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Explicit Formulation of Double Exponential Smoothing and its Consequences in the Memory of the Linear Homoscedastic Signals

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Abstract. Double exponential smoothing is an effective tool of denoising used in signals. In the present work an effort has been put up to structure the formulation of double exponential smoothing in the form of matrix equation from which some important observations can be made which gives an inner vista of the process. In addition to this an analytic study has been incorporated to identify whether there is any change in the memory of a discrete signal with linear homoscedastic memory after employing double exponential smoothing in it. It has been observed that the memory of a discrete signal governed by autoregressive method of order 1 becomes heterogeneous while order 2 autoregressive method probably becomes random on application of double exponential smoothing.

Keywords: Double exponential smoothing, Memory of a signal, Linear homoscedastic signal, Autoregressive method.

1. Introduction

The physics behind the business of signal processing is to explore the information about the physical source from which signal is being generated. The study of a signal not only explores the internal dynamics of the source but also can predict the future prospects of it. It provides the validation of theories and models as well as their improvements. Analysis of signals sometimes can give birth to a new theory or model. Now-a-days it is a burning deal in statistical signal processing to comprehend and explore the memory, self affinity, nonlinear and complex phenomena in signals. But in reality the problem is that any kind of signal either from an experiment or from a dynamical system, or from any economic, sociological or biological aspect, usually contains systemic or manual error. This error is coined as 'noise'. Noise can be treated as the unwanted part of a signal. Analysis of such signal in presence of noise often leads to a wrong interpretation of it. So we need to develop an initial platform by denoising the signal from which we can start the extensive study on it. Filtering/smoothing a signal is always an indispensable task to deal with in signal processing.

Double exponential smoothing [1, 2, 3, 4, 5, 6] is a useful tool to denoise a signal. Basically it is very much useful in those signals where somehow trend is very much present. Trend in a signal is a slow, gradual change in some property of the signal over the whole interval under investigation. Double exponential smoothing can remove the trend efficiently to give a good platform for further analysis from the Detrended signal.

The motto of our work is to represent the process of double exponential smoothing in a convenient matrix equation form from which we can explore different important properties involving the parameters. In addition to this the present work focuses on the application of double exponential smoothing on linear homoscedastic signals (kind of homogeneous discrete signal in which functional representation of memory remains the same throughout the signal) [7, 8, 9, 10, 11, 12]. As autoregressive model [13] is a very good example of linear homoscedasticity we have considered discrete signals governed by autoregressive method of order *1* and *2* only in the present study to understand the influence on the memory.

2. Theory

2.1. Matrix formulation of double exponential smoothing

The method of double exponential smoothing [1, 2, 3, 4, 5, 6] is governed by the following system of equations shown below

$$\left.\begin{aligned} x_1^{(p)} &= x_1 \;;\; b_1 = x_2 - x_1 \\ x_i^{(p)} &= \alpha x_i + (1-\alpha)(x_{i-1}^{(p)} + b_{i-1}) \\ \text{and}\quad b_i &= \beta\,(x_i^{(p)} - x_{i-1}^{(p)}) + (1-\beta) b_{i-1} \\ & \qquad (i = 2,3,4,\ldots\ldots,N) \end{aligned}\right\} \tag{1}$$

where $\{x_i\}_{i=1}^{N}$ is the observed discrete signal, $\{x_i^{(p)}\}_{i=1}^{N}$ is the smoothed signal and $\{b_i\}_{i=1}^{N}$ is the trend traced in the signal, α and β are the 'signal smoothing parameter' and 'trend smoothing parameter' respectively. We have $0 < \alpha < 1$ and $0 < \beta < 1$.

