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Some Results Concerning Stability Problem in Fredholm Theory and Applications

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Abstract. In this paper, we give some results concerning stability in the Fredholm and Browder operators, via the concept of measures of weak noncompactness. Moreover, we investigate perturbation of left (right) Fredholm and Browder operators by polynomially weak compact operators. Finally, we establish the invariance, under weakly compact perturbations, of various essential spectra of a closed densely defined operators. These results are exploited to describe the essential spectra of a multidimensional neutron transport operator.

Keywords: Measures of weak noncompactness in Banach spaces, Fredholm operators, Essential spectra, Transport theory.

1. Introduction

Let X and Y be two complex Banach spaces. We denote by $\mathcal{C}(X,Y)$ (resp., $\mathcal{L}(X,Y)$) the space of all closed densely defined linear operators acting from X into Y (resp., the space of all bounded linear operators acting from X into Y). The closed subspace of all compact (resp., weakly compact) operators of $\mathcal{L}(X,Y)$ is designed by $\mathcal{K}(X,Y)$ (resp., $\mathcal{W}(X,Y)$). For $T\in\mathcal{C}(X,Y)$, we write $\mathcal{N}(T)\subseteq X$ for the null space, $\mathcal{R}(T)\subseteq Y$ for the range of T and $\mathcal{D}(T)$ for the domain of T. We denote $\mathcal{N}^{\infty}(T)=\bigcup_n \mathcal{N}(T^n)$ and $\mathcal{R}^{\infty}(T)=\bigcap_n \mathcal{R}(T^n)$. We set $\alpha(T):=dim\mathcal{N}(T)$ and $\beta(T):=codim\mathcal{R}(T)=dimY/\mathcal{R}(T)$. The sets of upper semi-Fredholm operators and lower semi-Fredholm operators are respectively defined by:

$$\Phi_+(X,Y)=\{T\in\mathcal{C}(X,Y)\ such that\ \alpha(T)<\infty and\ \mathcal{R}(T)\ closed\ in\ Y\},$$

$$\Phi_-(X,Y)=\{T\in\mathcal{C}(X,Y)\ such that\ \beta(T)<\infty (then\ \mathcal{R}(T)\ closed\ in\ Y)\}.$$

$\Phi(X,Y):=\Phi_+(X,Y)\cap\Phi_-(X,Y)$ is the set of Fredholm operators in $\mathcal{C}(X,Y)$. If $X=Y$, the sets $\mathcal{L}(X,X)$, $\mathcal{C}(X,X)$, $\mathcal{K}(X,X)$, $\mathcal{W}(X,X)$, $\Phi_+(X,X)$, $\Phi_-(X,X)$ and $\Phi(X,X)$ are replaced respectively by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\mathcal{W}(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi(X)$. If $T\in\Phi_+(X,Y)\cup\Phi_-(X,Y)$, the number $i(T):=\alpha(T)-\beta(T)$ is called the index of T. $\rho(T)$ is the resolvent set of T. Recall that, for $T\in\mathcal{C}(X)$, $X_T:=\mathcal{D}(T)$ (the domain of T) endowed with the graph norm

$\|.\|_T$ (i.e., $\|x\|_T = \|x\| + \|Tx\|$) is a Banach space and we have $T \in \mathcal{L}(X_T, X)$. We denote $\hat{T}$ the restriction of T to $\mathcal{D}(T)$. Let J be a linear operator on X. If $X_T \subset \mathcal{D}(J)$, then J will be called T-defined. If J is T-defined, we will denote be $\hat{J}$ its restriction to X_T. Moreover, if $\hat{J} \in \mathcal{L}(X_T, X)$, we say that J is T-bounded.

Notice that if $T \in \mathcal{C}(X)$ and J a T-bounded, then we get the obvious relations:

$$\begin{cases} \alpha(\hat{T}) = \alpha(T), \beta(\hat{T}) = \beta(T), \mathcal{R}(\hat{T}) = \mathcal{R}(T), \\ \alpha(\hat{T}+\hat{J}) = \alpha(T+J), \beta(\hat{T}+\hat{J}) = \beta(T+J), \mathcal{R}(\hat{T}+\hat{J}) = \mathcal{R}(T+J). \end{cases} \tag{1}$$

Hence, $T \in \Phi(X)$ (resp., $\Phi_+(X)$) if and only if $\hat{T} \in \Phi(X_T, X)$ (resp., $\Phi_+(X_T, X)$).

