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Approximate Solutions of Second Order Strongly and High Order Nonlinear Duffing Equation with Slowly Varying Coefficients in Presence of Small Damping

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Abstract. Based on the He's homotopy perturbation (HP) and the extended form of the Krylov-Bogoliubov- Mitropolskii (KBM) method, an approximate technique has been developed for obtaining the approximate solutions of second order strongly and high order nonlinear Duffing equation with slowly varying coefficients in presence of small damping. The first approximate solutions obtained by the presented method show a good agreement with the corresponding numerical solutions for the several damping effects. The implementation and efficiency of the presented method has been explained by an example.

Keywords: HP and KBM methods, Duffing equation with slowly varying coefficients, high order nonlinearity and example.

AMS Mathematics Subject Classification (2010): 34E05

1. Introduction

The study of nonlinear differential systems is of great importance not only in all areas of physics but also in engineering and other disciplines, since most of the phenomena in our world are nonlinear and are described by nonlinear differential equations. Most of these nonlinear differential systems occur in nature with slowly varying coefficients in presence of small damping. The common perturbation methods for constructing the approximate analytical solutions to the nonlinear differential equations are the Krylov-Bogoliubov- Mitropolskii (KBM) [1-3] method, the Lindstedt-Poincare (LP) method [4-5] and the method of multiple time scales [4]. Almost all perturbation methods are based on an assumption that small parameter must exist in the equations. Lim *et al.* [7] have presented a new analytical approach to the Duffing- harmonic oscillator. He [8] has investigated the homotopy perturbation technique. In another paper, He [9] has developed a coupling method of a homotopy perturbation technique and a perturbation technique for strongly nonlinear problems. Recently, He [10] has also presented a new interpretation of homotopy perturbation method for solving strongly nonlinear differential systems. Belendez *et al.* [11] have presented the application of He's homotopy perturbation method to the Duffing harmonic oscillators. Hu [12] has obtained the solution of a quadratic nonlinear oscillator by the method of harmonic balance. Roy *et al.* [13] have presented the effects of higher approximation of Krylov-Bogoliubov-Mitropolskii solution and matched asymptotic differential systems with slowly varying coefficients and damping near to a turning point for weakly nonlinear differential systems. Arya and

Bojadziev [14] have presented the analytical technique for time depended oscillating systems with slowly varying parameters, damping, and delay Alam *et al.* [16] have developed the general Struble technique for weakly nonlinear systems with large damping. Uddin *et al.* [17] have presented an approximate technique for solving strongly nonlinear differential systems with cubic nonlinearity in presence of damping effects. Uddin and Sattar [18] have presented an approximate technique to Duffing` equation with small damping and slowly varying coefficients for cubic nonlinearity. Uddin and Sattar [19] have developed an approximate technique for solving strongly nonlinear biological systems with small damping effects. Uddin *et al.* [20] have presented an approximate analytical technique for solving a certain type of fourth order strongly nonlinear oscillatory differential systems with cubic nonlinearity in presence of small damping. Recently Ghosh *et al.* [21] have presented an approximate technique for solving second order strongly and high order nonlinear differential systems in presence of small l damping without slowly varying coefficients. The authors [8-12] have studied the differential systems with cubic nonlinearity in absence of damping effects. But most of the physical and oscillating systems encounter in presence of small damping in nature and it plays an important role to the nonlinear differential systems. In this article, we have developed an approximate analytical technique for solving second order strongly and high order nonlinear differential systems with slowly varying coefficients in presence of small damping based on the He's homotopy perturbation [8-11] and the extended form of the KBM [1-3] methods. Figures are provided to compare between the solutions obtained by the presented method with the corresponding numerical (considered to be exact) solutions obtained by fourth order Runge-Kutta method.

2. The method

We are interested to consider a strongly and high order nonlinear differential system [13] with slowly varying coefficients in presence of small damping in the following form

$$\ddot{x} + 2k(\tau)\dot{x} + e^{-\tau}x = -\varepsilon_1 f(x), \quad k << 1, \tag{1}$$

subject to the initial conditions

$$x(0) = b_0, \ \dot{x}(0) = 0, \tag{2}$$

where $k > 0$ and $2k$ is the linear damping coefficient which varies slowly with time t, $\tau = \varepsilon t$ is the slowly varying time, ε is a small parameter, ε_1 is parameter not necessarily small, b_0 is positive constant and known as initial amplitude of the systems and $f(x)$ is a given high order nonlinear function which satisfies the following condition

$$f(-x) = -f(x). \tag{3}$$

For simplicity, we are going to use the following transformation

$$x = y(t)e^{-kt}. \tag{4}$$

Differentiating Eq. (4) twice with respect to time t and substituting $\ddot{x}, \dot{x}$ together with x into Eq. (1) and then simplifying them, we obtain the following equation

$$\ddot{y} + (e^{-\tau} - k^2)y = -\varepsilon_1 e^{kt} f(y e^{-kt}). \tag{5}$$

According to the homotopy perturbation method [8-11, 17-21], Eq. (5) can be re-written as the following form