(1) can be conveniently written in the following form

$$\left.\begin{aligned} x_1^{(p)} &= x_1 \;;\; b_1 = x_2 - x_1 \\ x_i^{(p)} &= (1-\alpha) x_{i-1}^{(p)} + (1-\alpha) b_{i-1} + \alpha x_i \\ \text{and}\quad b_i &= -\alpha\beta x_{i-1}^{(p)} + (1-\alpha\beta) b_{i-1} + \alpha\beta x_i \end{aligned}\right\} \tag{1A}$$

$(i = 2,3,4,\ldots\ldots,N)$

(1A) can be suitably written in the following matrix recurrence form

$$X_i = AX_{i-1} + BY_i \tag{2}$$

where, $$X_i = \begin{pmatrix} x_i^{(p)} \\ b_i \end{pmatrix};\; A = \begin{pmatrix} 1-\alpha & 1-\alpha \\ -\alpha\beta & 1-\alpha\beta \end{pmatrix};\; B = \begin{pmatrix} \alpha & 0 \\ \alpha\beta & 0 \end{pmatrix} \text{ and } Y_i = \begin{pmatrix} x_i \\ 0 \end{pmatrix} \tag{3}$$

Next, we put

$$X_i - BY_i = Z_i \tag{4}$$

Using (4) in (2) we get

$$Z_i = AX_{i-1} \tag{5}$$

(5) can be elaborately expressed as below using (3) and (4)

$$\begin{pmatrix} x_i^{(p)} - \alpha x_i \\ b_i - \alpha\beta x_i \end{pmatrix} = \begin{pmatrix} 1-\alpha & 1-\alpha \\ -\alpha\beta & 1-\alpha\beta \end{pmatrix} \begin{pmatrix} x_{i-1}^{(p)} \\ b_{i-1} \end{pmatrix} \tag{6}$$

We can have the following interesting and important observations from the present analysis.

Observation 1: The matrix '*A*' can be expressed as a linear combination of the three 2X2 matrices$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$; $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ [with corresponding scalars as $(1-\alpha), (-\alpha\beta)$and 1] respectively which are singular as well as linearly independent of each other although the matrix A is non-singular as det A=$1-\alpha \neq 0$ (as $\alpha < 1$).

Observation 2: If we consider $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$; $M_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $M_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then we can have $M_1^2 = M_1$; $M_2^2 = M_2$ and $M_3^2 = M_3$ i.e. all these three matrices are idempotent matrices.

Observation 3: We can also observe $M_1M_2 = M_1$; $M_2M_1 = M_2$; $M_2M_3 = M_3$; $M_3M_2 = M_2$; $M_1M_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $M_3M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, for the present choice these three matrices appear to be non-commutative with respect to usual matrix multiplication. Moreover the present set of three singular and linearly independent matrices viz. M_1, M_2 and M_3 is not closed with respect to matrix multiplication.

2.2. Impact of double exponential smoothing on discrete signal

The present work focuses on the effect of double exponential smoothing on the memory of a linear homoscedastic discrete signal. To serve the present purpose we have taken into account discrete signals governed in particular by autoregressive methods of order *1* and *2*.

2.2.1. Case study on a discrete signal governed by autoregressive method of order *1*

In this section we investigate the effect of double exponential smoothing on the memory of a signal governed by autoregressive method of order *1*.

Let the observed discrete signal be $\{x_i\}_{i=1}^{N}$ and smoothed signal be$\{x_i^p\}_{i=1}^{N}$

As $\{x_i\}_{i=1}^{N}$ is governed by autoregressive of order *1* we have [13]

$$x_i = w_1 x_{i-1} \tag{7}$$

for $i=2, 3, ..., N$

where w_1 is the process parameter (ignoring the random shock).

Now by applying double exponential smoothing as in (1) in the present signal we have by using (7)

$$\left.\begin{aligned} b_1 &= (w_1-1)x_1 \\ x_2^{(p)} &= w_1 x_1 \\ b_2 &= (w_1-1)x_1 \end{aligned}\right\} \tag{8}$$

$$x_3^{(p)} = \alpha x_3 + (1-\alpha)(x_2^{(p)} + b_2) \tag{9}$$

Using (7) and (8) in (9) we get

$$x_3^{(p)} = \{\alpha w_1 + 2(1-\alpha)\}x_2 - (1-\alpha)x_1 \quad (10)$$

From (10) it is clear that $x_3^{(p)}$ shows a memory of length *2*.