Definition 1.1. Let X and Y be two Banach spaces. 1. Let $T \in \mathcal{L}(X,Y)$. 0.5 cm (i) T is said to have a left Fredholm inverse if there exists $T_l \in \mathcal{L}(Y,X)$ and $K \in \mathcal{K}(X)$ such that $T_lT = I_X - K$. The operator T_l is called left Fredholm inverse of T. 0.5 cm (ii) T is said to have a right Fredholm inverse if there exists $T_r \in \mathcal{L}(Y,X)$ such that $I_Y - TT_r \in \mathcal{K}(Y)$. The operator T_r is called right Fredholm inverse of T. (iii) T is said to have a Fredholm inverse if there exists a map which is both a left and a right Fredholm inverse of T. 2. Let $T \in \mathcal{C}(X)$. T is said to have a left Fredholm inverse (resp., right Fredholm inverse, Fredholm inverse) if $\hat{T}$ has a left Fredholm inverse (resp., right Fredholm inverse, Fredholm inverse). ♦

The sets of left and right Fredholm inverses are respectively defined by:

$$\Phi_l(X) := \{T \in \mathcal{C}(X) such that T has a left Fredholm inverse\}$$

$$\Phi_r(X) := \{T \in \mathcal{C}(X) such that T has a right Fredholm inverse\}.$$

It should notice, by the classical theory of Fredholm operators (see for example [10]), that $\Phi(X) = \Phi_r(X) \cap \Phi_l(X)$.

In this paper, we are concerned, for $T \in \mathcal{C}(X)$, with the following essential spectra:

$\sigma_W(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi(X)\}$,

$\sigma_S(T) := \mathbb{C} \setminus \{\lambda - T \in \Phi(X) \text{ such that } i(\lambda - T) = 0\} = \mathbb{C} \setminus \rho_S(T)$,

$\sigma_B(T) := \mathbb{C} \setminus \{\lambda \in \rho_S(T) \text{ such that all scalars near } \lambda \text{ are in } \rho(T)\} = \mathbb{C} \setminus \rho_B(T)$,

$\sigma_{le}(T) := \{\lambda \in \mathbb{C}; \lambda - T \notin \Phi_l(X)\}$, $\rho_{le}(T) := \mathbb{C} \setminus \sigma_{le}(T)$,

$\sigma_{re}(T) := \{\lambda \in \mathbb{C}; \lambda - T \notin \Phi_r(X)\}$, $\rho_{re}(T) := \mathbb{C} \setminus \sigma_{re}(T)$. The subsets $\sigma_W(.)$ is the Wolf essential spectrum [8, 17], $\sigma_S(.)$ is the Schechter essential spectrum [14], $\sigma_B(.)$

denotes the Browder essential spectrum [8] and $\sigma_{le}(.)$ (resp., $\sigma_{re}(.)$) is the left (resp., right) essential spectrum [10]. Note that, in general, we have $\sigma_{le}(T)\cup\sigma_{re}(T)=\sigma_W(T)\subset\sigma_S(T)\subset\sigma_B(T)$. Let $T\in\mathcal{L}(X)$. Recall that $a(T)$ (resp. $d(T)$), the ascent (resp. the descent) of T, is the smallest non-negative integer n such that $\mathcal{N}(T^n)=\mathcal{N}(T^{n+1})$ (resp. $\mathcal{R}(T^n)=\mathcal{R}(T^{n+1})$). If no such n exists, then $a(T)=+\infty$ (resp. $d(T)=+\infty$). The set of left (right) Browder operators are defined respectively by:

$$\mathcal{B}_l(X)=\{T\in\mathcal{L}(X)\ suchthat\ T\in\Phi_l(X)\ and\ a(T)<\infty\},$$
$$\mathcal{B}_r(X)=\{T\in\mathcal{L}(X)\ suchthat\ T\in\Phi_r(X)\ and\ d(T)<\infty\}.$$

The set of Browder operators on X is $\mathcal{B}(X)=\mathcal{B}_l(X)\cap\mathcal{B}_r(X)$.

Definition 1.2. A Banach space X is said to have the Dunford-Pettis property (for short property DP) if for each Banach space Y every weakly compact operator $T:X\to Y$ takes weakly compact sets in X into norm compact sets of Y. ♦

It is well known that any L_1-space has the property DP [5]. For further examples we refer to [4].

Definition 1.3. Let X and Y be two complex Banach spaces. An operator $T:X\to Y$ is said to be a Dunford-Pettis operator (for short property DP operator) if T maps weakly compact sets onto compact sets. ♦

An important question is to characterize, for given $T\in\Phi(X)$, the class of $S\in\mathcal{C}(X)$, such that $T+S$ still belongs to $\Phi(X)$. A lot of work, devoted to this subject, has been done. We refer, for examples, to [1, 2, 3, 8, 9] and the references therein. The purpose of this work is to pursue the analysis started in [1, 3, 9]. More precisely, in section 2, by the use of the concept of measures weak of noncompactness, we study the stability problem in Fredholm and semi-Fredholm operators sets. The section 3 is devoted to perturbations of left (right) Fredholm and Browder operators by polynomially weak compact operators. Moreover, we apply the obtained results in section 2 to discuss their incidence on the behavior of essential spectra of operators belonging to $\mathcal{C}(X)$, where X has the Dunford-Pettis property. Finally, we give more precise description of the essential spectra of multidimensional neutron transport equation on L_1-spaces.