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon_1 e^{kt} f(y e^{-kt}), \tag{6}$$

where

$$\omega^2 = e^{-\tau} - k^2 + \lambda. \tag{7}$$

Herein λ is an unknown constant which can be determined by eliminating the secular terms. However, for a damped nonlinear differential system ω is a time dependent function and it varies slowly with time *t*. To handle this situation, we need to use the extended form of the KBM [1-3] method. According to this technique, we are going to choose the first approximate analytical solution of Eq. (6) in the following form

$$y = b\cos\psi, \tag{8}$$

where b and ψ represent the amplitude and phase variable respectively and they vary slowly with time t. According to the KBM [1-2] method b and ψ satisfy the following first order differential equations

$$\begin{aligned} \dot{b} &= \varepsilon A_1(b,\tau) + \varepsilon^2 A_2(b,\tau) + \cdots, \\ \dot{\psi} &= \omega(\tau) + \varepsilon B_1(b,\tau) + \varepsilon^2 B_2(b,\tau) + \cdots, \end{aligned} \tag{9 a, b}$$

where ε is a small positive parameter and A_j and B_j are unknown functions. Now differentiating Eq. (8) twice with respect to time *t*, utilizing the relations Eq. (9) and substituting $\ddot{y}$ and y into Eq. (6) and then equating the coefficients of $\sin\psi$ and $\cos\psi,$ we obtain the value of the unknown functions A_1 and B_1 as the form

$$A_1 = -\omega' b/(2\omega),\ B_1 = 0, \tag{10}$$

where prime denotes differentiation with respect to slowly varying time τ. Now putting Eq. (8) into Eq. (4) and Eq. (10) into Eq. (9) we obtain the following equations

$$x = b\, e^{-k\,t}\cos\psi, \tag{11}$$

and

$$\begin{aligned} \dot{b} &= -\varepsilon\,\omega' b/(2\omega), \\ \dot{\psi} &= \omega(\tau). \end{aligned} \tag{12 a, b}$$

Thus, the first approximate solution of Eq (1) is performed by Eq. (11) with the assist of Eqs. (7) and (12). Usually, the integration of Eq. (12 a, b) is performed by well-known techniques of calculus [4-5], but sometimes they are solved by a numerical procedure [12-21].

3. Example

For implementing and justifying the above procedure, we are going to assume the following Duffing equation with slowly varying coefficients in presence of small damping as the form

$$\ddot{x} + 2k(\tau)\dot{x} + e^{-\tau}x = -\varepsilon_1 x^5, \tag{13}$$

where $f(x) = x^5$. Now using the transformation Eq. (4) into Eq. (13) and then simplifying them, we obtain

$$\ddot{y} + (e^{-\tau} - k^2)y = -\varepsilon_1 e^{-4k\,t} y^5. \tag{14}$$

According to the homotopy perturbation [8-11, 17-20] method, Eq. (14) can be re-written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon_1 e^{-4k\,t} y^5, \tag{15}$$

where

$$\omega^2 = e^{-\tau} - k^2 + \lambda. \tag{16}$$

According to the extended form of the KBM [1-3] method, the solution of Eq. (15) is given by Eq. (8). In presence of secular terms, the solution will be non-uniform and break down. So, researchers must be needed to remove the secular terms from their obtained solutions for finding the uniform solutions. For avoiding the secular terms in particular solution of Eq. (15), setting the coefficients of the $\cos\psi$ terms is zero, we obtain

$$\lambda b - \frac{5\varepsilon_1 b^4 e^{-4kt}}{8} = 0, \tag{17}$$

For the nontrivial solution $i.e., b \neq 0,$ Eq. (17) leads to

$$\lambda = \frac{5\varepsilon_1 b^3 e^{-4kt}}{8}. \tag{18}$$

Substituting the value of λ from Eq. (18) into Eq. (16), it yields

$$\omega^2 = e^{-\tau} - k^2 + \frac{5\varepsilon_1 b^3 e^{-4kt}}{8}. \tag{19}$$

This is a time dependent frequency equation of the given nonlinear differential systems. As $t \to 0$, Eq. (19) reduces to

