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Progress in Nonlinear Dynamics and Chaos

Coding Theorems on New Fuzzy Information Theory of Order α and Type β

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Abstract. In this paper, we propose a new two parametric fuzzy average code-word length $L_\alpha^\beta(A)$ for a fuzzy set '*A*' and its relationship with fuzzy information measure $H_\alpha^\beta(A)$ is discussed. Also, the bounds of proposed average code-word length, in terms of fuzzy information measure, are obtained.

Keywords: Coding Theorem, Code-Word Length, Holder's Inequality, and Kraft's Inequality.

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1. Introduction

Due to lack of sharp distinction whether a particular item belongs to a set or not, a concept of imperfect information arises i.e., fuzziness. The concept of fuzzy sets introduced by Zadeh [10] uses imprecise knowledge to define an event. A fuzzy subset '*A*' of universe $X=\{x_1,x_2,...,x_n\}$ is defined as

$$A=\{(x_i,\mu_A(x_i)): x_i\in X,\ \mu_A(x_i)\in[0,1]\ \&\ i=1,2,...,n\}$$

where $\mu_A(x_i)$ is a membership function which gives the degree of belongingness of the element 'x_i' to the set '*A*'. In case, $\mu_A(x_i)=0$ or $\mu_A(x_i)=1$ for all 'x_i' then '*A*' is called a crisp set.

We take $X=\{x_1,x_2,...,x_n\}$ a discrete random variable with respective probabilities $p_1,p_2,...,p_n$, $p_i\geq 0$ and $\sum_{i=1}^{n}p_i=1$. Shannon [9] introduced the following measure of information and named it entropy. $H(P)=-\sum_{i=1}^{n}p_i\log_D p_i$.

Let us consider 'n' code words $c_1, c_2, ..., c_n$ of lengths $l_1, l_2, ..., l_n$ with probabilities $p_1, p_2, ..., p_n$ satisfying the Kraft [5] Inequality $\sum_{i=1}^{n} D^{-l_i} \leq 1$, where '$D$' is the size of code alphabet, then the lower bound of mean code word length i.e., $L = \sum_{i=1}^{n} p_i l_i$ lies between $H(P)$ and $H(P)+1$ as proved by Shannon [9] in his noiseless coding theorem.

The concept of fuzziness, because of its property to consider inaccurate and ambiguous information has vast applicability that covers fields of engineering, computer science, medicine, fuzzy aircraft control and so on. Different information measures along with the basic noiseless coding theorems have been given by several authors who include Kapur [3,4], Renyi [7], Ashiq et.al. [1,2], Baig et. al. [6] etc. The lower bounds for the average code-word length of a uniquely decipherable code have been obtained in terms of Shannon's [9] measure of entropy. Kapur [4] propounded the relation between the probabilistic entropy and coding. The probabilistic measures of entropy do not work in all situations; in such cases the idea of fuzziness can be considered.

2. Generalized fuzzy average code word length

Consider the measure proposed by Safeena [8] et.al. as

$$H_\alpha^\beta(A) = \frac{\beta}{1-\alpha} \log_D \left[\frac{1}{n} \sum_{i=1}^{n} \left\{ \mu_A^{\beta(1-\alpha)}(x_i) + (1-\mu_A(x_i))^{\beta(1-\alpha)} \right\} \right]; 0<\alpha<1, 0<\beta\leq 1 \,\&\, \beta>\alpha. \quad (1)$$

Corresponding to the above measure (1), we propose the following average code-word length

$$L_\alpha^\beta(A) = \frac{\alpha\beta}{1-\alpha} \log_D \left[\frac{1}{n} \sum_{i=1}^{n} \left\{ \mu_A^{\beta(1-\alpha)}(x_i) + (1-\mu_A(x_i))^{\beta(1-\alpha)} \right\} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} \right];$$
$$0<\alpha<1, 0<\beta\leq 1 \,\&\, \beta>\alpha. \quad (2)$$

In equation (2), 'D' is the size of code alphabet. Now, we obtain the bounds of (2) in terms of (1) under the following condition

$$\frac{1}{n} \sum_{i=1}^{n} \left\{ \mu_A^{\beta(1-\alpha)}(x_i) + (1-\mu_A(x_i))^{\beta(1-\alpha)} \right\} D^{-l_i} \leq 1. \quad (3)$$

