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Progress in Nonlinear Dynamics and Chaos

A Weighted Sobolev Space Theory of Parabolic Stochastic Partial Differential System on Non-Smooth Domains

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Abstract. Scholars usually focus on the single valued stochastic differential equations in the study of Sobolev space theory. In this paper, we consider the non-divergence form of stochastic parabolic systems on arbitrary domains $\mathcal{O} \subset \mathbb{R}^d$. By localization technique and continuity method the existence and uniqueness are proved in the weighted Sobolev space allowing the derivatives of the solutions to blow up near the boundary. Furthermore, an a priori estimate of the solution is also obtained.

Keywords: Stochastic differential system; non-smooth domain; weighted Sobolev space; Itô-Wentzell formula

AMS Mathematics Subject Classification (2010): 35R60, 60H15

1. Introduction

The second order parabolic equations on smooth domains have been almost completely studied over the last couple of decades. Recently, scholars are trying to study the partial differential problem under the minimal smoothness assumption of the domains. We only refer to [1] for a brief survey of recent works on non-smooth domains such as Lipshitz domains, non-tangentially accessible domains twisted holder domains and John domains.

Inspired by such works on non-smooth domains, in this paper, the authors consider a certain type of stochastic partial differential problem. At the same time, the authors also find that more and more scholars who study the partial differential equation (not stochastic type) are interested in extending their theories to $\mathbb{R}^{d_1}$-valued version (see [2-5]). Hence, we consider the following $\mathbb{R}^{d_1}$-valued stochastic parabolic equation (called stochastic parabolic system)

$$\begin{cases} \mathrm{d}u^k = \left(L_k u + D_i f_k^i + f_k^0\right)\mathrm{d}t + \left(\Lambda_{k,m} u + g_m^k\right)\mathrm{d}w_t^m, \quad (x,t)\in \mathcal{O}\times[0,T], \\ u(x,0) = u_0(x), \quad x\in \mathcal{O}, \end{cases} \tag{1}$$

where

$$L_k u = a_{kr}^{ij} D_i D_j u^r + b_{kr}^i D_i u^r + c_{kr} u^r, \quad \Lambda_{k,m} u = \sigma_{kr,m}^i D_i u^r + \mu_{kr,m} u^r .$$

Here, the summation convention with respect to $i, j = 1,2,\cdots,d$, $r = 1,2,\cdots,d_1$ and $m = 1,2,\cdots$ is enforced. $(\Omega, \mathcal{F}, P)$ is a complete probability space and $\{\mathcal{F}_t, t\ge 0\}$ be a filtration such that $\mathcal{F}_0$ contains all P-null sets of Ω. Denote by $\mathcal{P}$ the predictable σ-algebra on $\Omega\times[0,T]$ associated with $\{\mathcal{F}_t, t\ge 0\}$. Let $\left\{w_t^m\right\}_{m=1}^{\infty}$ be one-dimensional $\left\{\mathcal{F}_t\right\}$-adapted Wiener processes and independent defined on $(\Omega, \mathcal{F}, P)$ and $C_0^\infty = C_0^\infty(\mathbb{R}^d, \mathbb{R}^{d_1})$ denote the set of all $\mathbb{R}_{d_1}$-valued infinitely differentiable functions with compact support in $\mathbb{R}^d$. Denote by $\mathcal{D}$ the space of $\mathbb{R}^{d_1}$-valued distributions on C_0^∞. Precisely, we define $(u,\phi)\in\mathbb{R}^d$ with components $(u,\phi)_k = (u^k, \phi^k)$, $k = 1,2,\cdots,d_1$ for $u\in\mathcal{D}$ and $\phi\in C_0^\infty$. Here, each u_k is a usual $\mathbb{R}$-valued distribution defined on $C^\infty(\mathbb{R}^d, \mathbb{R})$. $\mathcal{O}\subset\mathbb{R}^d$ is arbitrary bounded domain (we do not assume that $\mathcal{O}$ is smooth).

If $d_1 = 1$, the system (1) on smooth domains degenerates to the stochastic parabolic equation which has been well studied by many authors (see [6-10]). An L^p-theory of this kind of stochastic parabolic equation with space domain $\mathbb{R}^d$ was first presented by Krylov in [7] (cf. see [8] for L^2-theory), and since then the results have been extended which were defined on arbitrary C_1-domains Ω in $\mathbb{R}^d$ by Krylov and many other mathematicians (see [11-12]). If $d_1 = 3$, the motion of a random string with a small mass can be modeled by a stochastic parabolic partial differential system (see [13-14]).

The main guidelines we follow are quite common: getting a priori estimates and using the localization technique and method of continuity. The method of continuity requires a starting point, which in our case is the solvability of the following single equations

$$\mathrm{d}u^k = \left[a_{kk}^{ij} D_i D_j u^k + f_k^0 + D_i f_k^i\right]\mathrm{d}t + \left(\sigma_{kk,m}^i D_i u^k + g_m^k\right)\mathrm{d}w_t^m. \tag{2}$$

Here, we only use summation notation on i, j. Denote $x_t^{ik} = \sum_{m=1}^{\infty}\int_0^t \sigma_{kk,m}^i(s)\mathrm{d}w_s^m$ for each

i,k and $x_t^k = (x_t^{1k}, x_t^{2k}, \cdots, x_t^{dk})$. By using the Itô-Wentzell formula (see Lemma 4.7 in [7]) to $\overline{u}^k(t,x) = u^k(t, x - x_t^k)$, one gets

$$\mathrm{d}\overline{u}^k = \left(\left(a_{kk}^{ij} - \frac{1}{2}(\sigma_{kk}^i, \sigma_{kk}^j)_{\ell_2}\right) D_i D_j \overline{u}^r + \overline{f}_k^0 + D_i \overline{f}_k^i - (\sigma_{kk}^i, D_i \overline{g}^k)_{\ell_2}\right) \mathrm{d}t + \overline{g}_m^k \mathrm{d}w_t^m , \tag{3}$$

where $\overline{f}_k^0(t,x) = f_k^0(t, x - x_t^k)$, $\overline{f}_k^i(t,x) = f_k^i(t, x - x_t^k)$, $\overline{g}^k(t,x) = g^k(t, x - x_t^k)$. The Eq.(3) can be decomposed into two parts(for details, see Reference [7], Definition 3.1 and Definition 3.5) . The first part is a set of single stochastic parabolic equation whose L^p theory have already been well studied. The second part is a deterministic system whose L^p theory can also be found in [12].One of the main difficulties is that most of the first derivatives of the solutions in the stochastic part still exist after using the Ito-Wentzell formula. With an extra condition imposed on Theorem 2.1 or Remark 3.5 we construct an L^p theory of the system by adopting the strategy from [7] in which the theory of stochastic partial differential equations is constructed.

