||y(t) - x(t)|| \leq \psi(\varepsilon); \qquad t \in J.$$

Definition 6. *Problem (1) is Ulam-Hyers-Rassias stable with respect to Φ, if there exists a real number $K_{f,\Phi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y: J \to \mathbb{R}^n$ in $\mathbb{B}$ of inequality (9), there exists a solution $x: J \to \mathbb{R}^n$ of Problem (1) in $\mathbb{B}$ with*

$$||y(t) - x(t)|| \leq \varepsilon K_{f,\Phi}\Phi(t); \qquad t \in J.$$

Definition 7. *Problem (1) is generalized Ulam-Hyers-Rassias stable with respect to* Φ, *if there exists a real number* $K_{f,\phi} > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $y: J \to \mathbb{R}^n$ *in* $\mathbb{B}$ *of inequality (7), there exists a solution* $x: J \to \mathbb{R}^n$ *of Problem (1) in* $\mathbb{B}$ *with*

$$||y(t) - x(t)|| \leq K_{f,\Phi}\Phi(t); \qquad t \in J.$$

Now we see Ulam-type stabilities for Problem (1) by using successive approximations.

Theorem 1. *Suppose that* f *satisfies assumption (H1). For every* $\varepsilon > 0$, *if* $y: J \to \mathbb{R}^n$ *in* $\mathbb{B}$ *satisfies inequality (7), then there exists a unique solution* $x: J \to \mathbb{R}^n$ *in* $\mathbb{B}$ *of Problem (1) with* $x^{(k)}(1) = y^{(k)}(1)$, *for* $k = 0,1,\cdots,m-1$. *Moreover, Problem (1) is Ulam-Hyers stable with*

$$||y(t) - x(t)|| \leq \left(\frac{E_\alpha(\theta(\log T)^\alpha) - 1}{\theta}\right)\varepsilon, \ t \in J, \text{ and } \theta = \left(\frac{M}{1-N}\right) > 0.$$

Proof: For every $\varepsilon > 0$, let $y: J \to \mathbb{R}^n$ in $\mathbb{B}$ satisfies inequality (7), then there exists a function $\sigma_y(t) \in \mathbb{B}$ (depending on y) such that

$$||\sigma_y(t)|| \leq \varepsilon, \quad \text{and} \quad \mathfrak{D}_1^\alpha y(t) = f(t, y(t), \mathfrak{D}_1^\alpha y(t)) + \sigma_y(t), \quad t \in J.$$

In the light of Lemma 5, y satisfies the fractional integral equation

$$y(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k + \mathfrak{I}_1^\alpha p^0(t) + \mathfrak{I}_1^\alpha \sigma_y(t), \ t \in J,$$

where $p^0 \in \mathbb{B}$ satisfies functional equation $p^0(t) = f(t, y(t), p^0(t))$ for $t \in J$.

Define $x^0(t) = y(t), t \in J$ and consider the sequence $\{x^j\} \subseteq \mathbb{B}$ given by

$$x^j(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p^{j-1}(s) \frac{ds}{s}, \ t \in J, \tag{9}$$

where $p^{j-1}(t) \in \mathbb{B}$ $(j \in \mathbb{N})$ is such that

$$p^{j-1}(t) = f(t, x^{j-1}(t), p^{j-1}(t)), \quad t \in J. \tag{10}$$

By using the principle of mathematical induction, we prove that

$$||x^j(t) - x^{j-1}(t)|| \leq \frac{\varepsilon}{\theta} \frac{[\theta(\log t)^\alpha]^j}{\Gamma(\alpha j + 1)}, \quad j \in \mathbb{N}, t \in J. \tag{11}$$

