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Cone Metric Spaces are not Generalized Metric Spaces: A Plenary Survey

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Abstract. We round up the concept of cone metric spaces, by giving major tries, to make general, the concept of metric spaces. Authors always reached the conclusion that they didn't really arrive at any clear generalizations. The last try was given in [5] where the paper set closed, the possibility for cone metrics to be real generalizations of metrics.

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1. Introduction

The concept of cone metric spaces goes back to 1980 where until that time, fixed point theory was systematically dealt with, essentially, through contractive conditions on maps. However, there are fixed points for maps which do not satisfy any contractive condition [1].The cause, therefore, was to find means, other than metrics, to check for fixed points for maps which do not satisfy regular contractive conditions. What should replace $[0,\infty)$ as a scale of distance, the question was. The answer was almost unanimous; Cones in real Banach spaces. We begin with the following background.

Definition 1.1. Let E be a real Banach space with norm $\|.\|$. A nonempty convex closed subset $P \subseteq E$ is called a cone if it satisfies:

i) $P \neq \{0\}$.

ii) $0 \leq a, b \in R$ *and* $x, y \in P$ *imply that* $ax + by \in P$.

iii) $x \in P$ *and* $-x \in P$ *imply that* $x = 0$.

The space E can be partially ordered by the cone $P \subseteq E$ as follows: $x \leq y$ if and only if $y - x \in P$. We write $x << y$ (x is way behind y) if $y - x \in P^{\circ}$, where P° denotes the interior of P.

Also, $x < y$ means that $x \leq y$ *but* $x \neq y$.

Definition 1.2. [8] A cone $P\ in\ (E,\|.\|)$ is called:

(N) **Normal**: if there exists a constant $k>0$ such that: $0\le x\le y$ implies that $\|x\|\le k\|y\|$.

The least positive integer k is called the normal constant of P. It is known that there are no cones with constant $k<1$, [9].

(R) **Regular**: if every increasing sequence which is bounded above is convergent. That is; if $\{x_n\}_{n\ge 1}$ is a sequence such that $x_1\le x_2\le ...\le y$ for some $y\in E$, then there is $x\in E$ such that $\lim_{n\to\infty}\|x_n-x\|=0$.

Equivalently; the cone P is regular if and only if every decreasing sequence which is bounded below is convergent.

(M) **Minihedral**: if $\sup\{x,y\}\ exists\ \ for\ all\, x,y\in E,$ and **strongly minihedral** if every subset of E which is bounded above has a supremum.

(S) **Solid** : if $P^\circ\ne\phi$..

In the following, we suppose that E is a real Banach space, P is a cone in E with nonempty interior and $\le$ is the partial ordering with respect to P.

Definition 1.3. [1] Let X be a nonempty set. Assume that the mapping $D:X\times X\to E\ satisfies$:

i) $0\le D(x,y)\ for\ all\ x,y\in X\ and\ D(x,y)=0\ if\ and\ only\ if\ x=y$.

ii) $D(x,y)=D(y,x)\ for\ all\ x,y\in E$.

iii) $\mathrm{D(x,y)}\le D(x,z)+D(z,y)\ for\ all\, x,y\ and\ z\in X$.

Then D is called a cone metric on X, and (X,D) is called a cone metric space.

We give the following examples:

Example1.4. Any metric space is a cone metric space with $P=[0,\infty)$.

Example1.5. [10] Let $q>0, E=l^q, and\ P=\{\{x_n\}_{n\ge 1}\in E: x_n\ge 0\ for\ all\, n.\}$

Let (X,ρ) be a metric space and $d:X\times X\to E$ be defined by:

$d(x,y)=\left(\left[\dfrac{\rho(x,y)}{2^n}\right]^{\frac{1}{q}}\right)_{n\ge 1}$. Then (X,d) is a cone metric space and the normal constant of P is equal to 1.

Now, from the top of One's head, replacing the interval [0,∞) with arbitrary cones, which are many, reveals obvious generality, but as we will see, the answer is not as (clear-cut) affirmative as One wishes. In fact, nobody has ever been able to explicitly define generality. In a sense, it should mean that we can completely describe fixed point occurrence in terms of cone metrics. Specifically, A map T has a fixed point if and only if it obeys some contractive condition subject to some cone metric D.