2. Main results

Throughout the article the coefficients $a_{kr}^{ij}(t;\omega)$, $a_{kr}^i(t;\omega)$, $b_{kr}^i(t;\omega)$, $\sigma_{kr,m}^i(t;\omega)$, $c_{kr}(t;\omega)$ and $\mu_{kr,m}(t;\omega)$ are assumed to be measurable with respect to $P \times \mathscr{B}(\mathbb{R}^d)$, where $\mathscr{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}^d$. And $\mathbb{R}^{d_1}$ is a Banach space with the norm $|u|_0 = \left(\sum_{k=1}^{d_1} (u^k)^p\right)^{\frac{1}{p}}$ which satisfies

$$N_2(d_1)\sum_{k=1}^{d_1} |u^k| \le |u|_0 \le N_1(d_1)\sum_{k=1}^{d_1} |u^k| . \tag{4}$$

To be more specific we will introduce some notations. $L_p(\mathbb{R}^d, \mathbb{R}^{d_1})$ denotes the space of all $\mathbb{R}^{d_1}$ -valued functions u which satisfies

$$\|u\|_{L_p(\mathbb{R}^d,\mathbb{R}^{d_1})}^p = \left\| |u|_0 \right\|_{L_p(\mathbb{R}^d)}^p = \sum_{k=1}^{d_1} \|u^k\|_{L_p(\mathbb{R}^d)}^p < \infty .$$

For $\gamma \in (-\infty, \infty)$, $1 \le p < \infty$, we set

$$H_{p,d_1}^\gamma = H_{p,d_1}^\gamma(\mathbb{R}^d; \mathbb{R}^{d_1}) = \left\{ u \,\middle|\, (1-\Delta)^{\gamma/2} u \in L_p(\mathbb{R}^d, \mathbb{R}^{d_1}) \right\} , \quad \|u\|_{H_{p,d_1}^\gamma} = \left\| (1-\Delta)^{\gamma/2} u \right\|_{L_p(\mathbb{R}^d,\mathbb{R}^{d_1})} . \tag{5}$$

H^{γ}_{p,d_1} is the Bessel potential Space. And H^{γ}_{p,d_1} equipped with the norm $\|\cdot\|_{H^{\gamma}_{p,d_1}}$ is a Banach space (see [7]). For a non-negative integer $\gamma = 0,1,2,\cdots$, we also have

$$H^{\gamma}_{p,d_1} = W^{\gamma}_{p}(\mathbb{R}^d;\mathbb{R}^{d_1}) = \left\{u \,\middle|\, D^{\alpha}u \in L_p(\mathbb{R}^d;\mathbb{R}^{d_1}), \forall|\alpha| \le \gamma\right\}. \tag{6}$$

By ℓ_2 we denote the set of all real-valued sequences $e = (e_1, e_2, \cdots)$ with the inner product $(e,f)_{\ell_2} = \sum_{m=1}^{\infty} e_m f_m$ and $|e|_{\ell_2} = (e,e)^{1/2}_{\ell_2}$. If $g = (g^1, g^2, \cdots, g^{d_1})$ and $g^k \in \ell_2$, we define

$$\|g\|^p_{H^{\gamma}_{p,d_1}(\ell_2)} = \left\| \left|(1-\Delta)^{\gamma/2} g^k\right|_{\ell_2} \right\|^p_{L_p(\mathbb{R}^d;\mathbb{R}^{d_1})}.$$

Denote $\rho(x) = dist(x, \partial O)$ and fix a bounded infinitely differentiable function ψ defined in O such that (see (2.6) in [15])

$$\rho(x) \le N\psi(x) \le N\rho(x), \quad \rho^m \left|D^m \psi_x\right| \le N(m) < \infty. \tag{7}$$

Let $\varsigma \in C^{\infty}_0(\mathbb{R}_+)$ be a nonnegative function satisfying

$$\sum_{n=-\infty}^{\infty} \varsigma(e^{n+t}) > c > 0, \quad \forall t \in \mathbb{R}. \tag{8}$$

Note that any nonnegative smooth $\varsigma \in C^{\infty}_0(\mathbb{R}_+)$ so that $\varsigma > 0$ on $[e^{-1}, e]$ satisfying (8). For $x \in O$ and $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$, we define

$$\varsigma_n(x) = \varsigma\left(e^n \psi(x)\right).$$

Then $\operatorname{supp}\varsigma_n \subset \left\{x \in O : e^{-n-k_0} < \rho(x) < e^{-n+k_0}\right\} =: G_n$ for some integer $k_0 > 0$,

$$\sum_{n=-\infty}^{\infty} \varsigma_n(x) \ge \delta > 0, \quad \varsigma_n \in C^{\infty}_0(G_n), \left|D^m \varsigma_n(x)\right| \le N(\varsigma, m)e^{mn}.$$

For $p \ge 1$ and $\gamma \in \mathbb{R}$. By $H^{\gamma,\theta}_{p,d_1}(O)$ we denote the set of all distributions u on O such that

$$\|u\|^p_{H^{\gamma,\theta}_{p,d_1}(O)} := \sum_{n=-\infty}^{\infty} e^{n\theta} \left\|\varsigma_{-n}(e^n\cdot)u(e^n\cdot)\right\|^p_{H^{\gamma}_{p,d_1}} < \infty.$$

We also use the above notation for ℓ_2-valued functions $g = (g_1, g_2, \cdots)$ that is

$$\|g\|^p_{H^{\gamma,\theta}_{p,d_1}(O,\ell_2)} := \sum_{n=-\infty}^{\infty} e^{n\theta} \left\|\varsigma_{-n}(e^n\cdot)u(e^n\cdot)\right\|^p_{H^{\gamma}_{p,d_1}(\ell_2)}.$$

Also if $\gamma = n$ is a nonnegative integer then

$$L_{p,\theta}(\mathcal{O},\mathbb{R}^{d_1}) := H^{0,\theta}_{p,d_1}(\mathcal{O}) = L_p(\mathcal{O},\mathbb{R}^{d_1},\rho^{\theta-d}\mathrm{d}x),$$

$$H^{n,\theta}_{p,d_1}(\mathcal{O}) := \left\{u \,\middle|\, u,\rho Du,\cdots,\rho^n D^n u \in L_{p,\theta}(\mathcal{O},\mathbb{R}^{d_1})\right\},$$

$$\|u\|^p_{H^{n,\theta}_{p,d_1}(\mathcal{O})} \sim \sum_{k=1}^{d_1}\sum_{|\alpha|\le n}\int_{\mathcal{O}}\left|\rho^{|\alpha|}D^\alpha u^k\right|^p \rho^{\theta-d}\mathrm{d}x.$$

Denote $\rho(x,y)=\rho(x)\wedge\rho(y)$. For $v\in(0,1]$ and $k=0,1,2,\cdots$ as in [16], we define

$$[f]^{(0)}_k = [f]^{(0)}_{k,\mathcal{O}} = \sup_{\substack{x\in\mathcal{O}\\ |\beta|=k}}\rho^k(x)\left|D^\beta f(x)\right|,$$

$$[f]^{(0)}_{k+v} = \sup_{\substack{x,y\in\mathcal{O}\\ |\beta|=k}}\rho^{k+v}(x,y)\frac{\left|D^\beta f(x)-D^\beta f(y)\right|}{|x-y|^v},$$

$$|f|^{(0)}_k = \sum_{j=0}^k [f]^{(0)}_{j,\mathcal{O}},\ |f|^{(0)}_{k+v} = \sum_{j=0}^k [f]^{(0)}_{j,\mathcal{O}} + [f]^{(0)}_{k+v}.$$

The above notations are used also for $\mathbb{R}^{d_1}$ valued functions $u=(u^1,u^2,\cdots,u^{d_1})$ and ℓ_2 valued functions $g=(g^1,g^2,\cdots)$. For instance,

$$[u]^{(0)}_k = \sup_{\substack{x\in\mathcal{O}\\ |\beta|=k}}\rho^k(x)\left|D^\beta u(x)\right|_0,\ [g]^{(0)}_k = \sup_{\substack{x\in\mathcal{O}\\ |\beta|=k}}\rho^k(x)\left|D^\beta g(x)\right|_{\ell_2}.$$

Here are some other properties of the space $H^{\gamma,\theta}_{p,d_1}(\mathcal{O})$ taken from [17].

Lemma 2.1. (i) The space $C_0^\infty(\mathcal{O},\mathbb{R}^{d_1})$ is dense in $H^{\gamma,\theta}_{p,d_1}(\mathcal{O})$.

