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Weak Convergence of Filters

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Abstract. In this paper, we introduce what we called weak convergence of filters and show that, in Uryson spaces, weak limits are unique. Moreover, we show that, in a regular space X, with $E \subseteq X$, $x \in \overline{E}\, if\ and\ only\ if$ there is a filter $\Im$ on X which converges weakly to x and $F \cap E \neq \phi \forall F \in \Im$ .We also prove that closure continuous maps preserve weak convergence of filters. As a main result, we prove that, in regular spaces, weak convergence of filters is equivalent to convergence of filters.

Keywords: Weakly convergent filter

1. Introduction

The concept of weak convergence of sequences and nets was studied long time ago [1,2]. In this paper, we study weak convergence of filters and obtain some useful results. In particular, and among other results, we prove here, that in regular spaces, convergence and weak convergence of filters are equivalent.

A filter $\Im^/$ is said to be finer than the filter $\Im$ if

$for\ all\ F \in \Im, \exists F^/ \in \Im^/ such\, that\ F^/ \subseteq F$. This is written $\Im \subseteq \Im^/\ or,\ \Im^/ \geq \Im$.

Let U_x be the neighborhood system of x in a topological space X on which a filter $\Im$ is given. Of course, U_x is a filter on X. We say that the filter $\Im$ converges to x,

and we write $\Im \to x\ \ if\ \Im \geq U_x$.

If β is a filter base on X, then the family $\Im = \{F : F \supseteq B\ for\ some\, B \in \beta\}$ is the filter generated by β . If $\Im$ is a filter on a topological space X then, by $\overline{\Im}$ we mean the filter generated by the filter base $\beta = \{\overline{F} : F \in \Im\}$, where $\overline{F}$ stands for the closure of F. It is clear that $\overline{\Im} \leq \Im$.

A function $f : X \to Y$ is said to be closure continuous at $x_\circ \in X$ if for every neighborhood $V of\ f(x)$ *there is a neighborhood* U *of* $x_\circ$ *with* $f(\overline{U}) \subseteq \overline{V}$. If this condition is satisfied at each point of X then f is called closure continuous on X [1].
It is not hard to show that continuous functions are closure continuous, for if V is a neighborhood of $f(x)$ take a neighborhood U of x such $f(U) \subseteq V$, and so $\overline{f(U)} \subseteq \overline{V}$. Now, by continuity of f, $f(\overline{U}) \subseteq \overline{f(U)} \subseteq \overline{V}$.
The converse, however is untrue.
For example, take $X = \{a,b,c\}$, $\tau_1 = \{\phi, X, \{a\}, \{a,b\}\}$, $\tau_2 = \{\phi, X, \{b\}, \{b,c\}\}$.
Then take the identity function $i : (X, \tau_1) \to (X, \tau_2)$ which is discontinuous, because $i^{-1}(\{b,c\}) = \{b,c\}$ which is not open in τ_1. In the meantime i is closure continuous. To see this, for any $x \in X$ and any neighborhood $V\ of\ \ f(x)$, $\overline{V_{\tau_2}} = X$, and so, for any neighborhood $U\ of\ \ x$ we have: $f(\overline{U}) \subseteq X = \overline{V_{\tau_2}}$.

Finally, a space X is called a Uryson space if whenever $x_1 \neq x_2\ in\ X$, there are open sets U and V in X containing x_1 *and* x_2 *respectively, such that* $\overline{U} \cap \overline{V} = \phi$ [3].

We close this introduction by noting that topologists nowadays prefer to insert their applications in generalized, or enlarged settings. This might help escaping tight limits and specifications. For this we refer interested readers to compare with [6,7]. In specific One can consult [8] for the general setting of Rough Set Theory.

2. Weak convergence of filtres

Definition 2.1. If $\Im$ is a filter on a topological space X and $x \in X$, then $\Im$ is said to converge weakly to x (written $\Im \xrightarrow{w} x$) if $\Im$ *is finer than* $\overline{U_x}$. That is,
$\Im \geq \overline{U_x}$ *where* $\overline{U_x}$ *is the filter generated by the collection* $\{\overline{U} : U \in U_x\}$.

Remark 2.2. It is easy check to see that *If* $\Im \to x$ *then* $\Im \xrightarrow{w} x$ but not conversely.
The following example shows this.

Example 2.2. Let $X = \{a,b,c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$.
Consider the filter $\Im = \{X, \{a,c\}\}$.
Now, $\Im \xrightarrow{w} a$ since the neighborhood system $U_a = \{\{a\}, \{a,b\}, \{a,c\}, X\}$, and $\overline{U_a} = \{\{a,c\}, X\}$, from which it clear that $\Im \geq \overline{U_a}$.
However, $\Im$ does not converge to a since $\Im$ is not finer than U_a .

Theorem 2.3. Let $f : X \to Y$ be a function, and let $x \in X$. Then f is closure continuous at x if and only if, whenever $\Im$ is any filter in X with $\Im \xrightarrow{w} x$ in X, we have : $f(\Im) \xrightarrow{w} f(x)$ in Y.

Proof: Suppose f is closure continuous at x, and suppose $\Im \xrightarrow{w} x$.

Let $U_{f(x)}$ be the neighborhood system of $f(x)$.

We claim that $f(\Im) \geq \overline{U}_{f(x)}$.

For this, let $\overline{V} \in \overline{U}_{f(x)}$ be arbitrary.

Now, V *is a neighborhood of* $f(x)$, and since f is closure continuous at x, there is a neighborhood U *of* x *such that* $f(\overline{U}) \subseteq \overline{V}$. Thus, $f(\overline{U}) \in f(\Im)$. Hence, $f(\Im) \geq \overline{U}_{f(x)}$, which means that $f(\Im) \xrightarrow{w} f(x)$.