Now from (1) we can have

$$b_3 = \beta(x_3^{(p)} - x_2^{(p)}) + (1-\beta)b_2 \quad (11)$$

Using (8), (9), (10) in (11) we get

$$b_3 = \beta(\alpha w_1 - 2\alpha + 2)x_2 + \{\beta(\alpha - 2w_1) + (w_1 - 1)\}x_1 \quad (12)$$

Again using (10) and (12) we can have from (1) the following

$$x_4^{(p)} = \alpha w_1 x_3 + (1-\alpha)(\alpha w_1 + 2 - 2\alpha + \alpha\beta w_1 - 2\alpha\beta + 2\beta)x_2 + (1-\alpha)(\alpha - 2 + \alpha\beta - 2\beta w_1 + w_1)x_1 \quad (13)$$

So $x_4^{(p)}$ shows a memory of length *3*.

Again using (10), (12) and (13) we can get from (1) the following

$$b_4 = \alpha\beta w_1 x_3 + (\alpha\beta w_1 - 2\alpha\beta + 2\beta - \alpha^2 w_1 - 2\alpha + 2\alpha^2 - \alpha^2\beta w_1 + 2\alpha^2\beta - 2\alpha\beta x2+1-\alpha(\alpha-1+\alpha\beta-2\beta w1+w1)x1 \quad (14)$$

Next using (13) and (14) we get

$$x_5^{(p)} = \alpha w_1 x_4 + (1-\alpha)(\alpha w_1 x_3) + (1-\alpha)(\alpha w_1 + 2 - 4\alpha + 2\alpha\beta w_1 - 6\alpha\beta + 4\beta+4\alpha2-2\alpha2\beta w1+4\alpha2\beta x2+1-\alpha2(2\alpha-3+2\alpha\beta-4\beta w1+2w1)x1 \quad (15)$$

So $x_5^{(p)}$ shows a memory of length *4*.

Altogether using the logic of induction it can be concluded that a discrete signal obeying AR(*1*) after smoothing turns out to be a signal with heterogeneous memory in which x_i^p will have memory length *(i-1); i=2,3,4…..N*.

2.2.2. Case study on a discrete signal governed by autoregressive method of order *2*

In this section we investigate the effect of double exponential smoothing on the memory of a signal governed by autoregressive method of order *2*.

As $\{x_i\}_{i=1}^N$ is governed by AR (2) we can have [13]

$$x_i = w_1 x_{i-1} + w_2 x_{i-2}$$

for *i=3,4,…..N* (16)

where w_1 and w_2 are process parameters (ignoring the random shock).

If possible let us assume that $\{x_i^{(p)}\}$ is also governed by AR (2).

So we can write

$$x_i^{(p)} = w_1' x_{i-1}^{(p)} + w_2' x_{i-2}^{(p)}$$

for *i=3,4,…..N* (17)

where w_1' and w_2' are the corresponding process parameters (ignoring the random shock).

Using (1) we have from (17) for *i=4*

$$\alpha x_4 = \alpha\{w_1' - (1-\alpha)(1+\beta)\}x_3 + \{(2w_1' - 3 + 2\alpha + 2\alpha\beta)(1-\alpha) + w_2'\}x_2 - (1-\alpha)\{w_1' + \alpha\beta + \alpha - 2\}x_1 \quad (18)$$

As x_4 is governed by AR (*2*) we must have

$$w_1' + \alpha\beta + \alpha - 2 = 0$$

i.e.