(ii) Assume that $\gamma-d/p=m+v$ for some $m=0,1,\cdots$ and $v\in(0,1]$, and i,j are multi-indices such that $|i|\le m$, $|j|=m$. Then for any $u\in H^{\gamma,\theta}_{p,d_1}(\mathcal{O})$, we have

$$\psi^{|i|+\theta/p}D^i u\in C(\mathcal{O},\mathbb{R}^{d_1}),\ \psi^{m+v+\theta/p}D^j u\in C^v(\mathcal{O},\mathbb{R}^{d_1}),$$

$$\left|\psi^{|i|+\theta/p}D^i u\right|_{C(\mathcal{O},\mathbb{R}^{d_1})} + \left|\psi^{m+v+\theta/p}D^j u\right|_{C^v(\mathcal{O},\mathbb{R}^{d_1})} \le C\|u\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O})}.$$

(iii) ψD, $D\psi$: $H^{\gamma,\theta}_{p,d_1}(\mathcal{O})\to H^{\gamma-1,\theta}_{p,d_1}(\mathcal{O})$ are bounded linear operators, and for any $u\in H^{\gamma,\theta}_{p,d_1}(\mathcal{O})$

$$\|u\|_{H^{\gamma,\theta}_{p,d_1}(O)} \le N\|\psi u_x\|_{H^{\gamma-1,\theta}_{p,d_1}(O)} + N\|u\|_{H^{\gamma-1,\theta}_{p,d_1}(O)} \le N\|u\|_{H^{\gamma,\theta}_{p,d_1}(O)},$$

$$\|u\|_{H^{\gamma,\theta}_{p,d_1}(O)} \le N\|(\psi u)_x\|_{H^{\gamma-1,\theta}_{p,d_1}(O)} + N\|u\|_{H^{\gamma-1,\theta}_{p,d_1}(O)} \le N\|u\|_{H^{\gamma,\theta}_{p,d_1}(O)}.$$

(iv) For any $v,\gamma\in\mathbb{R}$, $\psi^{v} H^{\gamma,\theta}_{p,d_1}(O) = H^{\gamma,\theta-pv}_{p,d_1}(O)$ and

$$\|u\|_{H^{\gamma,\theta-pv}_{p,d_1}(O)} \le N\|\psi^{-v}u\|_{H^{\gamma,\theta}_{p,d_1}(O)} \le N\|u\|_{H^{\gamma,\theta-pv}_{p,d_1}(O)}.$$

(v) If $\gamma\in(\gamma_0,\gamma_1)$ and $\theta\in(\theta_0,\theta_1)$, then

$$\|u\|_{H^{\gamma,\theta}_{p,d_1}(O)} \le \varepsilon\|u\|_{H^{\gamma_1,\theta}_{p,d_1}(O)} + N(\gamma,p,\varepsilon)\|u\|_{H^{\gamma_0,\theta}_{p,d_1}(O)},$$

$$\|u\|_{H^{\gamma,\theta}_{p,d_1}(O)} \le \varepsilon\|u\|_{H^{\gamma,\theta_0}_{p,d_1}(O)} + N(\gamma,p,\varepsilon)\|u\|_{H^{\gamma,\theta_1}_{p,d_1}(O)}.$$

Lemma 2.2. (i) let $s=|\gamma|$ if γ is an integer, and $s>|\gamma|$ otherwise, then

$$\|au\|_{H^{\gamma,\theta}_{p,d_1}(O)} \le N(d,s,\gamma)|a|^{(0)}_s\|u\|_{H^{\gamma,\theta}_{p,d_1}(O)}.$$

(ii) If $\gamma=0,1,2,\cdots$, then

$$\|au\|_{H^{\gamma,\theta}_{p,d_1}(O)} \le N\left(\sup_O|a|\right)\|u\|_{H^{\gamma,\theta}_{p,d_1}(O)} + N_0|a|^{(0)}_\gamma\|u\|_{H^{\gamma-1,\theta}_{p,d_1}(O)},$$

where $N_0=0$ if $\gamma=0$.

(iii) If $0\le r\le s$, then

$$|a|^{(0)}_r \le N(d,r,s)\left(\sup_O|a|\right)^{1-r/s}\left(|a|^{(0)}_s\right)^{r/s}.$$

The assertions also holds for ℓ_2-valued functions a (see [18, 19]).

Remark 2.3. By Lemma 2.2 for any $v\ge 0$, ψ^{v} is a point-wise multiplier in $H^{\gamma,\theta}_{p,1}(O)$. Thus if $\theta_1\le\theta_2$ then

$$\|u\|_{H^{\gamma,\theta_2}_{p,d_1}(O)} \le N\|\psi^{(\theta_2-\theta_1)/p}u\|_{H^{\gamma,\theta_1}_{p,d_1}(O)} \le N\|u\|_{H^{\gamma,\theta_1}_{p,d_1}(O)}.$$

Lemma 2.4. Let $\{\xi_n\}$ be a sequence of $C^\infty_0(O)$ functions such that

$$|D^m\xi_n|\le C(m)e^{mn},\quad \mathrm{supp}\,\xi_n\subset\{x\in O: e^{-n-k_0}<\rho(x)<e^{-n+k_0}\}$$

for some $k_0>0$. Then for any $u\in H^{\gamma,\theta}_{p,d_1}(O)$,

$$\sum_n e^{n\theta}\|\xi_{-n}(e^n x)u(e^n x)\|^p_{H^{\gamma}_{p,d_1}} \le N\|u\|^p_{H^{\gamma,\theta}_{p,d_1}(O)}.$$

If in addition

$$\sum_n |\xi_n|^p > \delta > 0,$$

then the inverse inequality also holds(see Theorem 2.2 in [19]).

For $0 < T < \infty$, we also use some common notation from Stochastic partial differential equations which were not mentioned here.

$$\mathbb{H}^{\gamma}_{p,d_1}(T) = \mathbb{H}^{\gamma}_{p,d_1}(\mathbb{R}^d, T) = L_p\left(\Omega\times(0,T], \mathcal{P}, H^{\gamma}_{p,d_1}\right),$$

$$\mathbb{H}^{\gamma}_{p,d_1}(T,\ell_2) = L_p\left(\Omega\times(0,T], \mathcal{P}, H^{\gamma}_{p,d_1}(\ell_2)\right), U^{\gamma}_{p,d_1} = L_p\left(\Omega, \mathcal{F}_0, H^{\gamma-2/p}_{p,d_1}\right),$$

$$\mathbb{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},T) = L_p\left(\Omega\times(0,T], \mathcal{P}, H^{\gamma,\theta}_{p,d_1}(\mathcal{O})\right),$$

$$\mathbb{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},T,\ell_2) = L_p\left(\Omega\times(0,T], \mathcal{P}, H^{\gamma,\theta}_{p,d_1}(\mathcal{O},\ell_2)\right),$$

$$\mathbb{L}_{0,d_1,\theta}(\mathcal{O},T) = \mathbb{H}^{0,\theta}_{p,d_1}(\mathcal{O},T), U^{\gamma,\theta}_{p,d_1}(\mathcal{O}) = \psi^{1-\frac{2}{p}} L_p(\Omega, \mathcal{F}_0, H^{\gamma-2/p}_{p,d_1}(\mathcal{O})),$$

$$\|u\|^p_{\mathbb{H}^{\gamma}_{p,d_1}(T)} = E\int_0^T \|u\|^p_{H^{\gamma}_{p,d_1}}\,\mathrm{d}t, \quad \|g\|^p_{\mathbb{H}^{\gamma}_{p,d_1}(T,\ell_2)} = E\int_0^T \|u\|^p_{H^{\gamma}_{p,d_1}(\ell_2)}\,\mathrm{d}t,$$

$$\|u\|^p_{\mathbb{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)} = E\int_0^T \|u\|^p_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O})}\,\mathrm{d}t, \quad \|g\|^p_{\mathbb{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)} = E\int_0^T \|u\|^p_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O},\ell_2)}\,\mathrm{d}t.$$

Finally, we show the following Banach space $\mathcal{H}^{\gamma+2}_{p,d_1}(T)$ which is modified from the $\mathbb{R}$-valued version in [7] to the $\mathbb{R}^{d_1}$-valued version.