First we show that inequality (12) is true for $j = 1$. By using successive approximations for any $t \in J$, we obtain

$$\begin{aligned}
||x^1(t) - x^0(t)|| &= \left\| \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k + \mathfrak{I}_1^\alpha p^0(t) - y(t) \right\| \\
&= \Big\| \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p^0(s) \frac{ds}{s} \\
&\quad - \Big(\sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k + \mathfrak{I}_1^\alpha p^0(t) - \mathfrak{I}_1^\alpha \sigma_y(t)\Big) \Big\| \\
&= ||\mathfrak{I}_1^\alpha \sigma_y(t)|| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} ||\sigma_y(s)|| \frac{ds}{s} \\
&\leq \varepsilon \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}, \quad t \in J,
\end{aligned}$$

which proves inequality (12) for $j = 1$. Now, we assume that the inequality (12) hold for $j = r, r \in \mathbb{N}$ and prove it for $j = r + 1$. Again by definition of successive approximations, for any $t \in J$, we have

$$||x^{r+1}(t) - x^r(t)|| \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} ||p^r(s) - p^{r-1}(s)|| \frac{ds}{s}. \tag{12}$$

Since $p^j(t) = f(t, x^j(t), p^j(t))$, $t \in J$ and using assumption (H1), we have

$$||p^r(t) - p^{r-1}(t)|| = ||f(t, x^r(t), p^r(t)) - f(t, x^{r-1}(t), p^{r-1}(t))||$$

$$\leq M||x^r(t) - x^{r-1}(t)|| + N||p^r(t) - p^{r-1}(t)||$$
$$= \theta||x^r(t) - x^{r-1}(t)||, \quad t \in J.$$

Using the above estimate in inequality (13), we obtain

$$||x^{r+1}(t) - x^r(t)|| \leq \frac{\theta}{\Gamma(\alpha)} \int_1^t (\log\frac{t}{s})^{\alpha-1} [||x^r(s) - x^{r-1}(s)||] \frac{ds}{s}$$
$$\leq \frac{\theta}{\Gamma(\alpha)} \int_1^t (\log\frac{t}{s})^{\alpha-1} [\frac{\varepsilon}{\theta} \frac{[\theta(\log s)^\alpha]^r}{\Gamma(r\alpha+1)}] \frac{ds}{s}$$
$$= \frac{\varepsilon\theta^r}{\Gamma(r\alpha+1)} (\int_1^t (\log\frac{t}{s})^{\alpha-1} \frac{(\log s)^{r\alpha}}{\Gamma(\alpha)} \frac{ds}{s})$$
$$= \frac{\varepsilon}{\theta} \frac{(\theta(\log t)^\alpha)^{(r+1)}}{\Gamma(\alpha(r+1)+1)}, \quad t \in J,$$

which is inequality (12) for $j = r + 1$. The proof of inequality (12) is completed by the principle of mathematical induction.

Furthermore, for any $t \in J$, from inequality (12), we obtain

$$||x^j(t) - x^{j-1}(t)|| \leq \frac{\varepsilon}{\theta} \sum_{j=1}^\infty \frac{(\theta(\log T)^\alpha)^j}{\Gamma(j\alpha+1)} \quad \text{and } j \in \mathbb{N}.$$

This gives

$$||x^j(t) - x^{j-1}(t)|| \leq \frac{\varepsilon}{\theta} (E_\alpha(\theta(\log T)^\alpha) - 1). \tag{13}$$

Hence the series $x^0(t) + \sum_{j=1}^\infty [x^j(t) - x^{j-1}(t)]$ converges absolutely and uniformly on J with respect to the norm $||\cdot||$. Consider

$$x(t) = x^0(t) + \sum_{j=1}^\infty [x^j(t) - x^{j-1}(t)], \quad t \in J. \tag{14}$$

Then

$$x^r(t) = x^0(t) + \sum_{j=1}^r [x^j(t) - x^{j-1}(t)]$$

is the r^{th} partial sum of the series (15), and gives

$$\lim_{r\to\infty} ||x^r(t) - x(t)|| = 0, \quad \text{for all } t \in J. \tag{15}$$

Since convergence is uniform, $x \in \mathbb{B}$. We prove that the limit function x is a solution of

$$x(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t (\log\frac{t}{s})^{\alpha-1} p(s) \frac{ds}{s}, \quad t \in J,$$

where $p \in \mathbb{B}$ satisfies the functional equation $p(t) = f(t, x(t), p(t))$, $t \in J$.

