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Two New Results on Permutation Graphs

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Abstract. The permutation graph is a very important subclass of intersection graphs. This graph class is used to solve many real life problems. In this article, an alternative proof is given to show that every path is a permutation graph. Also, it is proved that a lobstar is a permutation graph.

Keywords: **I**ntersection graph, permutation graph

1. Introduction

A graph *G=(V,E)* is called an intersection graph for a finite family *F* of a non empty set if there is a one to one correspondence between *F* and *V* such that two sets in *F* have non empty intersection if and only if there corresponding vertices in *V* have non empty intersection. For the terminologies of the graphs see [5,6,18,25].

Any graph can be represented as intersection graph. One of its classification is permutation graph.

Let *G= (V,E)* be a simple and undirected graph with n vertices and m edges and let *V*={1,2,…,*n*}. G is said to be a permutation graph if and only if there exists a permutation $\pi = \{\pi(1), \pi(2), ..., \pi(n)\}$ on {1,2,…,n} such that for all i,j∈ *V*, (i ,j) ∈ *E* if and only if $(i-j)(\pi^{-1}(i) - \pi^{-1}(j) < 0$ for each i∈ *V*, denote the position of the no. i in π. We assume that the graph is connected. Permutation graphs can be visualized as a class of intersection graphs.

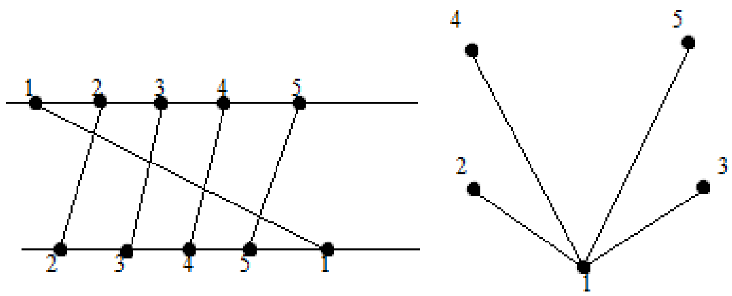


Figure 1: Permutation representation and the corresponding graph

Even et al. [4,12] showed the permutation graphs are exactly those comparability graphs whose complements are also comparability graphs. Chen solved the shortest path problem on bipartite permutation graph. The longest path problem was solved in linear time. A polynomial time solution for the weighted feedback vertex set problem in permutation graph is also developed. Permutation graphs are similar to interval graphs.

An O($n^2 logn$) time algorithm was presented for the generalized M2IS problems in permutation graphs.

The maximum two chain problem in permutation graphs was solved in $\theta(n \log n)$ time. The minimum cardinality connected dominating set in a permutation graph can be solved in O(n^2) time when the permutation module is not given. Many works have been done on permutation graphs [1-4,7,9-17,19-24].

1.1. Some important definitions

Circle graphs

In graph theory, a circle graph is the intersection graph of a set of chords of a circle i.e. it is an undirected graph whose vertices can be associated with chords of a circle such that two vertices are adjacent if and only if the corresponding chords cross each other.
A circle graph is also an intersection graph.

Comparability graphs

A non empty set A together with a relation of partial order $\leq$ on A is called a poset (partially ordered set) and is denoted by $(A,\leq)$. Let $(A,\leq)$ be a poset. Two elements a and b of A are said to be comparable if $a\leq b$ or $b\leq a$. otherwise a and b are non comparable.

A comparability graph is an undirected graph that connects pairs of elements that are comparable to each other in a partial order relation.

Perfect graph

A perfect graph is a graph in which the chromatic number of every induced subgraph equals the size of its largest clique of that subgraph.

Transitive orientation

A transitive orientation is an orientation such that the resulting directed graph is its own transitive closure. Given a directed graph find out if a vertex j is reachable from the vertex i for all pair (i,j) in the given graph. Here reachable means a path from i to j. the reachability matrix is called transitive closure of a graph G and is denoted by *C(G).* its corresponding diagram contains an edge(u,v) whenever there exist a path from u to v.