$$\omega_0 = \omega(0) = \sqrt{1 - k^2 + \frac{5\varepsilon_1 b_0^3}{8}}, \tag{20}$$

which is known as the constant frequency equation of the given nonlinear differential systems. Now integrating the Eq. (12 a), we get

$$b = b_0\sqrt{\frac{\omega_0}{\omega}}, \tag{21}$$

where b_0 is a constant of integration which represents the initial amplitude of the nonlinear differential systems. Now putting Eq. (21) into Eq. (19), we obtain the following bi-quadratic equation in ω

$$\omega^4 - q\omega^2 - r = 0, \tag{22}$$

where

$$q = e^{-\tau} - k^2, \quad r = \frac{5\varepsilon_1 b_0^4 \omega_0^2 e^{-4kt}}{4}. \tag{23}$$

Solving Eq. (22) for the real angular frequency ω, we obtain

$$\omega = \sqrt{\frac{q + \sqrt{q^2 + 4r}}{2}}, \tag{24}$$

The solution of Eq. (12 b) becomes

$$\psi = \psi_0 + \int_0^t \omega(t)dt, \tag{25}$$

where ψ_0 is the initial phase and ω is given by Eqs. (24). Therefore, the first approximate solution of Eq. (13) is obtained by Eq. (11) and the amplitude b and the phase ψ are calculated from Eq. (21) and Eq. (25) respectively with the help of Eq. (23) and Eq.(24). Thus, the determination of the first order analytical approximate solution of Eq. (13) is completed by the presented approximate analytical technique.

4. Initial conditions

The initial conditions of $\ddot{x} + 2k(\tau)\dot{x} + e^{-\tau}x = -\varepsilon_1 x^5$ are obtained as

$$x(0) = b_0 \cos\psi_0,$$
$$\dot{x}(0) = \left(\frac{b_0(2\varepsilon + 5\varepsilon_1 k b_0^4)}{4\varepsilon\,\omega_0(4\omega_0^2 + 2)} - k b_0 \right)\cos\psi_0 - b_0\,\omega_0 \sin\psi_0. \tag{26}$$

In general, the initial conditions $[x(0), \dot{x}(0)]$ are specified. Then one has to solve nonlinear algebraic equation in order to determine the initial amplitude b_0 and the initial phase ψ_0 that appear in the solutions, from the initial conditions equation (26).

5. Results and discussion

In this article, He's homotopy perturbation technique has been extended for solving second order strongly and high order nonlinear differential systems with slowly varying coefficients in presence of small damping based on the extended form of the KBM [1-3] method. From our results, it is seen that the first order approximate analytical solutions show a good agreement with the corresponding numerical solutions for the several damping effects. The approximate analytical solutions of Eq. (13) is computed by Eq. (11) with slowly varying coefficients in presence of small damping and high order nonlinearity and the corresponding numerical solutions are obtained by using the fourth order Runge-Kutta method. The presented method is very simple in its principle, and is very easy to implement for strongly $(\varepsilon_1 = 1.0)$ and high order nonlinear differential systems with slowly varying coefficients in presence of small damping as well as weakly $(\varepsilon_1 = 0.1)$. The variational equations of the amplitude and phase variables appear in a set of first order nonlinear ordinary differential equations. The integrations of these variational equations are obtained by well-known techniques of calculus [4-5]. In a lack of analytical solutions, they are solved by numerical procedure [4, 12-21]. The amplitude and phase variables change slowly with time t. The behavior of amplitude and phase variables characterizes the oscillating processes. Moreover, the variational equations of amplitude and phase variables are used to investigate the stability of the nonlinear differential equations. Ji-Huan He [8, 9] has developed homotopy perturbation for conservative nonlinear differential systems. But the presented method is valid for non-conservative nonlinear differential systems. The presented method can also overcome some limitations of the classical perturbation techniques; it does not require a small parameter $(i.e,\ \varepsilon_1 = 1.0)$ in the equations. The advantage of the presented method is that the first order approximate analytical solutions show a good agreement with the corresponding numerical solutions for small amplitude and damping. The method has been successfully implemented for solving the second order slowly varying differential systems with high order nonlinearity in presence of small damping for both strongly and weakly nonlinear cases. Comparisons are made between the solutions obtained by the presented coupling analytical technique and those obtained by the numerical (considered to be exact) solutions in figures.

