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Oscillation of Solutions of Certain Nonlinear Difference Equations

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Abstract. In this paper, some sufficient conditions for the oscillation of all solutions of
the nonlinear difference equation of the form
\[ \Delta^2 x_n + f(x_n) = 0, \quad n = 0,1,2, \ldots \]
are obtained. Examples are given to illustrate the results.

Keywords: Difference equations, oscillation, nonlinear.

1. Introduction
The notion of nonlinear difference equation was studied intensively by Agarwal [1] and
oscillatory properties were discussed by Agarwal et al. [2]. Recently there has been a lot
of interest in the study of oscillatory behavior of solutions of nonlinear difference
equations. We can see this in [3]-[9].

Szafranski and Szmanda [10] considered the second order nonlinear difference
equation of the form
\[ \Delta (r_n \Delta x_n) + q_n f(x_{n-\tau_n}) = 0, \quad n = 0,1,2, \ldots \]
and gave the sufficient conditions for the oscillation of solutions of the equation
considered. Motivated by this article, we consider the nonlinear difference equation of the form
\[ \Delta^2 x_n + f(x_n) = 0, \quad n = 0,1,2, \ldots \] (1)
where \( \Delta \) is the forward difference operator defined by \( \Delta x_n = x_{n+1} - x_n \) and \( f : \mathbb{R} \to \mathbb{R} \)
is continuous with \( xf(x) > 0 \) for \( x \neq 0 \). Our aim in this paper is to obtain some new
oscillation criteria for the solution of equation (1).

By a solution of equation (1), we mean a real sequence \( \{x_n\} \) which satisfies
equation (1) for all large \( n \). A solution \( \{x_n\} \) is said to be oscillatory if it is neither
eventually positive nor eventually negative. Otherwise it is called non-oscillatory.
2. Main results
In this section, we present some sufficient conditions for the oscillation of all the solutions of the equation (1).

**Theorem 2.1.** Every solution of equation (1) is oscillatory, if the sufficient condition \( \liminf_{n \to \infty} f(n) > 0 \) holds.

**Proof:** Assume that equation (1) has non-oscillatory solution \( \{x_n\} \) and we assume that \( \{x_n\} \) is eventually positive.

Then there exists a positive integer \( n_0 \) such that
\[
x_n > 0 \quad \text{for} \quad n \geq n_0.
\] (2)
From (1),
\[
\Delta^2 x_n = -f(x_n) \leq 0, \quad n \geq n_0
\]
Therefore \( \Delta x_n \) is an eventually non-increasing sequence.
We first show that \( \Delta x_n \geq 0 \) for \( n \geq n_0 \).
In fact, if there is an \( n_1 \geq n_0 \) such that \( \Delta x_{n_1} = c < 0 \) and \( \Delta x_n \leq c \) for \( n \geq n_1 \).
Summing the last inequality from \( n_1 \) to \( n-1 \), we get
\[
\sum_{k=n_1}^{n-1} \Delta x_k \leq \sum_{k=n_1}^{n-1} c,
\]
which implies, \( x_n - x_{n_1} \leq (n-n_1)c \).
Therefore, \( x_n \leq x_{n_1} + (n-n_1)c \to -\infty \) as \( n \to \infty \),
which is a contradiction to the fact that \( x_n > 0 \) for \( n \geq n_0 \).
Hence \( \Delta x_n \geq 0 \) for \( n \geq n_0 \).
Therefore we have \( x_n > 0 \), \( \Delta x_n \geq 0 \), \( \Delta^2 x_n \leq 0 \) for \( n \geq n_0 \).

Let \( L = \lim_{n \to \infty} x_n \).
Then \( L > 0 \) is finite or infinite.