2. Perturbation results by means measures of weak noncompactness

The purpose of this section is to establish some results concerning stability in the class of Fredholm operators via the concept of measures of weak noncompactness. First, we will adopt the following definitions:

Definition 2.1. Let X and Y be two Banach spaces, $\overline{B}_X$ the closure of the unit ball of X and let μ be a measure of weak noncompactness in Y. We define the function

$$\psi_\mu:\mathcal{L}(X,Y)\to[0,+\infty[\qquad T\to\psi_\mu(T)=\mu(T(\overline{B}_X)).$$

ψ_μ is called a measure of weak noncompactness of operators associated to μ. ♦

In what follows, consider X a Banach space, $\mathcal{M}_X$ be the family of all nonempty and bounded subsets of X, $T \in \mathcal{L}(X)$, T^* is its adjoint, μ (resp., μ^*) a measure of weak noncompactness in X (resp., in X^*) and Ψ_μ (resp., Ψ_{μ^*}) a measure of weak noncompactness of operators associated to μ (resp., to μ^*). We will make the following assumption: (H) : $\mu(T(A)) \le \Psi_\mu(T)\mu(A)$, for every $A \in \mathcal{M}_X$. We begin with the following preparating result which is crucial for our purposes.

Theorem 2.1. Let X be a complex Banach space. Suppose that, for $T \in \mathcal{L}(X)$,

$$\begin{cases} \bullet\ (H)\ \textit{holds true}, \\ \bullet\ \lim\limits_{n\to+\infty} \left(\psi_\mu(T^n)\right)^{\frac{1}{n}} = 0, \\ \bullet\ \textit{There exists } m \in \mathbb{N}^* \textit{such that} T^m \textit{ is DP operator}. \end{cases}$$

Then

$$I - T \in \Phi(X) \text{ and } i(I-T) = 0. \qquad ♦$$

Proof. Since $\lim_{n\to+\infty}\left(\psi_\mu(T^n)\right)^{\frac{1}{n}} = 0$, then there exists $n_0 \in \mathbb{N}^*$ such that for all $n \ge n_0, \psi_\mu(T^n) < 1$. On the other hand, for $n \in \mathbb{N}^*, I - T^n = R(T)(I-T), where R(T) := I + T + ... + T^{n-1}$. By [13, Lemma 4.3], it suffice to prove that, for any $K \in \mathcal{K}(X)$, $\alpha(I-T-K) < \infty$. To do so, it suffice to establish that the set $A := \mathcal{N}(I-T-K) \cap \overline{B}_X$ is compact. Consider $x \in A$, then $R(T)(I-T)(x) = R(T)K(x)$. Hence, for $n \ge n_0$, $x = T^n(x) + R(T)K(x)$. Obviously $A \subset T^n(A) + R(T)K(A)$. Applying $\mu(.)$ and taking account the hypothesis (H_1), we infer that

$$\mu(A) \le \mu(T^n(A)) \le \Psi_\mu(T^n)\mu(A).$$

Since $\Psi_\mu(T^n) < 1$, then $\mu(A) = 0$ and therefore A is relatively weakly compact including in $T^n(A) - R(T)K(A)$. We treat two cases:

Case 1: If $n_0 \ge m$, then T^{n_0} is DP operator. Hence, $T^{n_0}(A)$ is compact.

Case 2: If $n_0 < m$, then $\psi_\mu(T^m) < 1$. Since T^m is DP operator, then $T^m(A)$ is compact. In both cases, we infer that, for $n \ge n_0$, $T^n(A)$ is compact. Hence A is compact and therefore $I - T \in \Phi_+(X)$. Next, note that for $t \in [0,1]$, we have $(\psi_\mu(tT))^{n_0} < 1$ and $(tT)^m$ is DP operator. Then, from the above, $(I - tT) \in \Phi_+(X)$. Now, by the continuity of the index on $\Phi_+(X)$, we get $i(I-T) = i(I-tT) = i(I) = 0$.