Or we can write (3) as

$$\frac{1}{n} \sum_{i=1}^{n} \left\{ f\left(\mu_A(x_i), \mu_{A'}(x_i)\right) \right\} D^{-l_i} \leq 1. \quad (4)$$

where

$$f\left(\mu_A(x_i), \mu_{A'}(x_i)\right) = \mu_A^{\beta(1-\alpha)}(x_i) + (1-\mu_A(x_i))^{\beta(1-\alpha)}. \quad (5)$$

which is the generalized fuzzy Kraft's [5] inequality.

Theorem 2.1. For all integers $(D>1)$, the code-word lengths $l_1, l_2, ..., l_n$ satisfying the condition (4), then the code-word length (2) satisfies the inequality

$$H_\alpha^\beta(A) \le L_\alpha^\beta(A);\ 0<\alpha<1, 0<\beta\le 1 \,\&\, \beta>\alpha. \tag{6}$$

The equality holds in (6) if

$$l_i = -\log_D \left[\frac{1}{\frac{1}{n}\sum_{i=1}^{n} \{ f(\mu_A(x_i), \mu_{A'}(x_i))\}} \right]. \tag{7}$$

where

$$f(\mu_A(x_i), \mu_{A'}(x_i)) = \mu_A^{\beta(1-\alpha)}(x_i) + (1-\mu_A(x_i))^{\beta(1-\alpha)}.$$

Proof: By Holder's inequality, we have

$$\sum_{i=1}^{n} x_i\, y_i \ge \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}};\ \forall x_i, y_i\ \ 0; i=1,2,...,n \,\&\, \frac{1}{p}+\frac{1}{q}=1. \tag{8}$$

$$(p<1(\neq 0), q<0) \text{ or } (q<1(\neq 0), p<0).$$

The equality holds if and only if there exists a positive constant 'c' such that

$$x_i^p = c y_i^q.$$

Substituting, $x_i = \left[\frac{1}{n} f(\mu_A(x_i), \mu_{A'}(x_i))\right]^{\left(\frac{\alpha}{\alpha-1}\right)} D^{-l_i}$.

$$y_i = \left[\frac{1}{n} f(\mu_A(x_i), \mu_{A'}(x_i))\right]^{\left(\frac{1}{1-\alpha}\right)},\ p = \frac{\alpha-1}{\alpha} \,\&\, q = 1-\alpha.$$

Using these values in (8) and after simplification, we get

$$\sum_{i=1}^{n} \left[\frac{1}{n} \{ f(\mu_A(x_i), \mu_{A'}(x_i))\} D^{-l_i}\right] \ge$$

$$\left[\sum_{i=1}^{n} \left[\frac{1}{n} \{ f(\mu_A(x_i), \mu_{A'}(x_i))\} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]\right]^{\left(\frac{\alpha}{\alpha-1}\right)} \left[\sum_{i=1}^{n} \left[\frac{1}{n} \{ f(\mu_A(x_i), \mu_{A'}(x_i))\}\right]\right]^{\left(\frac{1}{1-\alpha}\right)}. \tag{9}$$

Using inequality (4), we get

$$\left[\sum_{i=1}^{n}\left[\frac{1}{n}\left\{ f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]\right]^{\left(\frac{\alpha}{\alpha-1}\right)}\left[\sum_{i=1}^{n}\left[\frac{1}{n}\left\{ f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}\right]\right]^{\left(\frac{1}{1-\alpha}\right)}\leq 1.$$

$$\Rightarrow\left[\sum_{i=1}^{n}\left[\frac{1}{n}\left\{ f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}\right]\right]^{\left(\frac{1}{1-\alpha}\right)}\leq\left[\sum_{i=1}^{n}\left[\frac{1}{n}\left\{ f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]\right]^{\left(\frac{\alpha}{1-\alpha}\right)}. \tag{10}$$

Applying logarithms with base '*D*' to the both sides of (10), we get

$$\frac{1}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{ f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}\right]\leq\frac{\alpha}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{ f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]. \tag{11}$$

As $0 < \beta \leq 1$, multiplying both sides of (11) by $\beta > 0$, we get

$$\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{ f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}\right]\leq\frac{\alpha\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{ f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]. \tag{12}$$

Using (5) in (12), we get

$$\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta(1-\alpha)}\right\}\right]\leq$$

$$\frac{\alpha\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\mu_A^{\beta(1-\alpha)}(x_i)+\left(1-\mu_A(x_i)\right)^{\beta(1-\alpha)}\right\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right].$$

We can write the above as

$$H_\alpha^\beta(A)\leq L_\alpha^\beta(A).$$

Hence, the result is established.