Definition 2.5. For a D-valued function $u = (u^1, u^2, \cdots, u^{d_1}) \in \mathcal{H}^{\gamma+2}_{p,d_1}(T)$, we say

$u \in \mathcal{H}^{\gamma+2}_{p,d_1}(T)$ if $u \in \psi\mathbb{H}^{\gamma+2}_{p,d_1}(\mathcal{O},T)$, $u(0,\cdot) \in U^{\gamma+2}_{p,d_1}(\mathcal{O})$, and there exist $f \in \psi^{-1}\mathbb{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)$, $g \in \mathbb{H}^{\gamma+1}_{p,d_1}(\mathcal{O},T,\ell_2)$ such that, for any $\phi \in C_0^\infty$

$$\left(u^k(t,\cdot),\phi\right) = \left(u^k(0,\cdot),\phi\right) + \int_0^t \left(f^k(s,\cdot),\phi\right)\mathrm{d}s + \sum_{m=1}^{\infty}\int_0^t \left(g^k_m(s,\cdot),\phi\right)\mathrm{d}w^m_s, \; k = 1,\cdots,d_1. \quad (9)$$

holds for all $t \in [0,T]$ with probability 1. In this case, we write $f = \mathbb{D}u$ and $g = \mathbb{S}u$. The norm of u in $\mathcal{H}^{\gamma+2}_{p,d_1}(T)$ is written as

$$\|u\|_{\mathcal{H}^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)} = \|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)} + \|\psi\mathbb{D}u\|_{\mathbb{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)} + \|\mathbb{S}u\|_{\mathbb{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)} + \|u(0,\cdot)\|_{U^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})}. \quad (10)$$

We also define $H^{\gamma+2,\theta}_{p,d_1,0}(\mathcal{O},T)=H^{\gamma+2}_{p,d_1}(\mathcal{O},T)\bigcap\{u\,|\,u(0,\cdot)=0\}$. Equation (9) can be written in the following simplified ways

$$\mathrm{d}u = f\mathrm{d}t + g_m \mathrm{d}w_t^m ,$$

and we say that $du = fdt + g_m dw_t^m$ holds in the sense of distributions.

Remark 2.6. (i) Remember that for any $\alpha,\gamma\in\mathbb{R}, \|\psi^\alpha u\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O})} \sim \|u\|_{H^{\gamma,\theta+p\alpha}_{p,d_1}(\mathcal{O})}$

Thus, the space $H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)$ is independent of the choice of ψ.

(ii)It is easy to check (see Remark 3.2 of [7] for details) that for any $\phi\in C_0^\infty(\mathcal{O})$ and $g\in H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)$, we have $\sum_{k=1}^\infty\int_0^T (g^k,\phi)^2 ds<\infty$, and therefore the series of stochastic integral $\sum_{k=1}^\infty\int_0^t (g^k,\phi)^2 dw_t^k$ converges in probability uniformly on $[0,T]$.

Lemma 2.7. Let $u_n\in H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T), n=1,2,\cdots$ and $\|u\|_{H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)}\le K$, where K is a finite constant. Then there exists a subsequence n_k and a function $u\in H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)$ so that

(i) u_{n_k}, $u_{n_k}(0,\cdot)$, Du_{n_k}, Su_{n_k} converges weakly to u, $u(0,\cdot)$, Du, Su in $H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)$, $U^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})$, $H^{\gamma,\theta}_{p,d_1}(\mathcal{O})$ and $H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},\ell_2)$, respectively.

(ii) For any $\phi\in C_0^\infty(\mathcal{O})$ and $t\in[0,T]$, we have $\left(u_{n_k}(t,\cdot),\phi\right)\to\left(u(t,\cdot),\phi\right)$ weakly in $L_p(\Omega)$.

Proof: The proof is identical to that of Theorem 3.11 in [7], where the theorem is proved when $\mathcal{O}=\mathbb{R}^d$, $d_1=1$. □

Lemma 2.8. (i) Let $2/p<\alpha<\beta\le 1$ and $u\in H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)$, then

$$E[\psi^{\beta-1}u]^p_{C^{\alpha/2-1/p}([0,T],H^{\gamma+2-\beta,\theta}_{p,d_1}(\mathcal{O}))}\le NT^{(\beta-\alpha)p/2}\|u\|_{H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)}$$

where $N=N(d,p,\gamma,\theta,\mathcal{O},T)$ is non-decreasing function of T. In particular for any $t\le T$,

$$\|u\|^p_{\mathbb{H}^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},t)} \le N\int_0^t \|u\|^p_{H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},s)}\,ds$$

Lemma 2.9. The space $\mathcal{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)$ and $\mathcal{H}^{\gamma,\theta}_{p,d_1,0}(\mathcal{O},T)$ are Banach space with norm (10). In addition if $t<T$, when T is a finite constant, then for $u\in \mathcal{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)$

$$\|u\|_{\mathbb{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},t)} \le N(d,T)\|u\|_{\mathcal{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},t)},\quad E\sup_{s\le t}\|u(s,\cdot)\|^p_{H^{\gamma-2,\theta}_{p,d_1}(\mathcal{O})} \le N(d,T)\|u\|_{\mathcal{H}^{\gamma,\theta}_{p,d_1}(\mathcal{O},t)}.$$

Proof: It is enough to repeat the proof of theorem 3.7 in [7], where the lemma is proved when $\mathcal{O}=\mathbb{R}^d_+$, $d_1=1$. □

This explains the sense in which Cauchy problem (1) is understood. Of course, we still need to impose the following assumption for the system (1).

Assumption 2.10. (i) The coefficients of L and Λ are measurable and dominated by a constant $K<\infty$. We also assume that the matrices $a_{kr}=(a^{ij}_{kr})$ are, perhaps, nonsymmetric and satisfy

$$a^{ij}_{kr}\lambda^i\lambda^j \ge \kappa|\lambda|^2,\ k,r=1,2,\cdots,d_1$$

for all $\lambda\in\mathbb{R}^d$ and all possible values of arguments. Here $\kappa>0$ is a fixed constant.

(ii) There exists a constant $\delta>0$ such that

$$\delta|\xi|^2 \le \xi^*_i(A^{ij}-\mathcal{A}^{ij})\xi_j,$$

where

$$A^{ij}=(a^{ij}_{kr}),\quad \mathcal{A}^{ij}=(\alpha^{ij}_{kr}),\quad \alpha^{ij}_{kr}=\frac{1}{2}\sum_{l=1}^{d_1}\left(\sigma^i_{lk},\sigma^j_{lr},\right)_{\ell_2},$$

ξ is any (real) $d_1\times d$ matrix, ξ_i is the ith column of ξ, $*$ denotes the matrix transpose, and again the summations on i,j are understood.

(iii) The coefficients a^{ij}_{kr}, σ^i_{kr} are uniformly continuous in x, that is, for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ so that for any $\omega,t>0$ ω, t > 0, i,j,k,r,

$$\left|a^{ij}_{kr}(\omega,t,x)-a^{ij}_{kr}(\omega,t,x)\right|+\left|\sigma^i_{kr}(\omega,t,x)-\sigma^i_{kr}(\omega,t,x)\right|_{\ell_2}<\varepsilon,\ \text{if}\ |x-y|<\delta.$$

Denote $\mathscr{M}^i=\left(\sigma^i_{kr}\right)$ with $\sigma^i_{kr}=\left(\sigma^i_{kr,m},m=1,2,\cdots\right)\in\ell_2$ and $\mathscr{G}^i=\mathscr{M}^i-\mathscr{C}^i$ where

$\mathscr{Q}^i$ is the diagonal part of $\mathscr{M}^i$, $\mathscr{Q}^i=\left(\delta_{kr}\sigma_{kr}^i\right)$. Using the assumption, we will elaborate our main result of the article.