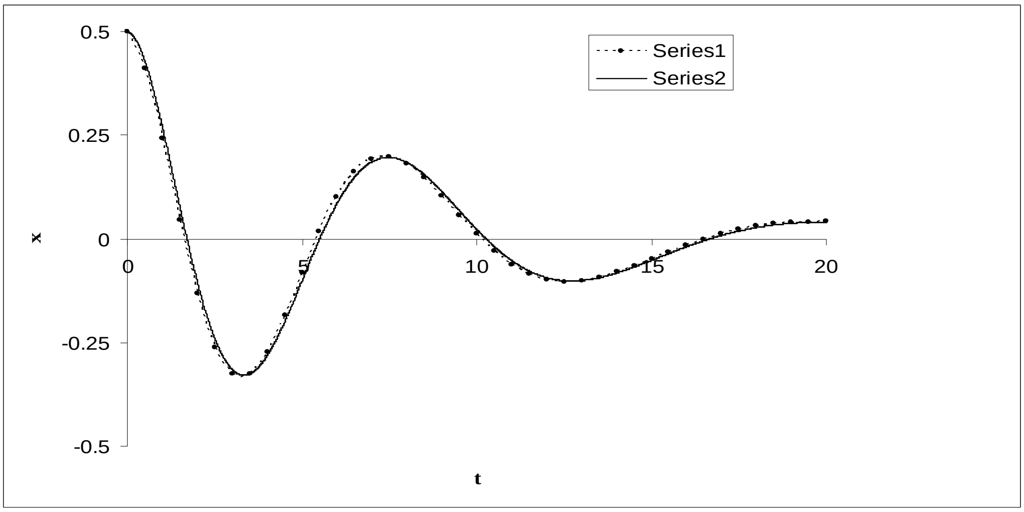

Figure 1: (a) First approximate solution of Eq. (13) is denoted by $-\bullet-$ (dotted lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \psi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.02339]$ with $k = 0.15,\ \varepsilon = 0.1,\ \varepsilon_1 = 1.0$ and $f = x^5$. Corresponding numerical solution is denoted by - (solid line).

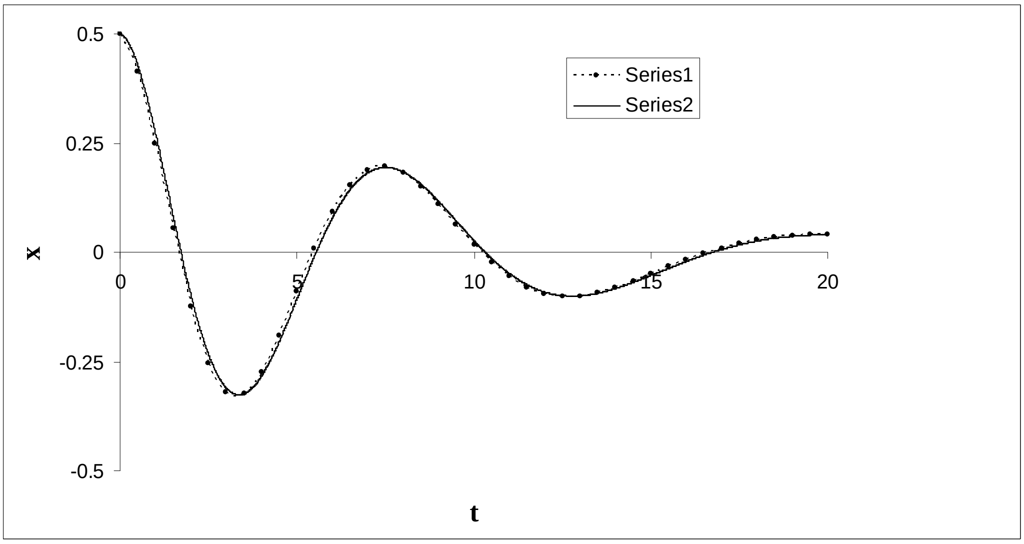

Figure 1: (b) First approximate solution of Eq. (13) is denoted by $-\bullet-$ (dotted lines) by the presented analytical technique with the initial conditions $b_0 = 0.5, \psi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.03131]$ with $k = 0.15,\ \varepsilon = 0.1,\ \varepsilon_1 = 0.1$ and $f = x^5$. Corresponding numerical solution is denoted by - (solid line).