Clique and stable set/ independent set

A clique in an undirected graph is a set of pair-wise adjacent vertices.
An independent set or a stable set in a graph is a set of pair-wise non adjacent vertices.
Let *G* be a graph of order n. an arrangement $l=(v_1,v_2,\ldots,v_n)$ of the vertices of G is called a cohesive vertex set order of G if the following conditions are satisfied:

a) If $i<k<j$ and v_iv_k, $v_kv_j \in E(G)$, then $v_iv_j \in E(G)$.
b) If $i<k<j$ and $v_iv_j \in E(G)$ then $v_iv_k \in E(G)$ or $v_kv_j \in E(G)$.

1.2. Some important properties of permutation graphs

1. A graph *G* is a permutation graph if and only if it is a circle graph that admits an equator i.e. an additional chord that represent every other chord.
2. A graph *G* is a permutation graph if and only if both *G* and its complement $\overline{G}$ are comparability graphs.
3. All permutation graphs are perfect graph.
4. A permutation graph is the complement of the comparability graphs.

5. Let π be the permutation on N corresponding to the permutation graph $G(\pi)$. If we reverse the sequence π then we obtain a graph which is also a permutation graph and it is nothing but the complement of the graph $G(\pi)$. In other word if π' be the reverse order of π then $G(\pi') = G'(\pi)$.
6. $G(\pi)$ is transitively orientable.
7. The decreasing subsequence of π and the cliques of $G(\pi)$ are in one to one correspondence. The increasing subsequence of π and the stable sets of $G(\pi)$ are in one to one correspondence.
8. The depth first tree of a permutation graph can be constructed in O(*n*) time.
9. Let *G* be a graph. Then the following are equivalent,
   a. *G* is a permutation graph.
   b. $\overline{G}$ is a permutation graph.
   c. Every induced subgraph of *G* is a permutation graph.
   d. Every connected component of *G* is a permutation graph.
10. A graph *G* is a permutation graph if and only if it has a cohesive order.
11. Let $\pi \in S_n$.Then $\pi = \{\pi(1), \pi(2), ..., \pi(n)\}$ is a cohesive order of the permutation

graph G_π.

2. Main results

Before going on main results some definitions have to be discussed.

A caterpillar is a tree having the property that removal of all pendant vertices results in a path.

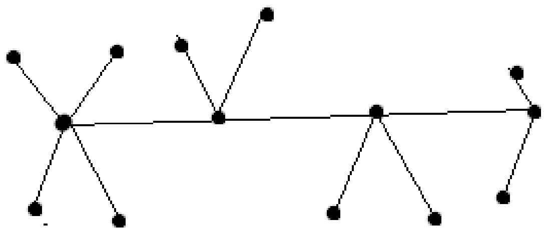


Figure 2: Caterpillar

The tree $K_{1,3}^*$ is formed by subdividing each edge of $K_{1,3}$ into two edges.

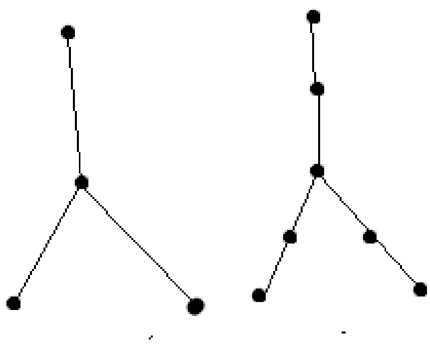


Figure 3: $K_{1,3}^*$

Lobster is a tree having the property that removal of all pendant vertices results a caterpillar.