For any $t \in J$, we prove $p^r \in \mathbb{B}, (r = 0,1,\cdots)$ generated in (10) satisfies

$$\lim_{r\to\infty} ||p^r(t) - p(t)|| = 0. \tag{16}$$

Using assumption (H1), we obtain

$$||p^r(t) - p(t)|| = ||f(t, x^r(t), p^r(t)) - f(t, x(t), p(t))||$$
$$\leq M||x^r(t) - x(t)|| + N||p^r(t) - p(t)||$$
$$= \theta||x^r(t) - x(t)||, \quad t \in J. \tag{17}$$

Further, using equation (16), equation (17) can be easily proved. Again, by definition of successive approximations

$$\| x(t) - \sum_{k=0}^{m-1} \frac{x_k}{\Gamma(k+1)} (\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t (\log\frac{t}{s})^{\alpha-1} p(s) \frac{ds}{s} \|$$
$$= \| x(t) - x^j(t) + \mathfrak{I}_1^\alpha p^{j-1}(t) - \mathfrak{I}_1^\alpha p(t) \|$$
$$\leq ||x(t) - x^j(t)|| + \frac{1}{\Gamma(\alpha)} \int_1^t (\log\frac{t}{s})^{\alpha-1} ||p^{j-1}(s) - p(s)|| \frac{ds}{s}.$$

Note that left hand side of above inequality is independent of j, taking limit as $j \to \infty$, we obtain

$$x(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t (\log\frac{t}{s})^{\alpha-1} p(s) \frac{ds}{s}, \quad t \in J. \tag{18}$$

This means $x(t)$ is solution of Problem (1) with initial condition

$$x^{(k)}(1) = y^{(k)}(1), \quad x^{(k)}(1), y^{(k)}(1) \in \mathbb{R}^n, k = 0,1,\cdots,m-1.$$

Lastly, from inequality (14) with series (15), it follows that Problem (1) is Ulam-Hyers stable with

$$||y(t) - x(t)|| \leq (\frac{E_\alpha(\theta(\log T)^\alpha) - 1}{\theta})\varepsilon, \quad t \in J. \tag{19}$$

To prove uniqueness of solution $x(t)$, assume that $\bar{x}(t)$ is another solution of Problem (1) with initial condition $\bar{x}^{(k)}(1) = y^{(k)}(1)$, $x^{(k)}(1), y^{(k)}(1) \in \mathbb{R}^n$, $k = 0,1,\cdots,m-1$. Then

$$\bar{x}(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^k + \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\bar{p}(s)\frac{ds}{s}, \quad t \in J,$$

where $\bar{p} \in \mathbb{B}$ satisfies $\bar{p}(t) = f(t, \bar{x}(t), \bar{p}(t))$. Therefore

$$||x(t) - \bar{x}(t)|| \leq \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}||p(s) - \bar{p}(s)||\frac{ds}{s}, \quad t \in J.$$

By hypothesis (H1),

$$||p(t) - \bar{p}(t)|| \leq \theta||x(t) - \bar{x}(t)||.$$

Hence

$$||x(t) - \bar{x}(t)|| \leq \frac{\theta}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}||x(s) - \bar{x}(s)||\frac{ds}{s}, \quad t \in J.$$

Applying Lemma 4 to above inequality with $u(t) = ||x(t) - \bar{x}(t)||$ and $a(t) = 0$, we obtain $||x(t) - \bar{x}(t)|| = 0$, for all $t \in J$. The proof is completed.