We have from equation (7),

$$l_i=-\log_D\left[\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\left\{ f\left(\mu_A(x_i),\mu_{A'}(x_i)\right)\right\}}\right].$$

Or $D^{-l_i} = \left[\dfrac{1}{\frac{1}{n}\sum_{i=1}^{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\}} \right].$ (13)

Raising both sides of (13) to the power $\left(\dfrac{\alpha-1}{\alpha}\right)$, we get

$$\text{Or } D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} = \left[\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\}} \right]^{\left(\frac{\alpha-1}{\alpha}\right)}.$$

$$\text{Or } D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} = \left[\frac{1}{n}\sum_{i=1}^{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} \right]^{\left(\frac{1-\alpha}{\alpha}\right)}. \quad (14)$$

Multiplying both sides of equation (14) by $\frac{1}{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\}$, and then summing over $i = 1, 2, ..., n$, we get

$$\sum_{i=1}^{n}\left[\frac{1}{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} \right] = \sum_{i=1}^{n}\left[\frac{1}{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} \right]^{\frac{1}{\alpha}}. \quad (15)$$

Applying logarithms to the base 'D' on both sides of (15), and then multiplying both sides by $\dfrac{\alpha\beta}{1-\alpha}$, we get

$$\frac{\alpha\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} \right] = \frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} \right].$$

We can write the above equation as

$$L_\alpha^\beta(A) = H_\alpha^\beta(A).$$

Hence, the result is established.

Theorem 2.2. For every code with lengths $l_1, l_2, ..., l_n$ satisfying the condition (4), $L_\alpha^\beta(A)$ can be made to satisfy the inequality

$$L_{\alpha}^{\beta}(A) < H_{\alpha}^{\beta}(A) + \beta;\ 0 < \alpha < 1, 0 < \beta \le 1 \,\&\, \beta > \alpha \ . \tag{16}$$

Proof: From thetheorem 2.1, we have

$$L_{\alpha}^{\beta}(A) = H_{\alpha}^{\beta}(A)\,. \tag{17}$$

(17) holds if and only if

$$D^{-l_i} = \left[\frac{1}{\frac{1}{n}\sum_{i=1}^{n} \{ f\,(\mu_A(x_i), \mu_{A'}(x_i))\}} \right];\ 0 < \alpha < 1, 0 < \beta \le 1 \,\&\, \beta > \alpha.$$

$$\Rightarrow l_i = \log_D \left[\frac{1}{n}\sum_{i=1}^{n} \{ f\,(\mu_A(x_i), \mu_{A'}(x_i))\} \right].$$

We chose the code-word lengths $l_i\ (i = 1, 2, ..., n)$ in such a way that they satisfy the inequality

$$\log_D \left[\frac{1}{n}\sum_{i=1}^{n} \{ f\,(\mu_A(x_i), \mu_{A'}(x_i))\} \right] \le l_i < \log_D \left[\frac{1}{n}\sum_{i=1}^{n} \{ f\,(\mu_A(x_i), \mu_{A'}(x_i))\} \right] + 1. \tag{18}$$

Considering the interval of length unity as

$$\delta_i = \left[\log_D \left[\frac{1}{n}\sum_{i=1}^{n} \{ f\,(\mu_A(x_i), \mu_{A'}(x_i))\} \right], \log_D \left[\frac{1}{n}\sum_{i=1}^{n} \{ f\,(\mu_A(x_i), \mu_{A'}(x_i))\} \right] + 1 \right].$$