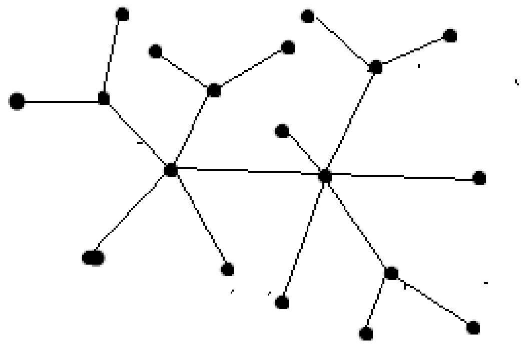


Figure 4: Lobster

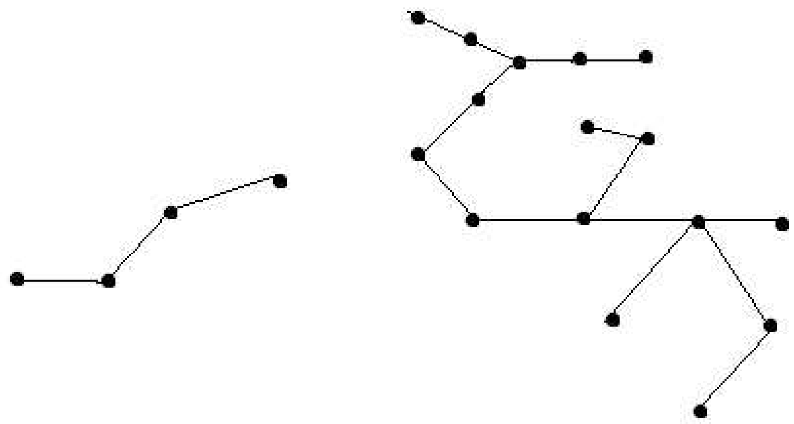
Theorem 1. A lobster is a permutation graph.

Proof: It is proved by the use of following two theorems.

I. A tree is a caterpillar if and only if it does not contain $K_{1,3}^{*}$ as a subgraph.

II. A tree is a permutation graph if and only if it is a caterpillar.

So, we can conclude that a tree is a permutation graph if and only if it does not contain $K_{1,3}^{*}$ as a subgraph.



(a) (b)

Figure 5: (a) Permutation graph, (b) not permutation graph

Lobster is a tree. So it is a permutation graph if it does not contain $K_{1,3}^{*}$ as a subgraph. And is not a permutation graph when it contains $K_{1,3}^{*}$ as a subgraph.

Theorem 2. Path P_n is a permutation graph.

Proof: It is proved by finding the permutation for different n by the use of mathematical induction because if we find out the permutation $\forall$ N then P_n is a permutation graph $\forall$ *N*.

For n=1, P_1 is a permutation graph.

Its corresponding permutation is, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

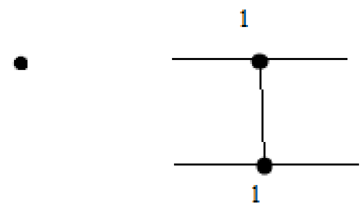


Figure 6: P_1

For n=2, P_2 is also a permutation graph because its corresponding permutation is, $\begin{pmatrix} 1\ 2 \\ 2\ 1 \end{pmatrix}$.

The corresponding picture is,

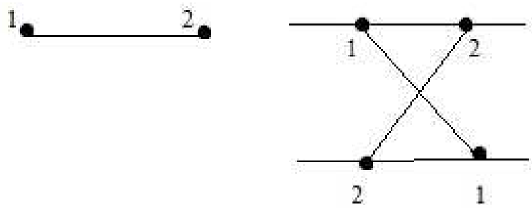


Figure 7: P_2

Let us assume that it is true for n=m.
We have to prove it for n=m+1.
Let m be odd, then m+1 is even.
Then the corresponding permutation is,

$$\begin{pmatrix} 2\ 1\ 4\ 3\ 6\ 5\ 8\ 7 \ldots\ldots\ldots\ldots m\text{-}2\ \ m\text{+}1\ \ m \\ 1\ 3\ 2\ 5\ 4\ 7\ 6 \ldots\ldots\ldots\ldots m\ \ \ m\text{-}1\ \ m\text{+}1 \end{pmatrix}.$$