The following list of definitions is essential and makes up exact mimics of their classical synonyms.

Definition 1.6. Let (X, d) be a cone metric space and (x_n) be a sequence in X , then:

(x_n) is said to be convergent to x if for every $e >> 0$ there is $n_\circ \in N$ such that: $n \geq n_\circ \Rightarrow d(x_n, x) << e$. In this case, we write $x_n \to x$.

(x_n) is called a Cauchy sequence in X whenever for every $e >> 0$ there is $n_\circ \in N$ such that: $m, n \geq n_\circ \Rightarrow d(x_m, x_n) << e$.

(X, d) is called a complete cone metric space if every Cauchy sequence is convergent.

Pathologically, The theory of cone metric spaces was motivated by the observations reviewed by the following examples :

Example 1.7. Let $E = R^2$ *and let* $P = \{(x, y) \in R^2 : x, y \geq 0\}$ **.**

Then P is a normal cone in E .

Let $X = \{(x,0) \in R^2 : 0 \leq x \leq 1\} \cup \{(0, y) \in R^2 : 0 \leq y \leq 1\}$.

Consider the mapping $d : X \times X \to E$ defined by :

$$d((x,0),(y,0)) = (\frac{4}{3}|x-y| \,, |x-y| \,),$$

$$d((0,x),(y,0)) = (|x-y| \,, \frac{2}{3}|x-y| \,),$$

$$d((x,0),(0,y)) = d((0,y),(x,0)) = (\frac{4}{3}x+y \,, x+\frac{2}{3}y).$$

Then (X, d) is a complete cone metric space.

Take the mapping $T : X \to X$ defined as :

$$T((x,0)) = (0,x) \text{ and } T((0,x)) = (\frac{1}{2}x,0) \ .$$

Now, T satisfies the contractive condition:

$d(T((x_1,x_2)),T((y_1,y_2))) \leq kd((x_1,x_2),(y_1,y_2))$ For all $(x_1,x_2),(y_1,y_2) \in X$,

with constant $k = \frac{3}{4} \in [0,1)$.

It is obvious that T has the unique fixed point $(0,0)$ of X . But, on the other hand, we see that T is not a contractive mapping in the Euclidean metric on X .

Example 1.8. [13] Let $E = C^1_R[0,1]$ be the space of all first differentiable functions on the interval [0,1] being equipped with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$.

Let $P=\{x\in E: x(t)\geq 0 \ for\ all\ t\in[0,1]\}$. Then P is a cone which is not normal.

Now, consider, for each $n\in N, x_n(t)=\frac{t^n}{n} and\ y_n=\frac{1}{n}$.

Then, For all $n\in N, 0\leq x_n\leq y_n$ and $\lim_{n\to\infty} y_n=0$. But we have :

$$\|x_n\|=\max\left\{\frac{t^n}{n}: t\in[0,1]\right\}+\max\{t^{n-1}: t\in[0,1]\}.$$

So, $\|x_n\|=\frac{1}{n}+1\succ 1.$

Hence, the sequence (x_n) does not converge to zero. Thus, the Sandwich theorem fails here.

What happened in this form of partial order is that the cone is not normal. The components of the string of inequalities do not advance consistently with the norm-needs when invoked.

Having noted this, we, on the one hand, have a seemingly new partial order which may host contractive conditions that metric spaces don't. But, on the other hand, unless the cone is normal, we lost one of the most important tools in analysis, the Sandwich theorem [.].

So the question has always been: Do cone metrics really generalize metrics? Recently this question has been investigated by many authors and was answered in the negative in many occasions. This article is intended to present the most recent major tries along those lines.