Hence, $I-T\in\Phi(X)$, which completes the proof.
The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.1. Let X be a complex Banach space and let $T\in\mathcal{L}(X)$ such that T^m is weakly compact for some $m\in\mathbb{N}^*$. (i) If T^m is DP operator, then $I-T\in\Phi(X)$ and $i(I-T)=0$. (ii) If X has the property of DP, then $I-T\in\Phi(X)$ and $i(I-T)=0$. ♦

In the rest of this section, consider $T\in\mathcal{C}(X)$ and S be T-bounded operator. The main result of this section is the following:

Theorem 2.2. Suppose that Ψ_μ satisfies the hypothesis (H) and assume that, for T_r a right Fredholm inverse of T,

$$\begin{cases} \bullet\ \textit{There exists } m\in\mathbb{N}^* \textit{ such that } (\hat{S}T_r)^m \textit{ is DP operatorin } \mathcal{L}(X) \\ \bullet\ \lim\limits_{n\to+\infty}\left(\psi_\mu(\hat{S}T_r)^n\right)^{\frac{1}{n}}=0. \end{cases}$$

Then the following statements hold. (i) $T+S\in\Phi_r(X)$ and $i(T+S)=i(T)$. (ii) If $\alpha(T)<\infty$, then $T+S\in\Phi(X)$ and $i(T+S)=i(T)$. ♦

Proof. (i) Keeping in mind Definition 1.1 and applying Theorem 2.1, we infer that $(\hat{T}+\hat{S}T_r\in\Phi(X)$ and $i((\hat{T}+\hat{S})T_r)=0$. Hence, $\hat{T}+\hat{S}\in\Phi_r(X_T,X)$. On the other hand, $i((\hat{T}+\hat{S})T_r)=i(T_r)+i(\hat{T}+\hat{S})=0$. Thus, $i(\hat{T}+\hat{S})=-i(T_r)=i(\hat{T})$. Finally the result follows from (1). (ii) We have $\hat{T}T_r=I_X-K\in\Phi(X)$. Since $\alpha(T)<\infty$, then the use of [15, Theorem 2.7 p. 171] leads to $T_r\in\Phi(X,X_T)$. Moreover, the fact that $(\hat{T}+\hat{S})T_r\in\Phi(X)$, then, by [15, Theorem 2.5 p.169], $\hat{T}+\hat{S}\in\Phi(X_T,X)$. Finally the result follows from (1).

In order to give a similar results to Theorem 2.2, consider μ_T a measure of weak noncompactness in X_T and Ψ_{μ_T} the measure of weak noncompactness of operators associated to μ_T. Now, arguing as in the proof of Theorem 2.2, we can prove the following:

Theorem 2.3. Suppose that Ψ_{μ_T} satisfies the hypotheses (H) and assume that, for T_l a left Fredholm inverse of T,

$$\begin{cases} \bullet\ \textit{There exists } m\in\mathbb{N}^* \textit{ such that } (T_l\hat{S})^m \textit{ is DP operatorin } \mathcal{L}(X_T) \\ \bullet\ \lim\limits_{n\to+\infty}\left(\psi_{\mu_T}(T_l\hat{S})^n\right)^{\frac{1}{n}}=0. \end{cases}$$

Then the following statements hold. (i) $T+S\in\Phi_l(X)$ and $i(T+S)=i(T)$. (ii) If

$\beta(T) < \infty$, then $T+S \in \Phi(X)$ and $i(T+S) = i(T)$. ♦

As an immediate consequence of theorems 2.2 and 2.3, we have:

Corollary 2.2. Assume that X has the property DP. Then (i) If there exists a right Fredholm inverse, T_r, of T such that $(\hat{S}T_r)^m$ is weakly compact in $\mathcal{L}(X)$ for some $m \in \mathbb{N}^*$, then $T+S \in \Phi_r(X)$ and $i(T+S) = i(T)$. Suppose moreover $\alpha(T) < \infty$, then $T+S \in \Phi(X)$. (ii) If there exists a left Fredholm inverse, T_l, of T such that $(T_l\hat{S})^m$ is weakly compact in $\mathcal{L}(X_T)$ for some $m \in \mathbb{N}^*$, then $T+S \in \Phi_l(X)$ and $i(T+S) = i(T)$. Suppose moreover $\beta(T) < \infty$, then $T+S \in \Phi(X)$.

3. Applications

3.1. Perturbation by polynomially weak compact operators

In this section, X designs a Banach space. Consider $\mathcal{P}(\mathcal{W}(X)) = \{S \in \mathcal{L}(X); \exists P \in \mathbb{C}[X] \setminus \{0\}, P(S) \in \mathcal{W}(X)\}$. The purpose of this subsection is to prove an important result about perturbation by polynomially weak compact operators. Recall that for $S \in \mathcal{P}(\mathcal{W}(X))$, there exists a unique nonzero complex polynomial m_S with leading coefficient 1 and of the minimal degree such that $m_S \in \mathcal{W}(X)$. In this text, m_S will be said the minimal polynomial of S.