Corollary 1. *Suppose that all the assumptions of Theorem 1 are satisfied. Then Problem* (1) *is generalized Ulam-Hyers stable.*

Proof: Let $\psi(\varepsilon) = (\frac{E_\alpha(\theta(\log T)^\alpha) - 1}{\theta})\varepsilon$ in (19) then $\psi(0) = 0$. Thus, Problem (1) is generalized Ulam-Hyers stable.

Theorem 2. *Suppose that (H1) and (H2) hold. Then for every* $\varepsilon > 0$ *and* $y: J \to \mathbb{R}^n$ *in* $\mathbb{B}$ *satisfying inequality* (9), *there exists a unique solution* $x: J \to \mathbb{R}^n$ *in* $\mathbb{B}$ *of Problem (1) with* $x^{(k)}(1) = y^{(k)}(1)$, $k = 0,1,\cdots,m-1$, *that satisfies*

$$||y(t) - x(t)|| \leq \varepsilon(\frac{K}{1-K\theta})\Phi(t), \quad t \in J.$$

Proof: For every $\varepsilon > 0$, let $y: J \to \mathbb{R}^n$ in $\mathbb{B}$ satisfies inequality (9). Then there exists a function $\sigma_y \in \mathbb{B}$ (depending on y) such that

$$||\sigma_y(t)|| \leq \varepsilon\Phi(t), \quad \text{and} \quad \mathfrak{D}_1^\alpha y(t) = f(t, y(t), \mathfrak{D}_1^\alpha y(t)) + \sigma_y(t), \quad t \in J.$$

By Lemma 5, y satisfies the fractional integral equation

$$y(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^k + \mathfrak{I}_1^\alpha p^0(t) + \mathfrak{I}_1^\alpha \sigma_y(t), \ t \in J,$$

where $p^0 \in \mathbb{B}$ satisfies functional equation $p^(t) = f(t, y(t), p^0(t)), t \in J$.

Consider the sequence $\{x^j\} \subseteq \mathbb{B}$ defined by (10) with $x^0(t) = y(t), t \in J$. By the principle of mathematical induction, we prove that

$$||x^j(t) - x^{j-1}(t)|| \leq \frac{\varepsilon}{\theta}(K\theta)^j\Phi(t), \quad j \in \mathbb{N}, t \in J. \tag{20}$$

First we show the inequality (21) is true for $j = 1$. For any $t \in J$, using definition of successive approximations and assumption (H2), we have

$$||x^1(t) - x^0(t)|| = ||x^1(t) - y(t)||$$
$$= ||\mathfrak{I}_1^\alpha \sigma_y(t)||$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}||\sigma_y(s)||\frac{ds}{s}$$
$$\leq \frac{\varepsilon}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\Phi(s)\frac{ds}{s}$$
$$= \varepsilon \parallel \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\Phi(s)\frac{ds}{s} \parallel$$
$$\leq \frac{\varepsilon}{\theta}(K\theta)\Phi(t), \quad t \in J.$$

Thus, inequality (21) holds for $j = 1$. Assume that inequality (21) is true for $j = r, r \in \mathbb{N}$ and using similar arguments as we presented in Theorem 1, we have

$$||x^{r+1}(t) - x^r(t)|| \leq \frac{\theta}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}||x^r(s) - x^{r-1}(s)||\frac{ds}{s}$$
$$\leq \frac{\varepsilon}{\Gamma(\alpha)}(K\theta)^r\int_1^t (\log\frac{t}{s})^{\alpha-1}\Phi(s)\frac{ds}{s}$$
$$= \varepsilon(K\theta)^r \parallel \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\Phi(s)\frac{ds}{s} \parallel$$
$$\leq \varepsilon(K\theta)^r K\Phi(t).$$