It remains to mention that it was Bogdan Rzepecki who appeared to us in the literature to introduce the concept of cone and cone metric (as a generalized metric). That was in 1980 [14]. Then in 1987 an article entitled "a common fixed point theorem in abstract spaces" came out by Lin, S.-D. However, in 2007, when L.G. Huang, and X. Zhang, wrote their article " Cone metric spaces and fixed point theorems for contractive mappings", researchers started to consider them as the founders of the topic. Readers are urged to see this development in [1]. Throughout this article, unless otherwise specified, we assume that E is a Banach space, and P is the cone in E which induces the partial order.

It is necessary for us to mention, here, that the first author was among those who were trying hard to introduce means of generalization for the concept of metric in the theory of cone metric spaces . But, as the current article shows, This attempt is an impossibility.

We would also like to point out that it is every mathematician's interest to find means of, relatively, weak, or sometimes strong conditions, under which One can embed certain aspects. For deeper study, the reader may consider [15], [16], and [17].

2. Cone metric toplogy

It was noted that metrics and cone metrics play an interchangeable role in the generation of topology, the same topology. Thus, it is impossible for a topologist to distinguish between the two terms. This fact disables the search for generalization means through topological tests. Here are the details.

Theorem 2.1. [2] For every cone metric $D: X \times X \to E$ there is a metric $d: X \times X \to [0, \infty)$ which is sequentially equivalent to D . This means that D and d have exactly the same convergent sequences.

One way to do this is to take $d(x, y) = \inf\{\|u\| : D(x, y) \le u\}$.

Theorem 2.2. [4] Every cone metric space (X, D) is a topological space which is first countable.

As usual, for local bases, One can take for balls about a point p of X , $B(P, \frac{1}{n}c)$.

Theorems [2,3] imply :

Theorem 2.3. [3] Suppose that (X, D) is a cone metric apace and T_D is the topology induced on X *by* D. Suppose that d is the metric induced on X by the cone metric D, and let T_d stand for the topology on X generated by the metric d . Then $T_D = T_d$.

In conclusion for this section, there will be no scope of generalization along topological occur ounces.

3. Cone normed spaces and best approximation

In the normed linear space $(X, \|.\|)$. We define, for $x \in X, d(x, G) = \inf\{\|x - g\| : g \in G\}$. If G is a subspace of X , an element $g_\circ \in G$ is called a best approximant of $x \, in \, G$ *if* $\|x - g_\circ\| = d(x, G)$. We usually denote the set of all best approximants of $x \, in \, G \, as \, P(x, G)$. If for each $x \in X$, the set $P(x, G) \neq \phi$, then G is said to be proximinal in X , and if $P(x, G)$ is a singleton for each $x \in X$ then G is called a Chebyshev subspace of *X*. A classical reference for best approximation theory in normed spaces we always consider [5]. We also refer readers to [8,9] for recent developments in the theory. Best cone approximation theory is now being embedded in cone metric spaces and hence in cone normed spaces which we. It is advisable that readers see [6] for classification means of best approximation in cone normed spaces.

Definition 3.1. Let X be a real vector space and E be a real Banach space ordered by the strongly minihedral cone P . Then the ordered pair $(X, \|.\|_c)$ is called a cone normed space when $\|.\|_c : X \to E$ is a function such that:

i) $\|x\|_c \ge 0 \, and \, \|x\|_c = 0$ *if and only if* $x = 0$.

ii) $\|ax\|_c = |a| \|x\|_c$ *for all* $a \in R$ *and all* $x \in X$.

iii) $\|x + y\|_c \le \|x\|_c + \|y\|_c$ *for all* $x, y \in Y$.

For a subspace G *of* X *and* $x \in X$, we define the cone metric distance between x and G as: $d_c(x, G) = \inf\{\|x - g\|_c : g \in G\}$.

The techniques of the following result are true mimics of parts of a master's thesis written back in the year 2000 at An-Najah National University by Dwaik and supervised jointly by the first author of this manuscript and Deeb. The thesis was considering classical best approximation in normed spaces, and entitled as "The S-Property and Best Approximation".