Definition 3.1. Let $S \in \mathcal{P}(\mathcal{W}(X))$, m_S be the minimal polynomial of S and let $T \in \mathcal{L}(X)$. We say that T is in communication with S if there exists a continuous map $\varphi : [0,1] \to \mathbb{C}$ for which $\varphi(0) = 0$, $\varphi(1) = 1$ and

$$\textit{for all } \lambda \textit{ zero of } m_S, \textit{ for all } t \in [0,1], \varphi(t)\lambda \notin \sigma_W(T). \tag{2}$$

If (2) holds for $\sigma_{le}(T)$ ($\sigma_{re}(T)$) instead of $\sigma_W(T)$, then we shall say that T is in left (right) communication with S. ♦

The main result of this section is the following:

Theorem 3.1. Let $T, S \in \mathcal{L}(X)$ such that the commutator $[S,T]$ is weakly compact and DP operator. Suppose that $S \in \mathcal{P}(\mathcal{W}(X))$ and $m_S(S)$ is DP operator. (i) If, $\forall \lambda \in \sigma_{le}(T)$, $m_S(\lambda) \neq 0$, then $T - S \in \Phi_l(X)$. If moreover T is in left communication with S, then $T \in \Phi_l(X)$ and $i(T-S) = i(T)$. (ii) If, $\forall \lambda \in \sigma_{re}(T)$, $m_S(\lambda) \neq 0$, then $T - S \in \Phi_r(X)$. If moreover T is in right communication with S, then $T \in \Phi_r(X)$ and $i(T-S) = i(T)$. (iii) If, $\forall \lambda \in \sigma_W(T)$, $m_S(\lambda) \neq 0$, then $T - S \in \Phi(X)$. If moreover T is in communication with S, then $T \in \Phi(X)$ and $i(T-S) = i(T)$. ♦

Proof. (i) Since $m_S(\lambda) \neq 0$ for all $\lambda \in \sigma_{le}(T)$, then we can write

$m_S(T)=\prod_{i=1}^{N}(T-\lambda_i)$, where, $\forall i\in\{1,...,N\}, \lambda_i\notin\sigma_{le}(T)$. Thus, $m_S(T)\in\Phi_l(X)$. On the other hand $m_S(S)$ is weakly compact and DP operator, then, by Theorem 2.3, $m_S(T)-m_S(S)\in\Phi_l(X)$. Since the commutator $[S,T]$ is weakly compact and DP operator, we can write

$$m_S(T)-m_S(S)=(T-S)L+W_1=L(T-S)+W_2,$$

with $L\in\mathcal{L}(X)$ and W_1,W_2 are two weakly compact and DP operators. Therefore, by Theorem 2.3, $T-S\in\Phi_l(X)$. Now, consider $Q_t(z)=\prod_{i=1}^{N}(z-\lambda_i\varphi(t))$, then $Q_t(\varphi(t)S)=(\varphi(t))^N m_S(S)$. Thus, $Q_t(\varphi(t)S)$ is weakly compact and DP operator. Moreover, for all $\lambda\in\sigma_{le}(T), Q_t(\lambda)\neq 0$. This yields

$$T-\varphi(t)S\in\Phi_l(X),\ forall\ t\in[0,1]. \tag{3}$$

By the continuity of the index function on $\Phi_l(X)$, we get $i(T-\varphi(t)S)$ is constant for all $t\in[0,1]$. In particular, $i(T-S)=i(T)$. (ii) Can be checked in the same way as (i) . (iii) Follows immediately from (i) and (ii).

Remark that if, for some $n\in\mathbb{N}^*$, $m_S(z)=z^n$, then, for all $T\in\Phi_l(X)$ (resp., $T\in\Phi_r(X)$), T is in left (resp., right) communication with S. Thus, we obtain the following:

Corollary 3.1. Let $T,S\in\mathcal{L}(X)$ such that the commutator $[S,T]$ is weakly compact and DP operator. Suppose that, for some $n\in\mathbb{N}^*$, S^n is weakly compact and DP operator. (i) If $T\in\Phi_l(X)$, then $T-S\in\Phi_l(X)$ and $i(T-S)=i(T)$. (ii) If $T\in\Phi_r(X)$, then $T-S\in\Phi_r(X)$ and $i(T-S)=i(T)$. (ii) If $T\in\Phi(X)$, then $T-S\in\Phi(X)$ and $i(T-S)=i(T)$. ♦

