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# On Super Standard Elements of a Lattice

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***Abstract.*** In this paper, authors introduced the notion of a super standard element of a lattice. They have given several characterizations of this element. For a fixed element $n$, a convex sublattice of $L$ containing $n$ is called an $n$-ideal. $n$-ideal generated by a single element $a$ is called a principle $n$-ideal, denoted by $<a>_n$. The set of all principal $n$-ideals is denoted by $P_n(L)$. They proved that when $n$ is super standard, $P_n(L)$ is a meet semi lattice. A meet semi lattice together with the property that any two elements possessing a common upper bound have a supremum is called a near lattice. Then the authors have introduced the concept of another type of element, known as nearly neutral element. They proved when $n$ is nearly neutral, then $P_n(L)$ is a near lattice, but not necessarily a lattice. Moreover, when $n$ is nearly neutral and complemented in any interval containing it, then $P_n(L)$ is a lattice. They preferred to call it as a nearly central element. At the end they have included a characterization of super standard elements.

***Keywords:*** Standard element, Super standard element, Neutral element, Nearly neutral element, Nearly central element.

***AMS Mathematics Subject Classification (2010):*** 06A12, 06A99, 06B10

## 1. Introduction

The $n$-ideal of a lattice have been studied by many authors including [1,2,3,4,5]. Let $n$ be an element of a lattice $L$. An $n$-ideal of $L$ is a convex sublattice of $L$ containing $n$. It is well known that this concept is a generalization of both ideals and filters of a lattice.

For two $n$-ideals $I$ and $J$, $I \wedge J = I \cap J$ and $I \vee J = \{x \in L : i_1 \wedge j_1 \le x \le i_2 \vee j_2$ *for some* $i_1, i_2 \in I,\ j_1, j_2 \in J\}$. The set of $n$-ideals in a lattice is denoted by $I_n(L)$. An $n$-ideal generated by a finite numbers of elements $a_1, a_2, \ldots\ldots, a_m$ is called a finitely generated $n$-ideal denoted by $<a_1, a_2, \ldots\ldots, a_m>_n$.

By [2], [3],
$< a_1, a_2, ..., a_m >_n = \{x \in L : a_1 \wedge a_2 \wedge .... \wedge a_m \wedge n \leq x \leq a_1 \vee a_2 \vee .... \vee a_m \vee n\}$.
Thus every finitely generated $n$-deal is 2-genrated and it is an interval $[a, b]$.
Moreover, it is well known,

$[a,b] \cap [c,d] = [a \vee c, b \wedge d]$ and
$[a,b] \cup [c,d] = [a \wedge c, b \vee d]$.

The set of all finitely generated $n$-ideals is again a lattice, denoted by $F_n(L)$. The $n$-ideal generated by a single element is called a principal $n$-ideal. Set of all principal $n$-ideals is denoted by $P_n(L)$. Of course $< a >_n = [a \wedge n, a \vee n]$. Thus

$$< a >_n \cap < b >_n = [a \wedge n, a \vee n] \cap [b \wedge n, b \vee n]$$
$$= [(a \wedge n) \vee (b \wedge n), (a \vee n) \wedge (b \vee n] \text{ and}$$
$$< a >_n \vee < b >_n = [a \wedge b \wedge n, a \vee b \vee n]$$

In this paper we have studied the super standard elements of a lattice. Then we have given some characterizations of $P_n(L)$ when $n$ is a super standard element.

By [7, 8], an element $s$ is called a standard element in a lattice $L$ if for all $x, y \in L$, $x \wedge (y \vee s) = (x \wedge y) \vee (x \wedge s)$.
In the pentagonal lattice $L$,

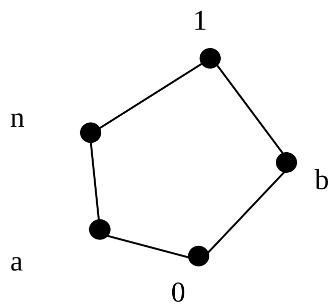



**Figure 1:**

element $n$ is standard but $a, b$ are not.

**2. Main results**

Recently we have characterized a standard element in the following way:

**Theorem 1.** Let $s$ be an element of a lattice $L$. Then the following conditions are equivalent:

i) $s$ is standard
ii) For all $t, x, y \in S$, $t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$.
iii) **Proof:** i)=>ii). Suppose $s$ is standard and $t, x, y \in L$.