**Case (i):** \( L > 0 \) is finite.
Since \( f \) is continuous function, we have
\[
\lim_{n \to \infty} f(x_n) = f(L) > 0. \quad \text{(A)}
\]
This implies we can choose a positive integer \( n_2 \geq n_0 \) such that
\[
f(x_n) > \frac{1}{2} f(L), \quad n \geq n_2 \quad \text{(3)}
\]
Substituting (3) in (1), we get
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\[ \Delta^2 x_n = -f(x_n) \]

\[ \leq -\frac{1}{2} f(L), \]

which implies,

\[ \Delta^2 x_n + \frac{1}{2} f(L) \leq 0, \quad n \geq n_2. \] (4)

Summing up both sides of (4) from \( n_2 \) to \( n \), we obtain

\[ \sum_{i=n_2}^n \Delta(\Delta x_i) + \frac{1}{2} f(L) \sum_{i=n_2}^n 1 \leq 0. \]

This implies,

\[ \Delta x_{n+1} - \Delta x_{n_2} + \frac{1}{2} f(L)(n+1-n_2) \leq 0. \]

Hence

\[ \frac{1}{2} f(L) \leq \frac{1}{(n+1-n_2)} \Delta x_{n_2}, \quad n \geq n_2. \]

That is,

\[ f(L) \leq 0 \quad \text{as} \quad n \to \infty, \]

which is a contradiction to (A).

**Case (ii):** \( L = \infty \)

Since \( \liminf_{n \to \infty} f(x_n) > 0 \), we may choose a positive constant \( c \) and a positive integer \( n_3 \) such that

\[ f(x_n) \geq c \quad \text{for} \quad n \geq n_3. \] (5)

Substituting (5) in (1), we get

\[ \Delta^2 x_n = -f(x_n) \leq -c. \]

This implies,

\[ \Delta^2 x_n + c \leq 0, \quad n \geq n_3. \]

Summing the last inequality from \( n_3 \) to \( n \), we get

\[ \Delta x_{n+1} - \Delta x_{n_3} + c(n+1-n_3) \leq 0. \]

So,

\[ \Delta x_{n+1} - \Delta x_{n_3} \leq -c(n+1-n_3). \]

Therefore, \( \Delta x_n \to -\infty \quad \text{as} \quad n \to \infty, \)

which is a contradiction to the fact that \( \Delta x_n > 0 \) and hence the proof.

**Corollary 2.2.** With the assumption of the above theorem, every bounded solution of equation (1) is oscillatory.

**Proof:** Proceeding as in the proof of Theorem 2.1 with assumption that \( \{x_n\} \) is bounded non-oscillatory solution of (1), we get the inequality (4)
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\[ \Delta^2 x_n + \frac{1}{2} f(L) \leq 0, \quad n \geq n_2. \]

Now,

\[ \Delta^2 x_n \geq \Delta^2 x_n - \Delta x_n. \quad (6) \]

From (4) and (6),

\[ \Delta^2 x_n - \Delta x_n + \frac{1}{2} f(L) \leq 0. \]

Summing the last inequality from \( n_2 \) to \( n \), we get

\[ \sum_{i=n_2}^{n} \Delta^2 x_i - \sum_{i=n_2}^{n} \Delta x_i + \frac{1}{2} f(L) \sum_{i=n_2}^{n} 1 \leq 0. \]

That is,

\[ \Delta x_{n+1} - \Delta x_{n_2} - x_{n+1} + x_{n_2} + \frac{1}{2} f(L)(n+1-n_2) \leq 0. \]

This implies,

\[ \frac{1}{2} f(L)(n+1-n_2) \leq x_{n+1} + \Delta x_{n_2} - x_{n_2}. \]

Since \( \{x_n\} \) is bounded, we may choose a positive constant \( c \) such that

\[ \frac{1}{2} f(L)(n+1-n_2) \leq c, \quad (7) \]

which implies,

\[ \frac{1}{2} f(L) \leq \frac{1}{(n+1-n_2)} c. \]

So,

\[ f(L) \leq 0 \quad \text{as} \quad n \to \infty, \]

which contradicts (A). Hence the proof is complete.

**Example 2.1.** Consider the difference equation \( \Delta^2 x_n + 4x_{n+1} = 0 \). The condition of Theorem 2.1 is satisfied. Hence all solutions of equation (1) are oscillatory. One of the solutions is \( (-1)^n \) which is oscillatory. The condition of corollary is also satisfied and the above solution is bounded one.

**REFERENCES**