Theorem 2.11. Let $u_0\in U_{2,d_1}^{\gamma+2}$, $f\in H_{2,d_1}^{\gamma}(T)$ and $g\in H_2^{\gamma+1}(T,\ell_2)$. Then under Assumption 2.10, there exists a constant $\varepsilon>0$ depending only on d_1, d, K, T and $\beta>0$ such that if

$$\sup_{\omega,t,x}\left|\mathscr{G}^i(\omega,t,x)\right|_{\ell_2}\le\varepsilon,\quad i=1,2,\cdots,d\ , \tag{11}$$

for any $\theta\in(d-\beta,d+\beta)$ the system (1) admits a unique solution $u\in H_{p,d_1}^{\gamma+2,\theta}(O,T)$ with the estimate

$$\|u\|_{H_{p,d_1}^{\gamma+2,\theta}(O,T)}\le N(\|\psi f^0\|_{H_{p,d_1}^{\gamma,\theta}(O,T)}+\sum_{i=1}^{d}\|f^i\|_{H_{p,d_1}^{\gamma+1,\theta}(O,T)}+\|g\|_{H_{p,d_1}^{\gamma+1,\theta}(O,T,\ell_2)}+\|u_0\|_{U_{p,d_1}^{\gamma+2,\theta}(O)}),$$

where N depends only on d_1, d, K, T, O.

Remark 2.12. In Remark 4.5 we will show that Theorem 2.1 can be extended to the case that $\mathscr{M}^i$s are diagonalizable by an orthogonal matrix $O(\omega,x)$. That is, the main result still holds if $O^*\mathscr{M}^iO$ is diagonal for each i, where exists $F_0\times B(\mathbb{R}_d)$-measurable $d_1\times d_1$ orthogonal matrix $O(\omega,x)$.

3. Auxiliary results

Before we start our main problem, we firstly consider the following Cauchy problem with the coefficients independent of x:

$$\begin{cases}\mathrm{d}u^k=\left(a_{kr}^{ij}D_iD_ju+D_if_k^i+f_k^0\right)\mathrm{d}t+\left(\sigma_{kr,m}^iD_iu^r+g_m^k\right)\mathrm{d}w_t^m,\\ u^k(0,\cdot)=u_0^k(\cdot).\end{cases} \tag{12}$$

The summation convention with respect to $i,j=1,2,\cdots,d$, $r=1,2,\cdots,d_1$, $m=1,2,\cdots$ is enforced which we have already mentioned previously . A W_p^γ-theory for the system (12) will be shown. We start with a theorem which easily follows from the results for single equations. Assume that A^{ij} is a $d_1\times d_1$ diagonal matrix and all entries of $\mathscr{M}^i$ are zero for each i,j. The system (12) can be written as

$$\begin{cases}\mathrm{d}u^k=\left(a_{kk}^{ij}D_iD_ju^k+D_if_k^i+f_k^0\right)\mathrm{d}t+g_m^k\mathrm{d}w_t^m,\\ u^k(0,\cdot)=u_0^k(\cdot).\end{cases} \tag{13}$$

We see that the system (13) is a set of d_1 number of independent single equations. Hence we can prove the Theorem 3.1

Theorem 3.1. Let $p\in[2,\infty)$, $\gamma\in[0,\infty)$, $T<\infty$. Then there exists $\beta>0$ for any $\theta\in(d-\beta,d+\beta)$, $u_0\in U^{\gamma+2,\theta}_{p,d_1}$, $f^0, f^i\in\psi^{-1}H^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)$ and $g\in H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)$ the system (13) with initial condition $u(0)=u_0$ has a unique solution $u\in H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)$ satisfying

$$\|u\|_{H^{\gamma+2,\theta}_{p,d_1}(T)}\le N(\|\psi f^0\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)}+\sum_{i=1}^{d}\|f^i\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T)}+\|g\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)}+\|u_0\|_{U^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})}), \tag{14}$$

where N only depends on d, d_1, p, γ, δ,κ, K, T.

Proof: By Theorem 5.1 in [7], one gets

$$\mathrm{d}u^k=\left[D_i\left(a^{ij}_{kk}D_ju^k\right)+D_if^i_k+f^0_k\right]\mathrm{d}t+g^k_m\mathrm{d}w^m_t,\quad u^k(0,\cdot)=u^k_0(\cdot)$$

has a unique solution $u^k\in H^{\gamma+2,\theta}_{p,1}(\mathcal{O},T)$ satisfying

$$\begin{aligned}\|u^k\|_{H^{\gamma+2,\theta}_{p,1}(\mathcal{O},T)}&\le N\left(\left\|\psi\left(f^0_k+\sum_{i=1}^{d}D_if^i_k\right)\right\|_{H^{\gamma,\theta}_{p,1}(\mathcal{O},T)}+\|g_k\|_{H^{\gamma+1,\theta}_{p,1}(\mathcal{O},T,\ell_2)}+\|u^k_0\|_{U^{\gamma+2,\theta}_{p,1}(\mathcal{O},)}\right)\\&\le N\left(\|\psi f^0_k\|_{H^{\gamma,\theta}_{p,1}(\mathcal{O},T)}+\sum_{i=1}^{d}\|\psi D_if^i_k\|_{H^{\gamma,\theta}_{p,1}(\mathcal{O},T)}+\|g_k\|_{H^{\gamma+1,\theta}_{p,1}(\mathcal{O},T,\ell_2)}+\|u^k_0\|_{U^{\gamma+2,\theta}_{p,1}(\mathcal{O},)}\right).\end{aligned} \tag{15}$$

Using Lemma 2.1 (iii), one gets

$$\|\psi D_if^i_k\|_{H^{\gamma,\theta}_{p,1}(\mathcal{O},T)}\le\|f^i_k\|_{H^{\gamma+1,\theta}_{p,1}(\mathcal{O},T)}. \tag{16}$$

(15) and (16) easily lead to

$$\|u^k\|_{H^{\gamma+2,\theta}_{p,1}(\mathcal{O},T)}\le N\left(\|\psi f^0_k\|_{H^{\gamma,\theta}_{p,1}(\mathcal{O},T)}+\sum_{i=1}^{d}\|f^i_k\|_{H^{\gamma+1,\theta}_{p,1}(\mathcal{O},T)}+\|g_k\|_{H^{\gamma+1,\theta}_{p,1}(\mathcal{O},T,\ell_2)}+\|u^k_0\|_{U^{\gamma+2,\theta}_{p,1}(\mathcal{O},)}\right).$$

Add up to the inequalities from 1 to d_1, one gets

$$\begin{aligned}\|u^k\|_{H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)}&\le\sum_{k=1}^{d_1}\|u^k\|_{H^{\gamma+2,\theta}_{p,1}(\mathcal{O},T)}\\&\le N\left(\|\psi f^0\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)}+\sum_{i=1}^{d}\|f^i\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T)}+\|g\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)}+\|u_0\|_{U^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},)}\right).\end{aligned}$$

Hence, the theory is proved. □

Next, we try to remove the restrictions that A^{ij}s are diagonal.

Theorem 3.2. The system (12) has a unique solution $u\in H^{\gamma+2}_{p,d_1}(\mathcal{O},T)$ with the estimate

(14) even if we remove the assumption that A^{ij} is a diagonal matrix for each i, j.