Theorem 3.2. Let $T,S\in\mathcal{L}(X)$ such that $ST=TS$. Suppose that $S\in\mathcal{P}(\mathcal{W}(X))$ and $m_S(S)$ is DP operator. (i) If T is in left communication with S and $a(T)<+\infty$, then T and $T-S$ are in $\mathcal{B}_l(X)$ and $i(T-S)=i(T)$. (ii) If T is in right communication with S and $d(T)<+\infty$, then T and $T-S$ are in $\mathcal{B}_r(X)$ and $i(T-S)=i(T)$. (iii) If T is in communication with S, $a(T)<+\infty$ and $d(T)<+\infty$, then T and $T-S$ are in $\mathcal{B}(X)$ and $i(T-S)=i(T)$.

Proof. (i) According to hypothesis and by the proof Theorem 3.1 (i), we get:

$T-\varphi(t)S\in\Phi_l(X)$ and $i(T-S)=i(T)$. Since $ST=TS$, then according to [7, Theorem 3], $\overline{\mathcal{N}^\infty(T-\varphi(t)S)}\cap\mathcal{R}^\infty(T-\varphi(t)S)$ is locally constant function on the set $[0,1]$ and therefore this function is constant on $[0,1]$. Since $a(T)<\infty$, then from [16, Proposition 1.6(i)] : $\mathcal{N}^\infty(T)\cap\mathcal{R}^\infty(T)=\overline{\mathcal{N}^\infty(T)}\cap\mathcal{R}^\infty(T)=\{0\}$ and hence,

$\overline{\mathcal{N}^{\infty}(T-S)}\cap\mathcal{R}^{\infty}(T-S)=\{0\}$. Thus, $\mathcal{N}^{\infty}(T-S)\cap\mathcal{R}^{\infty}(T-S)=\{0\}$, and again by [16, Proposition 1.6(i)], $a(T-S)<\infty$. (ii) Since $\sigma_{le}(T^*)\subset\sigma_{re}(T)$, then T^* is in left communication with S^*. Thus, applying (i), we get $d(T-S)=a(T^*-S^*)<\infty$. (iii) Follows immediately from (i) and (ii).

3.2. Invariance of essential spectra on spaces with the Dunford-Pettis property

Consider X a Banach space which has the property DP, $T\in\mathcal{C}(X)$ and assume that T satisfies the hypothesis $(\mathcal{A})$, that is:

$$(\mathcal{A})\begin{cases} (i)\ \textit{For all } T-\textit{bounded operator } K, \textit{ there exists } \lambda\in\mathbb{R} \textit{ such that} \\ \quad]\lambda,+\infty[\subset\rho(T+K). \\ \\ (ii)\ \rho_S(T) \textit{ is a connected set of } \mathbb{C}. \end{cases}$$

In order to study the invariance of the essential spectra of $T\in\mathcal{C}(X)$, we need first to establish the following useful lemma:

Lemma 3.1. Let $T\in\mathcal{C}(X)$ satisfying the hypothesis $(\mathcal{A})$ and let S be a T-bounded operator on X. Then the following assertions hold. (i) $\rho(T)$ is a connected set of $\mathbb{C}$. (ii) Assume that $\rho(T+S)$ is a connected set of $\mathbb{C}$. Then
$[S(\lambda-T)^{-1}]^n\in\mathcal{W}(X), \forall\lambda\in\rho(T)\Leftrightarrow[S(\lambda-T-S)^{-1}]^n\in\mathcal{W}(X), \forall\lambda\in\rho(T+S)$. ♦

Proof. (i) Since $\rho_S(T)$ is a connected set, then from [9, Lemma 3.1], $\sigma_B(T)=\sigma_S(T)$ and the result follows from the following identity:

$$\rho(T)=\rho_B(T)\setminus\{\lambda\in\sigma(T);\ \lambda \textit{ is an isolated eigenvalue of finite algebraic multiplicity}\}.$$

(ii) Since T satisfies the hypothesis $(\mathcal{A})$, there exists $\lambda_1,\lambda_2\in\mathbb{R}$ such that $]\lambda_1,+\infty[\subset\rho(T)$ and $]\lambda_2,+\infty[\subset\rho(T+S)$. If we take $\overline{\lambda}=\max\{\lambda_1,\lambda_2\}$, we have necessarily $]\overline{\lambda},+\infty[\subset\rho(T+S)\cap\rho(T)$. Hence, the set $\rho(T+S)\cap\rho(T)$ has a point of accumulation. For all $\lambda>\overline{\lambda}$, we have

$$[I+S(\lambda-T-S)^{-1}][I-S(\lambda-T)^{-1}]=[I-S(\lambda-T)^{-1}][I+S(\lambda-T-S)^{-1}]=I.$$