The
$t \wedge [(x \wedge y) \vee (x \wedge s)] = t \wedge [x \wedge (y \vee s)] = (t \wedge x) \wedge (y \vee s) = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$

(ii)=>(i) Suppose (ii) holds. Let $y \vee s = r$. Then $r \wedge y = y$ and $r \wedge s = s$.

Thus, $x \wedge (y \vee s) = x \wedge [(r \wedge y) \vee (r \wedge s)]$

$$= (t \wedge r \wedge y) \vee (x \wedge r \wedge s)$$

$$= (x \wedge y) \vee (x \wedge s)$$ and so $s$ is standard. □

An element $d$ of a lattice $L$ is called a distributive element if $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$ for all $x, y \in L$. It is well known that every standard element is distributive, but the converse is not necessarily true. For example, in Figure-1, element $b$ is distributive but not standard.

By [9], an element $m$ of a lattice $L$ is called modular if for all $x, y \in L$ with $y \leq x$, $x \wedge (m \vee y) = (x \wedge m) \vee y$.

In Figure 1, element $a$ is modular but $b$ is not. Of course every standard element is modular.

Following result is due to [9].

**Theorem 2.** An element $s$ of a lattice $L$ which is both modular and distributive is standard.

**Super standard element:** An element $n$ of a lattice $L$ is called a super standard element if

(i) $n$ is standard and

(ii) $n \wedge m(x,n,y) = n \wedge [(x \wedge y) \vee (x \wedge n) \vee (y \vee n)] = (x \wedge n) \vee (y \wedge n)$

In Figure 1, $n$ is not only standard but it is easy to verify that it is also super standard. But in Figure 2, $n$ is standard but not super standard.

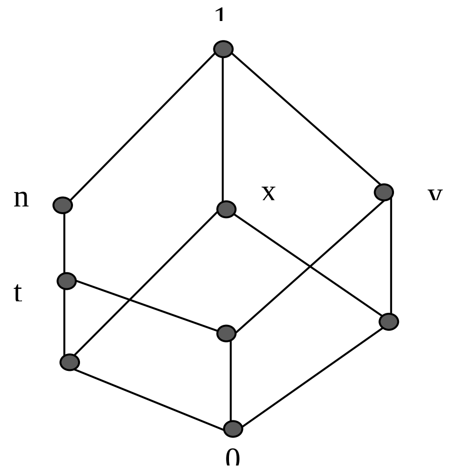



**Figure 2:**

Observe that $n \wedge m(x,n,y) = n \wedge 1 = n > t = (x \wedge n) \vee (y \wedge n)$.

Now we include a characterization of super standard element.

**Theorem 3.** An element $n$ of a lattice $L$ is super standard if and only if
$\forall \quad t, x, y \in L,$
$t \wedge m(x,n,y) = (t \wedge x \wedge y) \vee (t \wedge ((x \wedge n) \vee (y \wedge n))).$

**Proof:** Suppose $n$ is super standard in $L.$ Then $n$ is standard and
$n \wedge m(x,n,y) = (x \wedge n) \vee (y \wedge n)$ for all $x, y \in L.$ Then
$t \wedge m(x,n,y) = t \wedge [(x \wedge y) \vee (x \wedge n) \vee (y \wedge n))]$
$= t \wedge [(m(x,n,y) \wedge x \wedge y) \vee (n \wedge m(x,n,y))]$
$= (t \wedge (x \wedge y) \wedge m(x,n,y)) \vee (t \wedge m(x,n,y) \wedge n)$ (by Theorem1) as $n$ is standard
$= (t \wedge x \wedge y) \vee (t \wedge ((x \wedge n) \vee (y \wedge n))).$
Conversely, let $t \wedge m(x,n,y) = (t \wedge x \wedge y) \vee (t \wedge ((x \wedge n) \vee (y \wedge n)))$ for all
$\forall \quad t, x, y \in L$.
Then putting $t = n,$ we have
$n \wedge m(x,n,y) = (n \wedge x \wedge y) \vee (n \wedge ((x \wedge n) \vee (y \wedge n)))$
$= (x \wedge y \wedge n) \vee (x \wedge n) \vee (y \wedge n))$
$= (x \wedge n) \vee (y \wedge n).$
Moreover, $t \wedge [(x \wedge y) \vee (x \wedge n)]$
$= t \wedge [(x \wedge (x \wedge y)) \vee (x \wedge n) \vee (x \wedge y \wedge n)]$
$= t \wedge m(x,n,x \wedge y)$
$= (t \wedge x \wedge (x \wedge y)) \vee (t \wedge [(x \wedge n) \vee (x \wedge y \wedge n)])$
$= (t \wedge x \wedge y) \vee (t \wedge x \wedge n),$ and so by Theorem-1, $n$ is standard. Thus, $n$ is super standard. □