Proof: Firstly, we assume $\gamma = 0$. For each fixed ω the system

$$\begin{cases} \mathrm{d}w = \left[(A^{ij} - \delta_{ij} I) D_i D_j w\right] \mathrm{d}t, \\ w(0) = u_0 \end{cases} \tag{17}$$

is a deterministic system. Hence, by Section 10, Chapter 7 in [4], the system (17) has a unique solution $w \in H^{2,\theta}_{p,d_1}(O, T)$ with

$$\|w\|_{H^{2,\theta}_{p,d_1}(T)} \le N \|u_0\|_{U^{2,\theta}_{p,d_1}(O)}. \tag{18}$$

Now, we introduce a change of variables to transform our problem into a set of d_1 number of independent single equations. If we let

$$v(x,t) = u(x,t) - w(x,t).$$

Then, v will satisfy the following stochastic partial differential equation

$$\begin{cases} \mathrm{d}v^k = \left(a^{ij}_{kk} D_i D_j v^k + D_i f^i_k + f^0_k\right) \mathrm{d}t + g^k_m \mathrm{d}w^m_t, \\ v(0) = 0. \end{cases} \tag{19}$$

By Theorem 3.1, Cauchy problem (19) admits the unique solution $v \in H^{2,\theta}_{p,d_1}(O, T)$ with the estimate

$$\|v\|_{H^{2,\theta}_{p,d_1}(T)} \le N\left(\|\psi f^0\|_{L_{p,d_1,\theta}(O,T)} + \sum_{i=1}^{d} \|f^i_k\|_{H^{1,\theta}_{p,d_1}(O,T)} + \|g\|_{H^{1}_{p,d_1,\theta}(O,T,\ell_2)} \right). \tag{20}$$

Hence, $u = v + w$ is the unique solution of our Cauchy problem and one gets the estimate (14) for $\gamma = 0$ by combining (18) and (20).

Secondly, we prove that the claim of Theorem 3.1 holds even if $\gamma \ne 0$. It is clear that $(1-\Delta)^{\mu/2} : H^{\gamma,\theta}_{p,d_1} \to H^{\gamma-\mu,\theta}_{p,d_1}$ is an isometry for any $\gamma, \mu \in \mathbb{R}$ when $p \in (2, \infty)$. Hence, $u \in H^{\gamma+2,\theta}_{p,d_1}(O, T)$ is a solution of system (12) if and only if $\bar{u} := (1-\Delta)^{\gamma/2} u \in H^{2,\theta}_{p,d_1}(O, T)$ is a solution of system (12) when $(1-\Delta)^{\gamma/2} f^0$, $(1-\Delta)^{\gamma/2} f^i$, $i = 1, 2, \cdots, d$, $(1-\Delta)^{\gamma/2} g$, $(1-\Delta)^{\gamma/2} u_0$ is used instead of f^0, f^i, g, u_0 respectively. And we have already proved that $\bar{u}$ is the unique solution in the case $\gamma = 0$. Since, one gets

$$
\begin{aligned}
&\|u\|_{H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)} = \|\bar{u}\|_{H^{2,\theta}_{p,d_1}(\mathcal{O},T)} \\
&\le N\left(\left\|\psi(1-\Delta)^{\gamma/2} f^0\right\|_{L_{p,d_1,\theta}(\mathcal{O},T)} + \sum_{i=1}^{d}\left\|(1-\Delta)^{\gamma/2} f^i_k\right\|_{H^{1,\theta}_{p,d_1}(\mathcal{O},T)} + \left\|(1-\Delta)^{\gamma/2} g\right\|_{H^{1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)} + \left\|(1-\Delta)^{\gamma/2} u_0\right\|_{U^{2,\theta}_{p,d_1}(\mathcal{O})} \right) \\
&= N\left(\left\|\psi f^0\right\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)} + \sum_{i=1}^{d}\left\| f^i\right\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T)} + \|g_k\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)} + \|u^k_0\|_{U^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})} \right).
\end{aligned}
$$

The theorem is proved. □

Previously, we always assume that $\mathscr{M}^i = 0$. Now, we try to weaken it. Recall that $\mathscr{G}^i = \mathscr{M}^i - \mathscr{Q}^i$ where $\mathscr{Q}_d^i$ is the diagonal part of $\mathscr{M}^i$, $\mathscr{Q}^i = \left(\delta_{kr}\sigma^i_{kr}\right)$.

Theorem 3.3. Assume that if $\mathscr{M}^i$ s are diagonal matrices. There exists a coefficient $\beta > 0$, the system (12) admits a unique solution $u \in H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)$ and the estimate (14) holds for all $\theta \in (d-\theta, d+\theta)$.

Proof: As in the proof of Theorem 3.2 we may assume $\gamma = 0$. According to the assumption, one gets $\mathscr{M}^i(t) = (\sigma^i_{kr}\delta_{kr})$ and system (12) can be changed to

$$
\mathrm{d}u^k = \left(a^{ij}_{kr} D_i D_j u^r + f^0_k + D_i f^i_k\right)\mathrm{d}t + \left(\sigma^i_{kk,m} D_i u^k + g^k_m\right)\mathrm{d}w^m_t, \tag{21}
$$

with the initial condition $u^k(x,0) = u^k_0(x), k = 1,2,\cdots,d_1$. Define the process $x^{ik}_t = \sum_{m=1}^{\infty}\int_0^t \sigma^i_{kk,m}(s)dw^m_s$ for each i,k and $x^k_t = (x^{1k}_t, x^{2k}_t, \cdots, x^{dk}_t)$. Also, we define

$$
\bar{u}^k(t,x) = u^k(t, x - x^k_t),\ \bar{u}^k_0(x) = u^k_0(x - x^k_t),\quad \bar{f}^0_k(t,x) = f^0_k(t, x - x^k_t),
$$

$$
\bar{f}^i_k(t,x) = f^i_k(t, x - x^k_t),\qquad \bar{g}^k(t,x) = g^k(t, x - x^k_t).
$$

Using the Itô-Wentzell formula (see Lemma 4.7 in [7]), system (21) can be written as

$$
\begin{cases}
\mathrm{d}\bar{u}^k = \left[\left(a^{ij}_{kr} - \dfrac{1}{2}(\sigma^i_{kk}, \sigma^j_{kk})_{\ell_2}\delta_{kr}\right) D_i D_j \bar{u}^r + \bar{f}^0_k + D_i \bar{f}^i_k - (\sigma^i_{kk}, D_i \bar{g}^k)_{\ell_2}\right]\mathrm{d}t + \bar{g}^k_m \mathrm{d}w^m_t \\
\bar{u}^k(0,x) = \bar{u}^k_0(x),\ k = 1,2,\cdots,d_1,
\end{cases}
$$

or

$$\begin{cases} \mathrm{d}\bar{u} = \left[\left(A^{ij} - \frac{1}{2}(\mathscr{M}^i)^* \mathscr{M}^j \right) D_i D_j \bar{u} + \bar{f}^0 + D_i \bar{f}^i - \mathscr{M}^i D_i \bar{g} \right] \mathrm{d}t + \bar{g}_m \mathrm{d}w_t^m, \\ \bar{u}(0,x) = \bar{u}_0(x), k = 1, 2, \cdots, d_1, \end{cases} \tag{22}$$

where $\bar{u} = (\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^{d_1})$. By Theorem 3.2, the problem (22) has a unique solution $\bar{u} \in H_{p,d_1}^{2,\theta}(O,T)$ with

$$\begin{aligned} \|\bar{u}\|_{H_{p,d_1}^{2,\theta}(T)} &\le N \left(\left\| \psi \left(\bar{f}^0 + D_i \bar{f}^i - \mathscr{M}^i D_i \bar{g} \right) \right\|_{L_{p,d_1,\theta}(O,T)} + \|\bar{g}\|_{H_{p,d_1}^{1,\theta}(O,T,\ell_2)} + \|\bar{u}_0\|_{U_{p,d_1}^{2,\theta}} \right) \\ &\le N \left(\left\| \psi \left(\bar{f}^0 + \sum_{i=1}^d D_i \bar{f}^i \right) \right\|_{L_{p,d_1,\theta}(O,T)} + \left\| \psi \mathscr{M}^i D_i \bar{g} \right\|_{L_{p,d_1,\theta}(O,T)} + \|\bar{g}\|_{H_{p,d_1}^{1,\theta}(O,T,\ell_2)} + \|\bar{u}_0\|_{U_{p,d_1}^{2,\theta}} \right). \end{aligned} \tag{23}$$

Using Lemma 2.1 (iii), one gets

$$\left\| \psi \mathscr{M}^i D_i \bar{g} \right\|_{L_{p,d_1,\theta}(O,T,\ell_2)} \le K \left\| \psi D_i \bar{g} \right\|_{L_{p,d_1,\theta}(O,T,\ell_2)} \le K \|\bar{g}\|_{H_{p,d_1}^{1,\theta}(O,T,\ell_2)}. \tag{24}$$