In every δ_i, there lies one positive integer l_i such that

$$0 < \log_D \left[\frac{1}{n}\sum_{i=1}^{n} \{ f\,(\mu_A(x_i), \mu_{A'}(x_i))\} \right] \le l_i < \log_D \left[\frac{1}{n}\sum_{i=1}^{n} \{ f\,(\mu_A(x_i), \mu_{A'}(x_i))\} \right] + 1. \tag{19}$$

We will first show that the sequence $l_1, l_2, ..., l_n$ satisfies the generalized fuzzy Kraft [5] inequality (4). Now, taking L.H.S. of (19), we have

$$\log_D \left[\frac{1}{n}\sum_{i=1}^{n} \{ f\,(\mu_A(x_i), \mu_{A'}(x_i))\} \right] \le l_i\,.$$

$$\Rightarrow D^{-l_i} \le \left[\frac{1}{\frac{1}{n}\sum_{i=1}^{n} \{ f(\mu_A(x_i), \mu_{A'}(x_i))\}} \right]. \tag{20}$$

Multiplying both sides of (20) by $\frac{1}{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\}$ and then summing over $i=1,2,...,n$ on both sides, we get

$$\frac{1}{n}\sum_{i=1}^{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} D^{-l_i} \le 1.$$

which is the generalized fuzzy Kraft [5] inequality.

Now, taking R.H.S. of (19), we have

$$l_i < \log_D \left[\frac{1}{n}\sum_{i=1}^{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} \right] + 1.$$

$$\Rightarrow D^{l_i} < \left[\frac{1}{n}\sum_{i=1}^{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} \right] D. \tag{21}$$

Since, $0<\alpha<1$ thus $(1-\alpha)>0$ and $\left(\frac{1-\alpha}{\alpha}\right)>0$. Now, raising both sides of (21) to the power $\left(\frac{1-\alpha}{\alpha}\right)$, we have

$$D^{l_i\left(\frac{1-\alpha}{\alpha}\right)} < \left[\frac{1}{n}\sum_{i=1}^{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} D \right]^{\left(\frac{1-\alpha}{\alpha}\right)}.$$

$$\Rightarrow D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} < \left[\frac{1}{n}\sum_{i=1}^{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} \right]^{\left(\frac{1-\alpha}{\alpha}\right)} D^{\left(\frac{1-\alpha}{\alpha}\right)}. \tag{22}$$

Multiplying both sides of equation (22) by $\frac{1}{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\}$, and then summing over $i=1,2,...,n$ on both sides, we get

$$\sum_{i=1}^{n}\left[\frac{1}{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} \right] < \sum_{i=1}^{n}\left[\left\{ \frac{1}{n}\{ f(\mu_A(x_i), \mu_{A'}(x_i))\} \right\}^{\frac{1}{\alpha}} D^{\left(\frac{1-\alpha}{\alpha}\right)} \right]. \tag{23}$$

Applying logarithm to the base '*D*' on both sides of (23), we get

$$\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A'}(x_i))\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]<\frac{1}{\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A'}(x_i))\}\right]+\left(\frac{1-\alpha}{\alpha}\right). \tag{24}$$

Since $0<\alpha<1$ & $0<\beta\le 1$ thus $(1-\alpha)>0$ & $\left(\frac{\alpha\beta}{1-\alpha}\right)>0$ .Multiplying both sides of equation (24) by $\left(\frac{\alpha\beta}{1-\alpha}\right)$, we get

$$\frac{\alpha\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A'}(x_i))\}D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]<\frac{\beta}{1-\alpha}\log_D\left[\frac{1}{n}\sum_{i=1}^{n}\{f(\mu_A(x_i),\mu_{A'}(x_i))\}\right]+\beta.$$

or $L_\alpha^\beta(A)<H_\alpha^\beta(A)+\beta;\ 0<\alpha<1, 0<\beta\le 1\ \&\ \beta>\alpha$.
Hence, the result is established.

3. Conclusion

In this article, we propose a new generalized fuzzy code word length $L_\alpha^\beta(A)$ and develop the coding theorems corresponding to this code word length and also show that

$$H_\alpha^\beta(A)\le L_\alpha^\beta(A)<H_\alpha^\beta(A)+\beta;\quad 0<\alpha<1, 0<\beta\le 1\ \&\ \beta>\alpha .$$

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