**Theorem 4.** Let $n$ be a standard element of a lattice $L$. Then the following conditions are equivalent:
(i) $n$ is a super standard element.
(ii) $< x >_n \cap < y >_n = < m(x,n,y) >_n$ for all $x, y \in L$.

**Proof:** (i)$\Rightarrow$(ii). Suppose $n$ is super standard.

Now,
$< x >_n \cap < y >_n = [x \wedge n, x \vee n] \cap [y \wedge n, y \vee n] = [(x \wedge n) \vee (y \wedge n), (x \vee n) \wedge (y \vee n)]$
$= [m(x,n,y) \wedge n, (x \wedge y) \vee n]$
$= [m(x,n,y) \wedge n, m(x,n,y) \vee n] = < m(x,n,y) >_n$ as $n$ is super standard and so distributive .
Conversely, let $< x >_n \cap < y >_n = < m(x,n,y) >_n$
Then $[(x \wedge n) \vee (y \wedge n), (x \vee n) \wedge (y \vee n)] = [m(x,n,y) \wedge n, m(x,n,y) \vee n]$.
This implies $n \wedge m(x,n,y) = (x \wedge n) \vee (y \wedge n)$ and so $n$ is super standard. □
Now we have a nice improvement of Theorem 4.

**Theorem 5.** Let $n$ be a modular element of a lattice $L$. The following conditions are equivalent:

(i) $n$ is supper standard

(ii) $< x >_n \cap < y >_n = < m(x,n,y) >_n$ for all $x, y \in L$.

**Proof:** (i)=>(ii). Suppose $n$ is super standard. Then,

$$< x >_n \cap < y >_n = [x \wedge n, x \vee n] \cap [y \wedge n, y \vee n]$$
$$= [(x \wedge n) \vee (y \wedge n), (x \vee n) \wedge (y \vee n)]$$
$$= [(x \wedge n) \vee (y \wedge n), n \vee (x \wedge y)], \text{ as } n \text{ is super standard.}$$
$$= < m(x,n,y) >_n$$

(ii)=>(i) let for all $x, y \in L$, $< x >_n \cap < y >_n = < m(x,n,y) >_n$.

Then $[(x \wedge n) \vee (y \wedge n), (x \vee n) \wedge (y \vee n] = [n \wedge m(x,n,y), n \vee m(x,n,y)]$ and so

$n \wedge m(x,n,y) = (x \wedge n) \vee (y \wedge n)$

Moreover, $(x \vee n) \wedge (y \vee n) = n \vee m(x,n,y)$

$= n \vee (x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$

$= n \vee (x \wedge y)$ which implies $n$ is distributive. Since $n$ is modular, so by Theorem 2, $n$ is standard. Thus by Theorem 4, $n$ is super standard. □

**Corollary 6.** $P_n(L)$ is a meet semi lattice if $n$ is a super standard element of $L$. □

**Corollary 7.** If $n$ is modular and $P_n(L)$ is a meet semilattice then $n$ is supper standard and hence $n$ is standard. □

A meet semi lattice $(S;\leq)$ is called a near lattice if any two elements possessing a common upper bound have a supremum. Any finite meet semi lattice is a near lattice.

[ 6, Fig 1], gives an example of a meet semilattice which is not a near lattice.