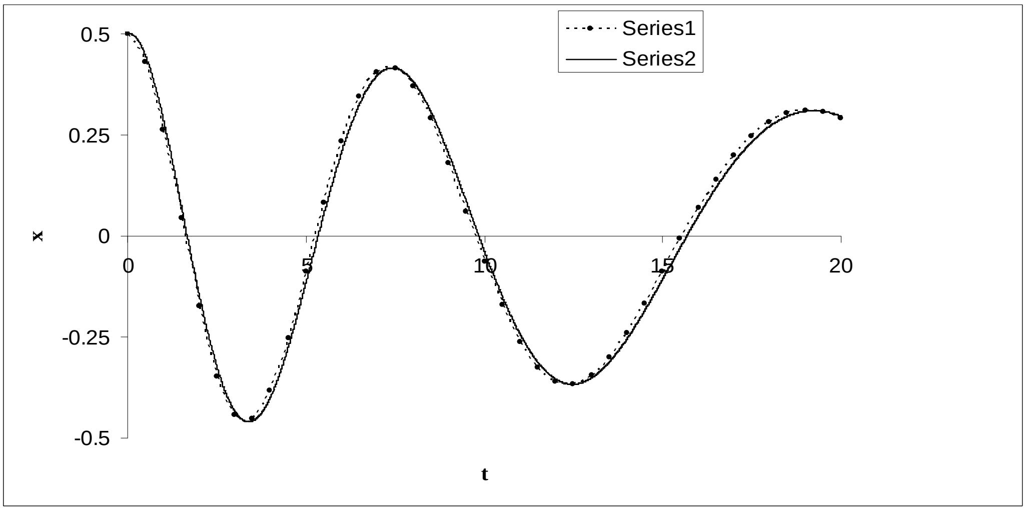

Figure 2: (a) First approximate solution of Eq. (13) is denoted by $-\bullet-$ (dotted lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \psi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = 0.01904]$ with $k = 0.05,\ \varepsilon = 0.1,\ \varepsilon_1 = 1.0$ and $f = x^5$. Corresponding numerical solution is denoted by - (solid line).

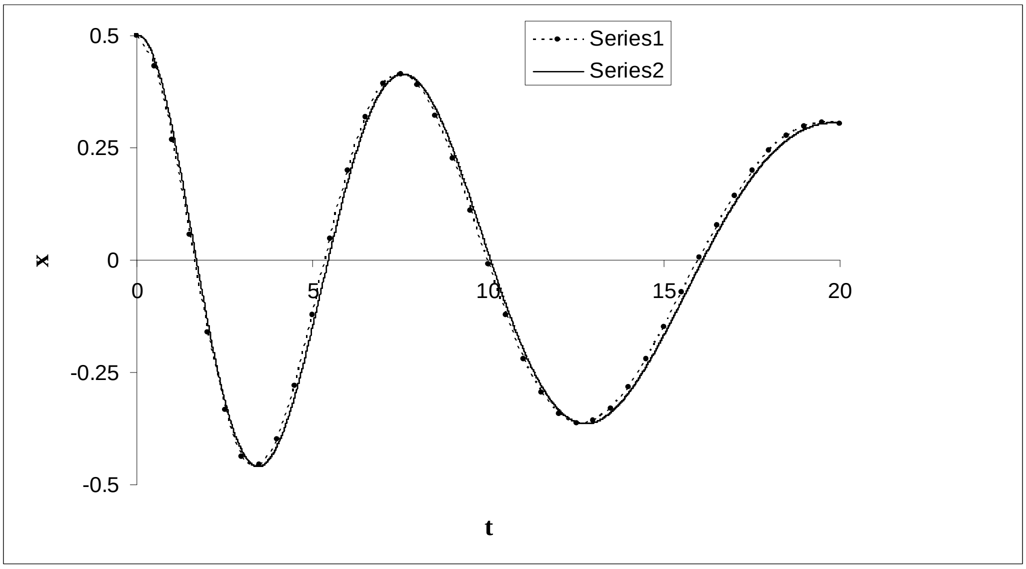

Figure 2: (b) First approximate solution of Eq. (13) is denoted by $-\bullet-$ (dotted lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \psi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = 0.01702]$ with $k = 0.05,\ \varepsilon = 0.1,\ \varepsilon_1 = 0.1$ and $f = x^5$. Corresponding numerical solution is denoted by - (solid line).

6. Conclusion

The great achievement of this article is that the presented approximate analytical technique is suitable for solving the second order nonlinear differential systems with slowly varying coefficients and high order nonlinearity in presence of small damping for strongly $(\varepsilon_1 = 1.0)$ as well as weakly $(\varepsilon_1 = 0.1)$ nonlinear cases but the classical perturbation and He's homotopy methods are not capable to handle for these situations.

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