Let $N_1 = \max\{N, KN\}$. (16), (23) and (24) easily lead to

$$\|\bar{u}\|_{H_{p,d_1}^{2,\theta}(T)} \le N_1 \left(\left\| \psi \bar{f}^0 \right\|_{L_{p,d_1,\theta}(O,T)} + \sum_{i=1}^d \left\| \bar{f}^i \right\|_{H_{p,d_1}^{1,\theta}(O,T,\ell_2)} + \|\bar{g}\|_{H_{p,d_1}^{1,\theta}(O,T,\ell_2)} + \|\bar{u}_0\|_{U_{p,d_1}^{2,\theta}} \right).$$

According to the definition of Bessel potential space and its norm, The upper inequality can be easily changed to the estimate (14). Hence the theory is proved. □

In the next theory, we only assume that $\mathscr{M}^i$s are close to diagonal matrices which satisfying: there exists a constant $\varepsilon > 0$ depending on d_1, p, γ, δ, κ, K, T such that for any $\omega \in \Omega$, $t \in [0,T]$ one gets

$$k_0 := \sup_{\substack{\omega \in \Omega, 1 \le i \le d, \\ t \in [0,T]}} \left| \mathscr{G}^i(\omega, t) \right| \le \varepsilon. \tag{25}$$

Theorem 3.4. Under the assumption (25), there exists a coefficient $\beta > 0$, the system (12) admits a unique solution $u \in H_{p,d_1}^{\gamma+2,\theta}(O,T)$, and the estimate (14) holds for all $\theta \in (d-\beta, d+\beta)$.

Proof: As in the proof of Theorem 3.2 we may assume $\gamma = 0$. Also, as usual we assume $u_0 = 0$ (see the proof of Theorem 5.1 in [7]). By Assumption 2.10, for any $d_1 \times d$ matrix ξ, one gets

$$2K|\xi|^2 \ge \xi_i^* \left(A^{ij} - \frac{1}{2}(\mathscr{M}^i)^* \mathscr{M}^j \right) \xi_j \ge \frac{\delta}{2} |\xi|^2$$

and on the other hand

$$\xi_i^*\left(A^{ij}-\frac{1}{2}(\mathscr{M}^i)^*\mathscr{M}^j\right)\xi_j$$
$$=\xi_i^*\left(A^{ij}-\frac{1}{2}(\mathscr{C}^i)^*\mathscr{C}^i\right)\xi_j-\frac{1}{2}\xi_i^*\left((\mathscr{C}^i)^*\mathscr{G}^i+(\mathscr{G}^i)^*\mathscr{C}^i+(\mathscr{G}^i)^*\mathscr{G}^i\right)\xi_j.$$

There exists a constant $\varepsilon_1>0$ such that if $k_0\le\varepsilon_1$, then

$$4K|\xi|^2\ge\xi_i^*\left(A^{ij}-\frac{1}{2}(\mathscr{M}^i)^*\mathscr{M}^j\right)\xi_j\ge\frac{\delta}{2}|\xi|^2$$

can be hold. Hence, by the result of Theorem 3.3, for each $u\in H^{2,\theta}_{p,d_1,0}(O,T)$ one can define $v:=\mathscr{R}u\in H^{2,\theta}_{p,d_1,0}(O,T)$ as the solution of

$$\begin{cases}\mathrm{d}v=\left(A^{ij}D_iD_jv+\overline{f}^{\,0}+D_i\overline{f}^{\,i}\right)\mathrm{d}t+(\mathscr{C}_m^{\,i}D_iu+\mathscr{G}_m^{\,i}D_iu+g_m)\mathrm{d}w_t^m,\\ v(0)=0.\end{cases}\tag{26}$$

If we assume $k_0\le\varepsilon_1$, the map $\mathscr{R}:H^{2,\theta}_{p,d_1,0}(O,T)\to H^{2,\theta}_{p,d_1}(O,T)$ is well defined and bounded. We plan to show that $\mathscr{R}^n$ is a contraction for some large integer n with a further restriction on k_0. Note that for $t\le T$ and any $u_1,u_2\in H^{2,\theta}_{p,d_1,0}(O,T)$ one gets

$$\|\mathscr{R}u_1-\mathscr{R}u_2\|^p_{H^{2,\theta}_{p,d_1}(O,T)}\le N_0\left\|\mathscr{G}^iD_i(u_1-u_2)\right\|^p_{H^{1,\theta}_{p,d_1}(O,t,\ell_2)},$$

where N_0 depends only on d_1, p, γ, δ, K, T. By the inequality $\|Du\|_{1,p}\le N(\|u_{xx}\|_p+\|u\|_p)$, for each t one gets

$$\begin{aligned}\left\|\mathscr{G}_e^{\,i}D_i(u_1-u_2)\right\|^p_{H^{1,\theta}_{p,d_1}(O,t,\ell_2)}&\le Nk_0^p\|(u_1-u_2)_{xx}\|^p_{L_{p,d_1,\theta}(O,T)}+Nk_0^p\|u_1-u_2\|^p_{L_{p,d_1,\theta}(O,T)}\\&\le Nk_0^p\|(u_1-u_2)_{xx}\|^p_{H^{2,\theta}_{p,d_1}(O,t)}+Nk_0^p\|u_1-u_2\|^p_{L_{p,d_1,\theta}(O,T)}.\end{aligned}$$

By the Lemma 2.8, one gets

$$\|u^k\|^p_{H^{\gamma+1,\theta}_{p,1}(O,t)}\le N\int_0^t\|u^k\|^p_{H^{\gamma+1,\theta}_{p,1}(O,s)}\,\mathrm{d}s,\ k=1,2,\cdots,d_1.$$

Hence, it follows that

$$\|u\|^p_{H^{\gamma+1,\theta}_{p,d_1}(O,t)}=\sum_{k=1}^{d_1}\|u^k\|^p_{H^{\gamma+1,\theta}_{p,1}(O,t)}\le N\sum_{k=1}^{d_1}\int_0^t\|u^k\|^p_{H^{\gamma+1,\theta}_{p,1}(O,s)}\,\mathrm{d}s=\int_0^t\|u^k\|^p_{H^{\gamma+1,\theta}_{p,d_1}(O,s)}\,\mathrm{d}s.\tag{27}$$

By (27), one gets

$$\begin{aligned}
&\left\|\mathscr{R} u_1-\mathscr{R} u_2\right\|_{H_{p,d_1}^{2,\theta}(O,t)}^p \\
&\leq N_0 N k_0^p\left\|u_1-u_2\right\|_{H_{p,d_1}^{2,\theta}(O,t)}^p+N_0 N k_0^p \int_0^t E\left\|u_1(s)-u_2(s)\right\|_{L_p}^p \mathrm{d}s \\
&\leq N_0 N k_0^p\left\|u_1-u_2\right\|_{H_{p,d_1}^{2,\theta}(O,t)}^p+N_0 N k_0^p N_1 \int_0^t E\left\|u_1-u_2\right\|_{H_{p,d_1}^{2,\theta}(O,s)} \mathrm{d}s,
\end{aligned} \tag{28}$$

where N_1 depends only on d_1, d, p, T. Hence, by introduction,

$$\begin{aligned}
&\left\|\mathscr{R}^n u_1-\mathscr{R}^n u_2\right\|_{H_{p,d_1}^{2,\theta}(O,t)}^p \leq (N_0 N k_0^p)^n\left\|u_1-u_2\right\|_{H_{p,d_1}^{2,\theta}(O,t)}^p \\
&+\sum_{k=1}^n \binom{n}{k}(N_0 N k_0^p)^{n-k}(N_0 N k_0^p N_1)^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!}\left\|u_1-u_2\right\|_{H_{p,d_1}^{2,\theta}(O,s)} \mathrm{d}s \left\|\mathscr{R}^n u_1-\mathscr{R}^n u_2\right\|_{H_{p,d_1}^{2,\theta}(O,T)}^p \\
&\leq \sum_{k=0}^n \binom{n}{k}(N_0 N k_0^p)^{n-k}(T N_0 N k_0^p N_1)^k / k!\left\|u_1-u_2\right\|_{H_{p,d_1}^{2,\theta}(O,T)} \\
&\leq 2^n (N_0 N k_0^p)^n \max_k\left\{(T N_1)^k / k!\right\}\left\|u_1-u_2\right\|_{H_{p,d_1}^{2,\theta}(O,T)}.
\end{aligned}$$