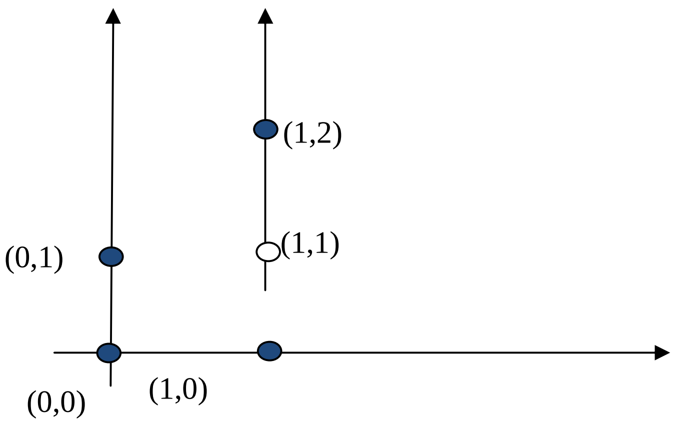



**Figure 3:**

In $R^2$, $S = \{(1,0),(0,1),(1,y) : y > 1, y \in R\}$ is a meet semilattice. $(1,2)$ is a common upper bound of both $(1,0)$ and $(0,1)$. But they don't have the supremum as $(1,1) \notin S$.

Now we know from Corollary 6 that $P_n(L)$ is a meet semilattice when $n$ is supper standard. But then $P_n(L)$ need not be a near lattice. In Figure-1, $n$ is supper standard. In $P_n(L)$, $<a>_n, <1>_n \subseteq <b>_n$ . But $<a>_n \vee <1>_n = \{a, n, 1\} \notin P_n(L)$. Hence $P_n(L)$ is not a near lattice.

An element $n \in L$ is called neutral if

(i) it is standard and

(ii) $n \wedge (x \vee y) = (n \wedge x) \vee (n \wedge y)$ for all $x, y \in L$ .

If $n$ is neutral then clearly $n \wedge m(x, n, y) = n \wedge [(x \wedge y) \vee (x \wedge n) \vee (y \wedge n)]$
$= (x \wedge y \wedge n) \vee (n \wedge ((x \wedge n) \vee (y \wedge n))) = (x \wedge y \wedge n) \vee (x \wedge n) \vee (y \wedge n)$
$= (x \wedge n) \vee (y \wedge n)$ implies it is super standard.

In figure 1, $n$ is super standard but $n \wedge (a \vee b) = n > a = (n \wedge a) \vee (n \wedge b)$ implies it is not neutral. By above theorem, $P_n(L)$ is a meet semi lattice when $n$ is super standard. But it is not necessarily a lattice even if $n$ is neutral. For example consider 3 element chain $\{0, n, 1\}$. That is, $0 \le n \le 1$. Then the elements of $P_n(L)$ are $\{n\}$, $<0>_n = \{0, n\}$, $<1>_n = \{1, n\}$

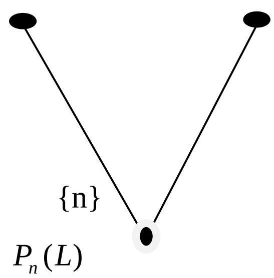



**Figure 4:**

which is not a lattice.

An element $n \in L$ is called nearly neutral if

(i) $n$ is standard and

(ii) For all $x, y \in L$, $n \wedge [x \vee (y \wedge n)] = (x \wedge y) \vee (y \wedge n)$

Of course every neutral element is nearly neutral. In figure 5, $n$ is nearly neutral, but $n \wedge (a \vee b) = n > 0 = (n \wedge a) \vee (n \wedge b)$ shows that $n$ is not neutral. Moreover, if $n$ is nearly neutral, then

$n \wedge m(x, n, y) = n \wedge [(x \wedge y) \vee (x \wedge n) \vee (y \wedge n)]$
$= n \wedge [(x \wedge y) \vee (((x \wedge n) \vee (y \wedge n)) \wedge n)]$
$= (n \wedge x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$
$= (x \wedge n) \vee (y \wedge n)$ implies $n$ is super standard.

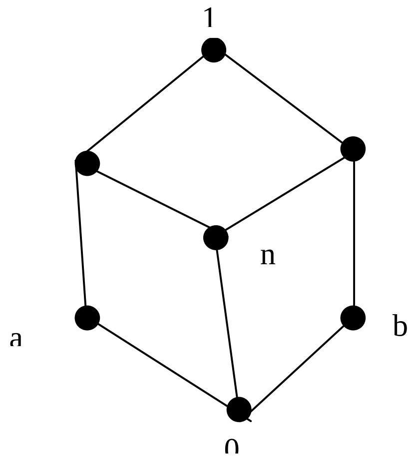



**Figure 5:**

**Theorem 8.** If $n$ is nearly neutral, then $P_n(L)$ is a near lattice.

**Proof:** We already know that $P_n(L)$ is a meet semi lattice. Moreover, for $x, y \in L$,
$< x >_n \cap < y >_n = < m(x,n,y) >_n$.