Conversely, suppose that whenever $\Im \xrightarrow{w} x$, *we have*: $f(x) \xrightarrow{w} f(x)$.

Let V *be a neighborhood of* $f(x)$. Since $\overline{U_x} \geq \overline{U_x}$, $\overline{U_x} \xrightarrow{w} x$. By hypothesis, we have: $f(\overline{U_x}) \xrightarrow{w} f(x)$. Thus $f(\overline{U}) \geq \overline{U}_{f(x)}$. Therefore, there is a neighborhood U *of* x *such that* $f(\overline{U}) \subseteq \overline{V}$, and hence, f is closure continuous at x.

Theorem 2.4. Let X be a Uryson's space, and $\Im$ be a filter on X.

If $\Im \xrightarrow{w} x$ *and* $\Im \xrightarrow{w} y$, *then* $x = y$.

Proof: Suppose that x *and* y are distinct elements in X. Since X is a Uryson's space, choose neighborhoods U *and* V *of* x *and* y *respectively with* $\overline{U} \cap \overline{V} = \phi$.

Now, since $\Im \xrightarrow{w} x$ we have: $\Im \geq \overline{U_x}$ and $\overline{U} \in \Im$.

Also, since $\Im \xrightarrow{w} y$ we have: $\Im \geq \overline{U_y}$ and $\overline{V} \in \Im$.

Thus, $\phi = \overline{U} \cap \overline{V} \in \Im$ which contradicts the fact that $\Im$ is a filter on X.

Therefore x *and* y could not have been distinct. We close this section with the following Remark.

Remark 2.5. If $\Im$ *and* $\Im'$ are filters on X with $\Im' \geq \Im$ and if $\Im \xrightarrow{w} x$, *then* $\Im' \xrightarrow{w} x$ as well.

Proof: Since $\Im \xrightarrow{w} x$, then $\Im \geq \overline{U_x}$. But since $\Im' \geq \Im$ we have that $\Im' \geq \overline{U_x}$.

Thus, $\Im' \xrightarrow{w} x$.

3. Main results

We begin with the following definition.

Definition 3.1. Let X be a topological space , and let $x \in X$ *and* $E \subseteq X$. Then x is called a weak-closure point of E if for all neighborhoods U *of* $x, \overline{U} \cap E \neq \phi$.

The set of all weak closure-points of E is denoted, here, by $\overline{E^w}$. It needs only a quick observation to see that the following Lemma is a true statement.

Lemma 3.2. Let E be a subset of a topological space X . Then $\overline{E} \subseteq \overline{E^w}$.
The converse of this lemma is false and here is an example.
Let $X = \{a,b,c,d\}$ with topology $\tau = \{\phi, X, \{a\}, \{a,b\}, \{c\}, \{a,c\}, \{a,b,c\}\}$.
Let $E = \{b\}$. Then $\overline{E} = \{b,d\}$, *but* $\overline{E^w} = \{a,b,d\}$.

Theorem 3.3. Let E be a subset of a topological space X and let $x \in X$. Then $x \in \overline{E^w}$ *if and only if there is a filter* $\Im$ *on* X such that $\Im \xrightarrow{w} x$ *and* $F \cap E \neq \phi$ *for all* $F \in \Im$.

Proof: Suppose $x \in \overline{E^w}$. So, for all neighborhoods U *of* $x, \overline{U} \cap E \neq \phi$.
Consider $\Im = \overline{U_x}$. It is clear that $\overline{U_x} \xrightarrow{w} x$. Moreover, we have:
for all $\overline{U} \in \overline{U_x}$, $\overline{U} \cap E \neq \phi$.
Conversely, let $\Im$ be a filter on X *with* $\Im \xrightarrow{w} x$ *and* $F \cap E \neq \phi$ *for all* $F \in \Im$.
We must show that $x \in \overline{E^w}$.
Let U be a neighborhood of x . Then $\Im \geq \overline{U_x}$ *because* $\Im \xrightarrow{w} x$.
But then by hypothesis, $\overline{U} \cap E \neq \phi$. Therefore $x \in \overline{E^w}$.
For our next result, we need to recall the following definition from [4].

Definition 3.4. A space X is said to be regular if $\overline{\Im} \to x$ *whenever* $\Im \to x$.

Lemma 3.5. Let X be a regular space, and let $\Im$ be a filter on X *and* $x \in X$.
If $\Im \xrightarrow{w} x$ *then* $\Im \to x$.

Proof: Suppose $\Im \xrightarrow{w} x$. Then, $\Im \geq \overline{U_x}$.
But $U_x \to x$ *and* X regular, so $\overline{U_x} \to x$ .Therefore, $\Im \to x$.
Combining Remark (2.5) and lemma (3.5), One gets the following :

Theorem 3.6. Let X be a regular space, and $\Im$ be a filter on X .
Then, $\Im \to x$ *if and only if* $\Im \xrightarrow{w} x$.
We close this paper with the following theorem.

Theorem 3.7. Let X be a regular space and $E \subseteq X$. Then $\overline{E^w} = \overline{E}$.

Proof: By lemma (3.2) $\overline{E} \subseteq \overline{E^w}$.

For the other inclusion, let $x \in \overline{E^w}$ be arbitrary. Then by theorem (3.3),

there is a filter $\Im$ *on* X such that $\Im \xrightarrow{w} x$ *and* $F \cap E \neq \phi$ *for all* $F \in \Im$.

But since X is regular, by lemma (3.5) $\Im \to x$ *and* $F \cap E \neq \phi$ *for all* $F \in \Im$.

Hence, $x \in \overline{E}$ [3, Theorem 12.6].

Thus, $\overline{E^w} \subseteq \overline{E}$. Thus equality is achieved.

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