Hence, $[I-S(\lambda-T)^{-1}]$ is invertible. Moreover, for all $\lambda\in\rho(T+S)\cap\rho(T)$:

$$\left[S(\lambda-T-S)^{-1}\right]^n=\left[S(\lambda-T)^{-1}\right]^n\left[I-S(\lambda-T)^{-1}\right]^{-n}.$$

Thus, for all $\lambda\in]\overline{\lambda},+\infty[$,

$$[S(\lambda-T-S)^{-1}]^n\in\mathcal{W}(X)\Leftrightarrow[S(\lambda-T)^{-1}]^n\in\mathcal{W}(X).$$

Finally, the result follows from [3, Lemma 4.2, Remark 4.3].

Now, we are ready to state and prove the main result of this subsection.

Theorem 3.3. Let S be T-bounded such that $\rho(T+S)$ is a connected set of $\mathbb{C}$.

Suppose that there exists $n \in \mathbb{N}^*$ such that, for all $\lambda \in \rho(T)$, $\left[S(\lambda - T)^{-1}\right]^n$ is weakly compact operator. Then the following assertions hold. (i) If $\rho_{re}(T)$ and $\rho_{re}(T+S)$ are connected sets of $\mathbb{C}$, then $\sigma_{re}(T+S) = \sigma_{re}(T)$. (ii) If $\rho_{le}(T)$ and $\rho_{le}(T+S)$ are connected sets of $\mathbb{C}$, then $\sigma_{le}(T+S) = \sigma_{le}(T)$. (iii) If $\rho_W(T)$ and $\rho_W(T+S)$ are connected sets of $\mathbb{C}$, then $\sigma_W(T+S) = \sigma_W(T)$. (iv) If $\rho_S(T+S)$ is a connected set of $\mathbb{C}$, then

$$\sigma_S(T+S) = \sigma_S(T) \text{ and } \sigma_B(T+S) = \sigma_B(T). \qquad \blacklozenge$$

Proof. (i) Let $F_n = \left\{\lambda \in \rho_{re}(T) \text{ such that} \left[\hat{S}(\lambda - T)_r\right]^n \in \mathcal{W}(X)\right\}$ Clearly we have $\rho(T) \subset F_n \subset \rho_{re}(T)$. From Lemma 3.1 (i), $\rho(T)$ has a point of accumulation. If $\rho_{re}(T)$ is a connected set, then by [3, Lemma 4.2, Remark 4.3], $F_n = \rho_{re}(T)$. Now, If we translate Theorems 2.2 (i) in terms of essential spectra, we get $\sigma_{re}(T+S) \subset \sigma_{re}(T)$. Conversely, by Lemma 3.1 (ii), $[S(\lambda - T - S)^{-1}]^n \in \mathcal{W}(X), \forall \lambda \in \rho(T+S)$. Since $\rho_{re}(T+S)$ is a connected set, then a similar reasoning as above leads to $\sigma_{re}(T) \subset \sigma_{re}(T+S)$. This proves (i). With the same argument we prove $(ii)-(iv)$ if we consider respectively:

$$G_n = \left\{\lambda \in \rho_{le}(T) \text{ such that} \left[\hat{S}(\lambda - T)_l\right]^n \in \mathcal{W}(X)\right\},$$
$$H_n = \left\{\lambda \in \rho_W(T) \text{ such that} \left[S(\lambda - T)^{-1}\right]^n \in \mathcal{W}(X)\right\} \text{ and}$$
$$Q_n = \left\{\lambda \in \rho_S(T) \text{ such that} \left[S(\lambda - T)^{-1}\right]^n \in \mathcal{W}(X)\right\}$$

Remark 3.1. (i) It should be observed that The results of Theorem 3.3 remaind valid if we suppose that, for all $\lambda \in \rho(T), [(\lambda - T)^{-1}S]^n$ is weakly compact for some $n \in \mathbb{N}^*$. (ii) If we suppose that there exists $\mu \in \rho(T)$ such that $S(\mu - T)^{-1} \in \mathcal{W}(X)$ then, for all $\lambda \in \rho(T)$, $S(\lambda - T)^{-1} \in \mathcal{W}(X)$ and therefore the results of Theorem 3.3 remaind valid. ♦