Now, let $< x >_n, < y >_n \subseteq < t >_n$.

Then $t \wedge n \leq x \wedge n \leq x \vee n \leq t \vee n$ and $t \wedge n \leq y \wedge n \leq y \vee n \leq t \vee n$

Thus, $t \wedge n \leq x \wedge y \wedge n \leq x \vee y \vee n \leq t \vee n.$

Now, $< x >_n \vee < y >_n = [x \wedge y \wedge n, x \vee y \vee n.]$

Let, $r = (t \wedge (x \vee y \vee n)) \vee (x \wedge y \wedge n)$

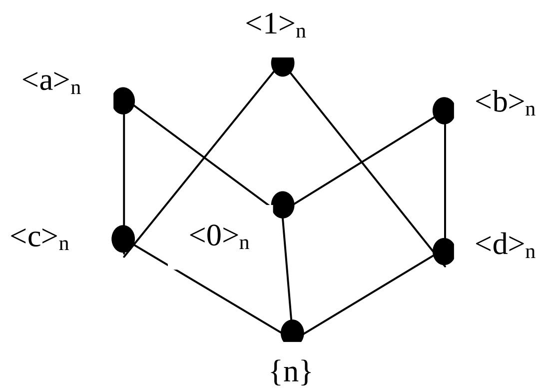
Then, $r \wedge n = n \wedge [(t \wedge (x \vee y \vee n)) \vee (x \wedge y \wedge n)]$
$= (n \wedge t \wedge (x \vee y \vee n)) \vee (x \wedge y \wedge n)$ as $n$ is a nearly neutral
$= (t \wedge n) \vee (x \wedge y \wedge n)$
$= x \wedge y \wedge n$

Also, $r \vee n = (t \wedge (x \vee y \vee n)) \vee (x \wedge y \wedge n) \vee n$
$= (t \wedge (x \vee y \vee n)) \vee n$
$= (t \vee n) \wedge (x \vee y \vee n)$ as $n$ is distributive
$= x \vee y \vee n$

Hence $< x >_n \vee < y >_n = < r >_n \in P_n(L).$

Therefore, $P_n(L)$ is a near lattice. □

In Fig-5, the elements of $P_n(L)$ are
$\{n\}, < c >_n = \{c, n\}, < a >_n = \{0, a, n, c\}, < 0 >_n = \{0, n\}$,
$< b >_n = \{0, b, n, d\}, < 1 >_n = \{n, c, d, 1\}$,
and $< d >_n = \{d, n\}$. The figure of $P_n(L)$ is shown in figure 6.




$P_n(L)$ **Figure 6:**

which is a near lattice (in fact, a semi Boolean lattice) but not a lattice.
We already know from [2] and [3] that if $n \in L$ is complemented in each interval containing it then $P_n(L)$ is always a lattice and in fact, then $P_n(L) = F_n(L)$.
An element $n$ is called a central element if

i) $n$ is neutral and
ii) It is complemented in each interval containing it.

Now we call an element $n \in L$ as nearly central element if

i) It is nearly neutral and
ii) It is complemented in each interval containing it.

In the following figure 7, $n$ is not central but it is nearly central.

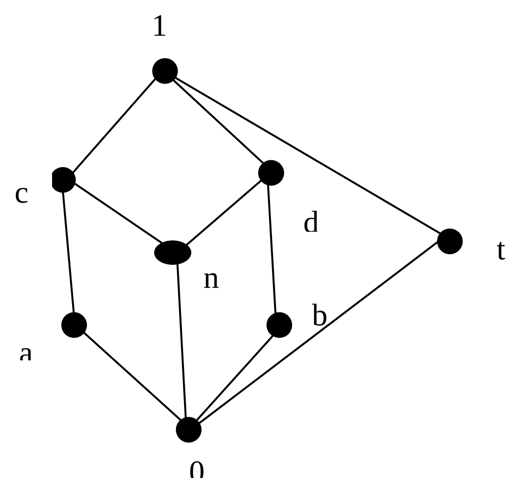



**Figure 7:**

Elements of $P_n(L)$ are

$\{n\}, <c>_n, <a>_n = \{0,a,n,c\}, <0>_n, <b>_n = \{0,b,n,d\},$

$<1>_n = \{n,c,d,1\}, L = <t>_n$.