Therefore

$$||x^{r+1}(t) - x^r(t)|| \leq \frac{\varepsilon}{\theta}(K\theta)^{r+1}\Phi(t), \quad t \in J,$$

which is inequality (21) for $j = r + 1$. By the principle of mathematical induction, inequality (21) is true for all j and the proof of inequality (21) is completed. Now using inequality (21) and assumption $0 < K\theta < 1$, we have

$$\sum_{j=1}^{\infty} ||x^j(t) - x^{j-1}(t)|| \leq \frac{\varepsilon}{\theta}(\sum_{j=1}^{\infty} (\theta K)^j)\Phi(t) = \frac{\varepsilon}{\theta}(\sum_{j=0}^{\infty} (\theta K)^j - 1)\Phi(t).$$

Therefore

$$\sum_{j=1}^{\infty} ||x^j(t) - x^{j-1}(t)|| \leq \frac{\varepsilon}{\theta}(\frac{1}{1-K\theta} - 1)\Phi(t) = \varepsilon(\frac{K}{1-K\theta})\Phi(t). \qquad (21)$$

Since $\Phi(t)$ is continuous on compact set J, it is bounded. Clearly, from above inequality (22), it follows that the series $x^0(t) + \sum_{j=1}^{\infty} [x^j(t) - x^{j-1}(t)]$ converges absolutely and uniformly on J, with respect to the norm $||\cdot||$. Define

$$x(t) = x^0(t) + \sum_{j=1}^{\infty} [x^j(t) - x^{j-1}(t)], \quad t \in J, \qquad (22)$$

and following the proof of Theorem 1, finally we obtain

$$||y(t) - x(t)|| \leq \varepsilon(\frac{K}{1-K\theta})\Phi(t), \quad t \in J.$$

Corollary 2. *Under hypothesis of Theorem 1, Problem (1) is generalized Ulam-Hyers -Rassias stable with respect to* $\Phi \in C(J, \mathbb{R}_+)$.

Proof: Set $\varepsilon = 1$ and $K_{f,\Phi} = \frac{K}{1-K\theta}$, it directly follows that Problem (1) is generalized Ulam-Hyers-Rassias stable.

4. An example

Let $\mathbb{R}^2$ be the normed space with the norm

$$||x|| = |x_1| + |x_2|, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Consider the following nonlinear implicit fractional initial value problem

$$\begin{cases} \mathfrak{D}_1^{\frac{5}{2}}x(t) = f(t, x(t), \mathfrak{D}_1^{\frac{5}{2}}x(t)), \quad t \in [1, e], \\ x^{(k)}(1) = x_k, \quad x_k \in \mathbb{R}^2, k = 0,1,2, \end{cases} \qquad (23)$$

where $x: [1, e] \to \mathbb{R}^2$ and a nonlinear function $f: [1, e] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ as

$$f(t,x(t),\mathfrak{D}_1^{\frac{5}{2}}x(t)) = f(t,(x_1(t),x_2(t)),(\mathfrak{D}_1^{\frac{5}{2}}x_1(t),\mathfrak{D}_1^{\frac{5}{2}}x_1(t)))$$
$$= \left(\frac{\log(2+t)}{1+|x_1(t)|+|x_2(t)|}, \frac{|\mathfrak{D}_1^{\frac{5}{2}}x_1(t)|+|\mathfrak{D}_1^{\frac{5}{2}}x_2(t)|}{e^{t^2+1}(1+|\mathfrak{D}_1^{\frac{5}{2}}x_1(t)|+|\mathfrak{D}_1^{\frac{5}{2}}x_2(t)|)}\right),\ t\in[1,e].$$