The figure is,

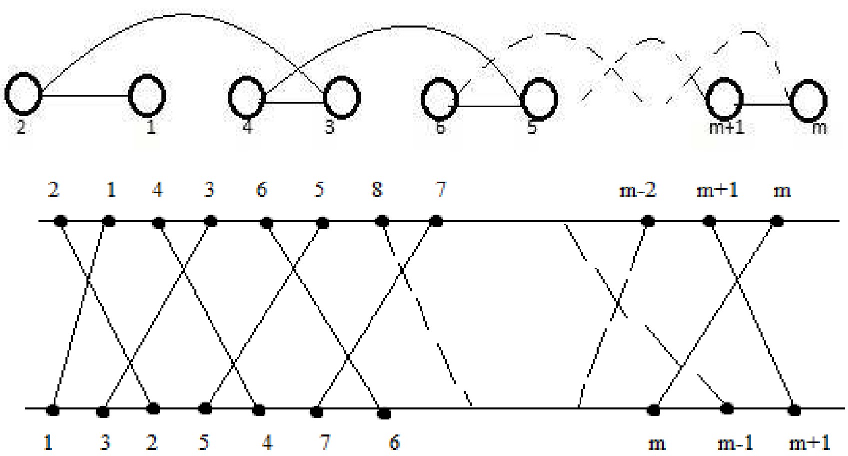


Figure 8: P_{m+1} (m+1 even) and its permutation representation

When m+1 odd i.e. m even, then the corresponding permutation becomes,

$$\begin{pmatrix} 2\ 1\ 4\ 3\ 6\ 5 \ldots\ldots\ldots m\ m\text{-}1\ m\text{+}1 \\ 1\ 3\ 2\ 5\ 4\ 7\ 6 \ldots\ldots\ldots m\text{-}2\quad m\text{+}1\ m \end{pmatrix}.$$

The figure is,

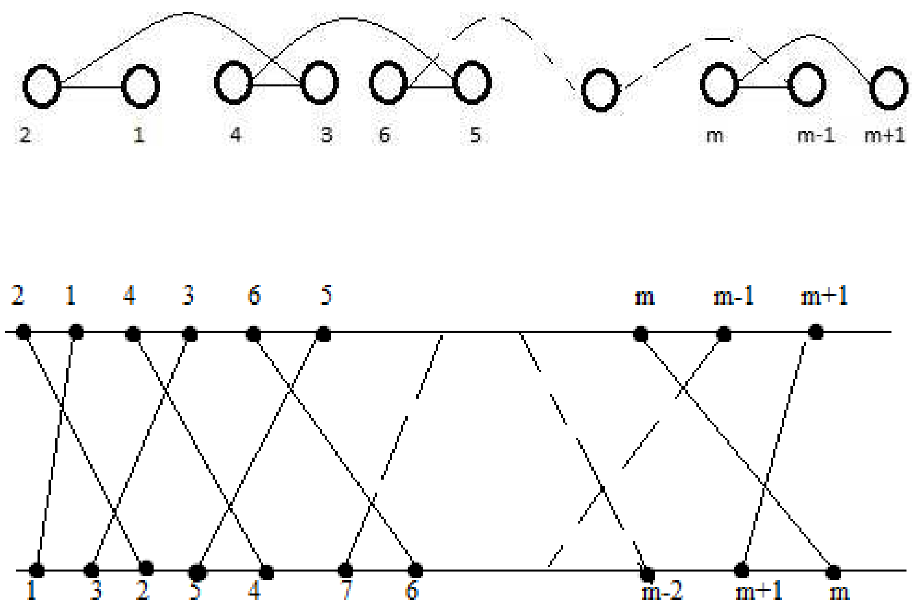


Figure 9: P_{m+1}(m+1 odd) and its permutation representation

The theorem is true for n=1,2,…,m+1.

So the theorem is true $\forall\ n \in N$.

3. Conclusion

Graph theory is a most useful tool to solve a lot of problems now a days. Permutation graph is also very important subclass of intersection graphs. As they are a subclass of perfect graphs many problem can be solved efficiently which are NP complete on arbitrary graphs. Many optimization problem becomes polynomial on permutation graphs

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