$$w_1' = 2 - \alpha\beta - \alpha \quad (19)$$

Using (1) we have from (17) for *i=5*

$$\alpha x_5 = \alpha\{w_1' - (1-\alpha)(1+\beta)\}x_4 + \alpha[(1-\alpha)\{(1+\beta)w_1' + \alpha + \alpha\beta^2 - 1 - 2\beta + 2\alpha\beta\} + w2']x3 + (1-\alpha)\{3-2\alpha-2\alpha\beta w1' + 2w2' + 5\alpha - 4 + 7\alpha\beta - 2\alpha2 - 4\alpha2\beta - 2\alpha2\beta2\}x2 + (1-\alpha)\{w_1'(\alpha\beta+\alpha-2) - w_2' - 4\alpha\beta - 3\alpha + 3 + 2\alpha^2\beta + \alpha^2 + \alpha^2\beta^2 + \alpha\beta^2\}x_1 \quad (20)$$

As by our assumption x_5 also obeys AR (*2*) coefficients of x_1 and x_2 both will be separately zero.

Hence we can have

$$\left.\begin{aligned} & w_1'(\alpha\beta+\alpha-2) - w_2' - 4\alpha\beta - 3\alpha + 3 + 2\alpha^2\beta + \alpha^2 + \alpha^2\beta^2 + \alpha\beta^2\} = 0 \\ & (3-2\alpha-2\alpha\beta)w_1' + 2w_2' + 5\alpha - 4 + 7\alpha\beta - 2\alpha^2 - 4\alpha^2\beta - 2\alpha^2\beta^2 = 0 \end{aligned}\right\} \quad (21)$$

Solving equation (21) we get

$$w_2' = \alpha - 1 + \alpha\beta^2 \quad (22)$$

Comparing (19), (21) and (22) we get

$$\alpha\beta^2 = 0 \quad (23)$$

So at least one of α and β is 0 which is not possible since α>0 as well as β>0.

So we arrive at a contradiction. Hence our previous assumption that $\{x_i^p\}$ is governed by AR (2) is wrong.

Next, if possible let us assume that $\{x_i^p\}$ is governed by AR (*1*).

Hence we can take $x_i^p = w_1' \, x_{i-1}^p$

for *i=2,3,4….N* (24)

where w_1' is the process parameter (ignoring the random shock).

Now using (1) in (24) for *i=4* we get

$$\alpha x_4 = \alpha\{w_1' - (1-\alpha)(1+\beta)\}x_3 + (1-\alpha)\{2w_1' - (3-2\alpha-2\alpha\beta)\}x_2 - (1-\alpha)\{w_1' + (\alpha\beta+\alpha-2)\}x_1 \quad (25)$$

As x_4 obeys AR (*2*) we must have

$$w_1' + (\alpha\beta+\alpha-2) = 0$$
$$\text{i.e. } w_1' = (2 - \alpha\beta - \alpha) \quad (26)$$

Again using (1) in (24) for *i=5* we get

$$\alpha x_5 = \alpha\{w_1' - (1-\alpha)(1+\beta)\}x_4 + \alpha(1-\alpha)\{(1+\beta)w_1' - (1-\alpha+2\beta-2\alpha\beta-\alpha\beta2\}x3 + 1-\alpha\{w1'3-2\alpha\beta-2\alpha-(4-5\alpha-7\alpha\beta+2\alpha2+4\alpha2\beta+2\alpha2\beta2\}x2 + 1-\alpha\{w1'\alpha\beta+\alpha-2-(4\alpha\beta+3\alpha-3-\alpha2-2\alpha2\beta-\alpha2\beta2-\alpha\beta2)\}x1$$
(27)

As by our assumption x_5 obeys AR (*2*) coefficients of x_2 and x_1 must be separately zero. So we get

$$w_1'(3 - 2\alpha\beta - 2\alpha) = 4 - 5\alpha - 7\alpha\beta + 2\alpha^2 + 4\alpha^2\beta + 2\alpha^2\beta^2 \quad (28)$$

Solving (26) and (28) we get *α=1* which gives rise to a contradiction as *0<α<1*. Hence our assumption that $\{x_i^p\}$ is governed by AR (*1*) is wrong.