3.3. Application to transport equation

In this section, we shall apply the results of the last subsection to give more precise description of the essential spectra to the multidimensional neutron transport equation which governs the time evolution of the distribution of neutrons in a nuclear reactor (cf., [6, 12]):

$$\begin{cases} \dfrac{\partial\psi}{\partial t}(x,v,t) & = & -v\dfrac{\partial\psi}{\partial x}(x,v,t)-\sigma(v)\psi(x,v,t)+\int_V k(x,v,v')\psi(x,v',t)dv' \\ & = & A_0\psi(x,v,t) \\ & = & T_0\psi(x,v,t)+R\psi(x,v,t) \\ \psi_{|\Gamma_-}=0 \quad and & & \psi(x,v,0)=\psi_0(x,v), \end{cases}$$

where T_0 is the streaming operator and R denotes the integral part of A_0 (the collision operator), $(x,v)\in D\times V$, where $D=\overset{\circ}{D}\subset \mathbb{R}^N$ and $V=\overset{\circ}{V}\subset\mathbb{R}^N$ $(N\geq 1)$. The unbounded operator A_0 is studied in the Banach space $X_1=L_1(D\times V,dxdv)$. Its domain is

$$\mathcal{D}(A_0)=\mathcal{D}(T_0)=\left\{\psi\in X_1,\ such\ that\ v\frac{\partial\psi}{\partial x}\in X_1,\psi_{|\Gamma_-}=0\right\},$$

where

$$\Gamma_-=\{(x,v)\in\partial D\times V\ such\ that\ v\ is\ ingoing\ at\ x\in\partial D\}.$$

The function $\sigma(.)$ is called the collision frequency. The scattering kernel $\kappa(.,.,.)$ define a linear operator R by : $R:X_1\to X_1,\psi\to\int_V k(x,v,v')\psi(x,v')dv'.$ Observe that the operator R acts only on the variables v'. So, x may be viewed merely as a parameter in $\mathcal{D}$. Hence, we may consider R as a function $R(.):x\in\mathcal{D}\to R(x)\in Z,$ where $Z=\mathcal{L}(L_1(V,dv))$ denotes the set of all bounded linear operators on $L_1(V,dv)$. In the following we will make the assumptions

$$(H_1)\qquad \begin{cases} -\textit{the function } R(.) \textit{ is strongly measurable,} \\ -\textit{there exists a compact subset } \mathcal{C}\subset\mathcal{L}(L_1(V,dv)) \textit{ such that}: \\ \ R(x)\in\mathcal{C} \textit{ a.e. on } \mathcal{D}, \\ -R(x)\in\mathcal{K}(L_1(V,dv)) \textit{ a.e. on } \mathcal{D}, \end{cases}$$

where $\mathcal{K}(L_1(V,dv))$ denotes the set of all compact operators on $L_1(V;dv)$.

Definition 3.2. A collision operator R is said to be regular if it satisfies (H_1). ♦

Notice that the spectrum of the operator T_0 was analyzed in [12]. In particular we have, $\sigma_W(T_0)=\sigma_S(T_0)=\sigma_B(T_0)=\sigma(T_0)=\{\lambda\in\mathbb{C} such\ that\ Re\lambda\leq-\lambda^*\},$ where $\lambda^*:=\liminf_{|\xi|\to 0}\sigma(\xi)$.

Lemma 3.2. ([12, Lemma 2.1]) Let K and H be two regular collision operators on X_1 and $Re\lambda>\eta,$ where η is the type of the C_0-semigroup generated by T_0.

(i) $K(\lambda - T_0)^{-1}H$ is weakly compact on X_1. If $\sigma(x,v) = \sigma(v)$ and if $\mathcal{D}$ is convex then $K(\lambda - T_0)^{-1}H$ is compact on X_1.

(ii) If $\omega > \eta$ then $\lim_{|Im\lambda|\to+\infty} \|K(\lambda - T_0)^{-1}H\| = 0$ uniformly in $\{\lambda \in \mathbb{C};\ Re\lambda \geq \omega\}$. ♦

The following theorem provides an extension and an improvement of some results (see for examples [3, 11]).

Theorem 3.4. Let R be a regular operator on X_1. (i) If $\rho_{le}(T_0 + R)$ is a connected set of $\mathbb{C}$, then $\sigma_{le}(T_0 + R) = \sigma_{le}(T_0)$. (ii) If $\rho_{re}(T_0 + R)$ is a connected set of $\mathbb{C}$, then $\sigma_{re}(T_0 + R) = \sigma_{re}(T_0)$. (iii) If $\rho_W(T_0 + R)$ is a connected set of $\mathbb{C}$, then $\sigma_W(T_0 + R) = \sigma_W(T_0)$. (iv) If $\rho_S(T_0 + R)$ is a connected set of $\mathbb{C}$, then

$$\sigma_S(T_0 + R) = \sigma_S(T_0) \text{ and } \sigma_B(T_0 + R) = \sigma_B(T_0). \qquad ♦$$

Proof. The results are consequence of Lemma 3.2 and Remark 3.1 (ii).

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