The figure of $P_n(L)$ is given in Figure 8.

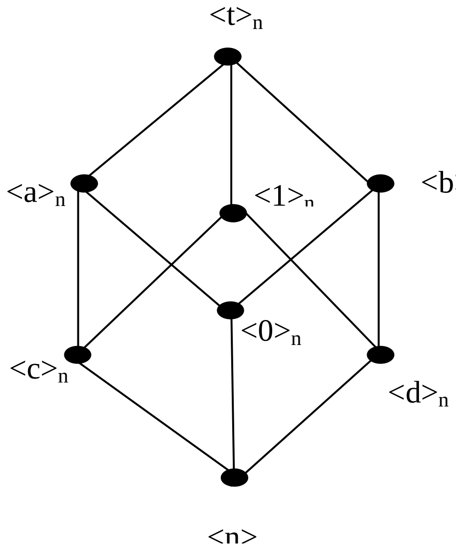



**Figure 8:**

We conclude the paper with the following theorem. This also gives a characterization of super standard element. To prove this we need the following result which is due to [2] and [3].

**Lemma 9.** Suppose in a lattice $L$, $n \in L$ is complemented in each interval containing it. Then $F_n(L) = P_n(L)$ and the map $\phi : P_n(L) \to (n]^d \times [n)$ defined by $\phi(<a>_n) = (a \wedge n, a \vee n)$ is an isomorphism. □

**Theorem 10.** Suppose in a lattice $L$, $n$ is complemented in each interval containing it. Then the following conditions are equivalent:

(i) $n$ is super standard

(ii) For all $a,b \in L$, $<a>_n = <b>_n$ implies $a = b$ is an isomorphism.

**Proof:** (i)=>(ii). Suppose (i) holds. Let $<a>_n = <b>_n$

$\Rightarrow [a \wedge n, a \vee n] = [b \wedge n, b \vee n]$

$\Rightarrow a \wedge n = b \wedge n, \quad a \vee n = b \vee n]$

$\Rightarrow a = b$ as $n$ is standard.

(ii)=>(i) By Lemma 9, $P_n(L)$ is a lattice. Now, $\phi : P_n(L) \to (n]^d \times [n)$ is an isomorphism, so $\phi$ is a meet homomorphism. Then $\phi(<a>_n \cap <b>_n) =$

$\phi(<a>_n) \wedge \phi(<b>_n)$. That is, $\phi(<m(a,n,b)>_n) = (a \wedge n, a \vee n) \vee (b \wedge n, b \vee n)$ That is $(n \wedge m(a,n,b), n \vee m(a,n,b)) = ((a \wedge n) \wedge_d (b \wedge n), (a \vee n) \wedge (b \vee n))$ That is $(n \wedge m(a,n,b), n \vee (a \wedge b) \vee (a \wedge n) \vee (b \wedge n)) = ((a \wedge n) \wedge_d (b \wedge n), (a \vee n) \wedge (b \vee n))$ That is. $(n \wedge m(a,n,b), (a \wedge b) \vee n) = ((a \wedge n) \wedge_d (b \wedge n), (a \vee n) \wedge (b \vee n))$. This implies $n \wedge m(a,n,b) = (a \wedge n) \wedge (b \wedge n)$ and $n \vee (a \wedge b) = (n \vee a) \wedge (n \vee b)$. Hence $n$ is distributive. Finally, let $a \wedge n = b \wedge n \text{ and } a \vee n = b \vee n.$ This implies $[a \wedge n, a \vee n] = [b \wedge n, b \vee n]$ That is $<a>_n = <b>_n$ and so by (ii) $a = b$. Therefore by [8, Theorem 3] $n$ is standard. Since we have already proved that $n \wedge m(a,n,b) = (a \wedge n) \vee (b \wedge n)$ so $n$ is super standard. □

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