Next if possible let us assume that $\{x_i^p\}$ is governed by AR (*3*).

So we can take

$$x_4^p = w_1' x_3^p + w_2' x_2^p + w_3' x_1^p \quad (29)$$

where w_1', w_2' and w_3' are the corresponding process parameters (ignoring the random shock).

Using (1) in (29) we get for *i=4*

$$\alpha x_4 = \alpha\{w_1' - (1-\alpha)(1+\beta)\}x_3 + 2\{(1-\alpha)w_1' + w_2 - (1-\alpha)(3-2\alpha-2\alpha\beta)\}x_2 - \{w_3' - (1-\alpha)w_1' - (1-\alpha)(\alpha\beta+\alpha-2)\}x_1 \quad (30)$$

As x_4 is governed by AR (*2*) we must have

$$w_3' - (1-\alpha)w_1' = (1-\alpha)(\alpha\beta+\alpha-2) \quad (31)$$

Using (1) in (29) we get for *i=5*

$$\alpha x_5 = \alpha\{w_1' - (1-\alpha)(1+\beta)\}x_4 + \alpha\{(1-\alpha)(1+\beta)w_1' + w_2' - (1-\alpha)(1-\alpha + 2\beta - 2\alpha\beta - \alpha\beta2\}x3 + \{1-\alpha3-2\alpha\beta-2\alpha w1' + 2(1-\alpha)w2' + w3' - (1-\alpha)(4-5\alpha-7\alpha\beta+2\alpha2 + 4\alpha2\beta + 2\alpha2\beta2\}x2 + 1-\alpha\{w1'\alpha\beta+\alpha-2-w2' - (4\alpha\beta+3\alpha-3-\alpha2-2\alpha2\beta-\alpha2\beta2-\alpha\beta2)\} x_1 \quad (32)$$

As x_5 is governed by AR (*2*) we must have the coefficients of both x_1 and x_2 in (32) as zero. This gives

$$w_1'(\alpha\beta+\alpha-2) - w_2' = 4\alpha\beta + 3\alpha - 3 - \alpha^2 - 2\alpha^2\beta - \alpha^2\beta^2 - \alpha\beta^2 \quad (33)$$

and

$$(1-\alpha)(3-2\alpha\beta-2\alpha)w_1' + 2(1-\alpha)w_2' + w_3' = (1-\alpha)(4-5\alpha-7\alpha\beta+2\alpha^2+4\alpha^2\beta+2\alpha^2\beta^2) \quad (34)$$

By applying Cramer's rule in (31), (33) and (34) we arrive at an inconsistency. Hence our assumption that $\{x_i^p\}$ is governed by AR (*3*) is wrong.

From the above analysis we can conclude that if a discrete signal be originally governed by AR(*2*) after the application of double exponential smoothing it possibly becomes random since it denies to exhibit any trace of memory with order *1*,*2* or *3* and as the original memory length is *2* we cannot expect to have a distant memory length like *4* or more after smoothing.

3. Discussion

The present work furnishes a matrix equation model homologous to the original form of double exponential smoothing. This analysis fetches some important observation from the viewpoint of matrix algebra giving a good insight of the smoothing process. Moreover the present study shows that on applying double exponential smoothing over a signal governed by AR(*1*) we can eventually find a sequential heterogeneous memory while AR(*2*) possibly transforms to random behaviour. This result seems to be interesting in view of the earlier conclusion made in [12] that a discrete signal obeying AR(*2*) after the application of simple exponential smoothing turns into a signal governed by AR(*3*). This phenomenon is not observed in the present application of double exponential smoothing. In future the present study can be extended to have general exclusive solution of the matrix formulation. Moreover study can be made to identify the possible change in memory of a discrete signal with higher linear homogeneous memory length (*3* or more than that) after the application of double exponential smoothing. Future study can also be performed to understand the influence of double exponential smoothing in discrete signals having nonlinear homoscedastic memory.

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