For any $x=(x_1,x_2), y=(y_1,y_2), \bar{x}=(\bar{x}_1,\bar{x}_2), \bar{y}=(\bar{y}_1,\bar{y}_2)\in\mathbb{R}^2$, we have

$$||f(t,x,y)-f(t,\bar{x},\bar{y})|| \le ||f(t,(x_1,x_2),(y_1,y_2),(\bar{y}_1,\bar{y}_2))||$$
$$=\| \left(\frac{\log(2+t)}{1+|x_1|+|x_2|}, \frac{|y_1|+|y_2|}{e^{t^2+1}(1+|y_1|+|y_2|)}\right) - \left(\frac{\log(2+t)}{1+|\bar{x}_1|+|\bar{x}_2|}, \frac{|\bar{y}_1|+|\bar{y}_2|}{e^{t^2+1}(1+|\bar{y}_1|+|\bar{y}_2|)}\right) \|$$
$$=\| (\log(2+t)[\frac{1}{1+|x_1|+|x_2|}-\frac{1}{1+|\bar{x}_1|+|\bar{x}_2|}], \frac{1}{e^{t^2+1}}[\frac{|y_1|+|y_2|}{1+|y_1|+|y_2|}-\frac{|\bar{y}_1|+|\bar{y}_2|}{1+|\bar{y}_1|+|\bar{y}_2|}]) \|$$
$$=\| (\log(2+t)[\frac{|\bar{x}_1|-|x_1|+|\bar{x}_2|-|x_2|}{(1+|x_1|+|x_2|)(1+|\bar{x}_1|+|\bar{x}_2|)}], \frac{1}{e^{t^2+1}}[\frac{|y_1|-|\bar{y}_1|+|y_2|-|\bar{y}_2|}{(1+|y_1|+|y_2|)(1+|\bar{y}_1|+|\bar{y}_2|)}]) \|$$
$$=\log(2+t)\left|\frac{|\bar{x}_1|-|x_1|+|\bar{x}_2|-|x_2|}{(1+|x_1|+|x_2|)(1+|\bar{x}_1|+|\bar{x}_2|)}\right| + \frac{1}{e^{t^2+1}}\left|\frac{|y_1|-|\bar{y}_1|+|y_2|-|\bar{y}_2|}{(1+|y_1|+|y_2|)(1+|\bar{y}_1|+|\bar{y}_2|)}\right|.$$

For any $a,b\ge 0$, we have $1\le(1+a+b)$. Therefore

$$||f(t,x,y)-f(t,\bar{x},\bar{y})|| \le \log(2+t)|(|\bar{x}_1|-|x_1|+|\bar{x}_2|-|x_2|)| + \frac{1}{e^{t^2+1}}|(|y_1|-|\bar{y}_1|+|y_2|-|\bar{y}_2|)|$$
$$\le \log(2+t)||(||\bar{x}||-||x||)|| + \frac{1}{e^{t^2+1}}||(||y||-||\bar{y}||)||$$
$$\le \log(2+e)||\bar{x}-x|| + \frac{1}{e^2}||y-\bar{y}||.$$

Thus, function f satisfies condition (H1) with $M=\log(2+e)>0$ and $0<N=\frac{1}{e^2}<1$. By Theorem 1 [11], Problem (24) has a unique solution on $[1,e]$.

Moreover, as shown in Theorem 1, for every $\varepsilon>0$ if $y:[1,e]\to\mathbb{R}^2$ satisfies

$$||\mathfrak{D}_1^{\frac{5}{2}}x(t)-f(t,x(t),\mathfrak{D}_1^{\frac{5}{2}}x(t))|| \le \varepsilon, \quad t\in[1,e], \tag{24}$$

there exists a unique solution $x:[1,e]\to\mathbb{R}^2$ such that

$$||y(t)-x(t)|| \le \left(\frac{E_{\frac{5}{2}}(\theta(\log e)^{\frac{5}{2}})-1}{\theta}\right)\varepsilon, \quad \text{for all } t\in[1,e],$$

where $\theta=\frac{M}{1-N}=\frac{e^2\log(2+e)}{(e^2-1)}$. Hence problem (24) is Ulam-Hyers stable.

Next, by corollary 2, $\psi(\varepsilon)=\frac{E_{\frac{5}{2}}(\theta)-1}{\theta}\varepsilon$ then $\psi(0)=0$ which means Problem (24) is generalized Ulam-Hyers stable.

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