Theorem 3.2. [3] Let $(X, \|.\|_c)$ be a cone normed space and G be a subspace of X. Then:

a) $d_c(x+g,G) = d_c(x,G)$ *for all* $x \in X$ *and* $g \in G$.

b) $d_c(x+y,G) \le d_c(x,G) + d_c(y,G)$ *for all* $x, y \in X$.

c) $d_c(\alpha x,G) = |\alpha| d_c(x,G)$ *for all* $\alpha \in R$ *and* $x \in X$.

d) $\|d_c(x,G) - d_c(y,G)\|_c \le \|x-y\|_c$.

We also have the following:

Theorem 3.3. [3] Let $(X, \|.\|_c)$ be a cone normed space and G be a subspace of X.

Then:

a) If $x \in G$ *then* $P_c(x,G) = \{x\}$.

b) If G is not closed then $P_c(x,G) = \phi$ *for all* $x \in X$.

c) *for all* $x \in X$, $P_c(x,G)$ is a convex set.

For the next result, we need to make the following definition.

Definition 3.4. [7] Let $(X, \|.\|_c)$ be a cone normed space. A subset A of X is said to be bounded in X if $\sup\{\|x-y\|_c : x, y \in A\}$ *exists in* E.

Theorem 3.5. [3] Let G be a subspace of a cone normed space $(X, \|.\|_c)$. Then :

(a) The set $P_c(x,G)$ *is bounded for all* $x \in X$.

(b) If G is closed then the set $P_c(x,G)$ is closed for all $x \in X$.

For a conclusion of this section, we adopt the following:

Definition 3.6. [3] Let $(X, \|.\|_c)$ be a cone normed space, $G \subseteq X$, *and* $x \in X$.

We say that x is orthogonal to G, written $x \perp G$ if

$$\|x\|_c \le \|x+\alpha g\| \text{ for all scalars } \alpha \text{ and } g \in G.$$

Theorem 3.7. [3] Let $(X, \|.\|_c)$ be a cone normed space, $G \subseteq X$, $x \in X \setminus G$ *and* $g \in G$.

Then: $g \in P_c(x,G)$ *if and only if* $(x-g) \perp G$.

It is therefore evident that, in terms of best approximation, cone metrics do not make any remarkable extension to cones.

4. A list of generalization thoughts

As we saw in example (1.7), L.G. Huang, X. Zhang constructed a map with a unique fixed point. That map did not obey any metric contraction conditions, but satisfied a cone metric contraction condition from which they drew their conclusion of the existence of the unique fixed point.

That was basically the motive to search for generalization means. But then researchers started to get negative results in this direction. Here are some tries.

Proposition 4.1. [4] We had already seen (Theorem 2.2) that cone metric spaces are first countable. Furthermore, sequentially compact sets are compact, just as in metric spaces.

Proposition 4.2. [2] Every cone metric space is metrizable (by Theorem 2.1) and the equivalent metric space satisfies the same contractive conditions as the cone metric. So most of the fixed point theorems which have been proved are straightforward results from the metric case.

Proposition 4.3. [11] Let $(E,\|.\|)$ be a real Banach space with a strongly minihedral normal cone P. Then there is a norm $[\|.\|]$ on E with respect to which P is a normal cone with normal constant 1.

Remark 4.4. [10] We find the following statement as the conclusion made in [10]:

Every theorem about Banach spaces is automatically true for the corresponding cone metric spaces, so it is redundant to prove results in cone metric spaces where the underlying space is a real Banach space.

Remark 4.5. The result of proposition (4.2) was first conjectured with proof in [10]. Authors of [11] proved the result in a totally different approach. This, now, makes Remark (4.4) stronger.

5. A concluding note

We remind with the following:

Definition 5.1. [14] Let (X,D) be a cone metric space. A function $f:X\to X$ is called a contraction if there is $0<c<1$ such that $D(f(x),f(y))\le cd(x,y)$ *for all* $x,y\in X$.

Remark 5.2. [14] In general, contractions are used in fixed point theory. One can easily show that if the function f is a contraction for the cone metric space (X,D) then it is also a contraction for the metric space (X,d), where d is the induced metric on X by the cone metric D as in (2.1).

Thus, after a sum of more than six hundred papers dealing with cone metric spaces have been published so far, the notion of cone metric spaces is not more general than that of a metric space [14].

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