This allows us to find k_0 depending only on d, p, δ, κ and K, so that the operator $\mathscr{R}^n$ is a contraction with coefficient $1/2$. Of course, this yields all our assertion. The theorem is proved. □

For the last theory of Cauchy problem (12), we remove the restriction that $\mathscr{M}^i$s are close to diagonal matrices. Assume $\bar{\mathscr{M}}^i(\omega,t)=\mathscr{T}^*(\omega)\mathscr{M}^i(\omega,t)\mathscr{T}(\omega)$ for some F_0-measurable orthogonal matrix $\mathscr{T}$, where $A^{ij}(\omega,t)$, $\bar{\mathscr{M}}^i(\omega,t)$ s satisfy assumption 2.10, $\mathscr{M}^i$ s satisfy the condition (25) in Theorem 3.4. Hence the following theory holds.

Corollary 3.5. Then the assertion of Theorem 3.3 holds for the system (6) with $\bar{\mathscr{M}}^i$ in place of $\mathscr{M}^i$, $i=1,2,\cdots,d$.

Proof: Firstly we consider the following problem

$$\mathrm{d}v=\left((\mathscr{T} A^{ij}\mathscr{T}^*)D_i D_j v+\mathscr{T} f^0+\mathscr{T} D_i f^i\right)\mathrm{d}t+\left(\mathscr{M}_m^i D_i v+\mathscr{T} g_m\right)\mathrm{d}w_t^m,\ v(0)=\boldsymbol{u}_0.$$

Because $\mathscr{T}$ is orthogonal, $(\mathscr{T} A^{ij}\mathscr{T}^*)$ and $\mathscr{M}^i$ ($i,j=1,2,\cdots,d$) satisfy the same conditions which A^{ij} and $\bar{\mathscr{M}}^i$ satisfy. Note that ε in (25) is independent of the choice of $\mathscr{T}$. By Theorem 3.4 there exists a unique solution $v\in H_{p,d_1}^{\gamma+2}(O,T)$ satisfying

$$\begin{aligned}\|v\|_{H^{\gamma+2}_{p,d_1}(\mathcal{O},T)} &\le N\left(\left\|\mathcal{T}\psi f^0+\mathcal{T}\psi\sum_{i=1}^{d}D_i f^i\right\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)}+\|\mathcal{T}g\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)}+\|\mathcal{T}u_0\|_{U^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})}\right)\\ &\le N\left(\|\mathcal{T}\psi f^0\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)}+\left\|\mathcal{T}\psi\sum_{i=1}^{d}D_i f^i\right\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)}+\|\mathcal{T}g\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)}+\|\mathcal{T}u_0\|_{U^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})}\right)\\ &\le N\left(\|\psi f^0\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)}+\sum_{i=1}^{d}\|f^i\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T)}+\|g\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)}+\|u_0\|_{U^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})}\right).\end{aligned} \tag{29}$$

Secondly, we define $u=\mathcal{T}^*v$. It is clear that u is the unique solution of

$$\begin{cases}\mathrm{d}u=\left(A^{ij}D_iD_ju+\overline{f}^0+D_i\overline{f}^i\right)\mathrm{d}t+(\mathcal{M}_m^iD_iu+g_m)\mathrm{d}w_t^m,\\ u(0)=\boldsymbol{u}_0\end{cases}$$

and the estimate (14) follows from (29). Hence the corollary is proved. □

4. Proof of Theorem 2.11

We closely follow the proof of Theorem 5.10 of [7]. As usual we assume $u_0=0$. For simple, let's us define

$$(f,g)\in \mathbb{F}_p^{\gamma}(\mathcal{O},T)=\psi^{-1}H^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)\times H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2),$$

and

$$\|(f,g)\|_{\mathbb{F}_p^{\gamma}(\mathcal{O},T)}=\|\psi f\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O},T)}+\|g\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},T,\ell_2)}.$$

Definition 4.1. Assume that for $\omega\in\Omega$ and $t\ge 0$, we give the following operators

$$L(\cdot,t):H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})\to H^{\gamma,\theta}_{p,d_1}(\mathcal{O}),\quad \Lambda(\cdot,t):H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})\to H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},\ell_2),$$

where $L(\cdot,t)=(L_1(\cdot,t),L_2(\cdot,t),\cdots,L_{d_1}(\cdot,t))$, $\Lambda(\cdot,t)=(\Lambda_1(\cdot,t),\Lambda_2(\cdot,t),\cdots,\Lambda_{d_1}(\cdot,t))$.

Assume that

(i) For any ω and t, the operators $L(u,t)$ and $\Lambda(u,t)$ are continuous with respect to u.

(ii) For any $u\in H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})$, the operators $L(u,t)$ and $\Lambda(u,t)$ are predictable.

(iii) For any $\omega\in\Omega$, $t\ge 0$ and $u\in H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})$, one gets

$$\|\psi L(u,t)\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O})}+\|\Lambda(u,t)\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O},\ell_2)}\le N_{L,\Lambda}(1+\|u\|_{H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})}),$$

where $N_{L,\Lambda}$ is a constant.

Then for a $\mathbb{R}^d$ valued function $u=(u_1,u_2,\cdots,u_{d_1})\in H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})$, we write

$$(L,\Lambda)u=-(f,g),$$

if $(f,g)\in \mathcal{F}_p^{\gamma}(\mathcal{O},T)$, and by the virtue of definition, for $t\in[0,T]$, one gets $\mathbb{D}u=L(u,t)+f$, $\mathbb{S}u=\Lambda(u,t)+g$.

Remark 4.2. According to our condition on L and Λ, one gets $(L(u,t),\Lambda(u,t))\in \mathcal{F}_p^{\gamma}(\mathcal{O},T)$ for any $u\in \mathcal{H}^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)$. Also, $(L,\Lambda)u=(L(u,t)-\mathbb{D}u,\Lambda(u,t)-\mathbb{S}u)$. In particular, the operator (L,Λ) is well defined on $\mathcal{H}^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)$, and , as follows easily from Definition 4.1 (iii)

$$\|(L,\Lambda)u\|_{\mathcal{F}_p^{\gamma}(\mathcal{O},T)}\le(1+2N_{L,\Lambda})\|u\|_{\mathcal{H}^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)}+2N_{L,\Lambda}T^{1/p}.$$

In term of Definition 4.1, Theorem 3.4 can be written with the following version

Theorem 4.3. Let a and σ satisfy the assumptions from the beginning section 2. Define

$$L^0(\cdot,t)=(L^0_1(\cdot,t),L^0_2(\cdot,t),\cdots,L^0_{d_1}(\cdot,t)),\ L^0_k=a^{ij}_{kr}D_iD_ju^r,$$

$$\Lambda^0(\cdot,t)=(\Lambda^{m,0}_1(\cdot,t),\Lambda^{m,0}_2(\cdot,t),\cdots,\Lambda^{m,0}_{d_1}(\cdot,t)),\quad \Lambda^0_{k,m}=\sigma^i_{kr,m}D_iu^r.$$

Then the operator (L^0,Λ^0) is a one-to-one operator from $\mathcal{H}^{\gamma+2,\theta}_{p,d_1}(\mathcal{O},T)$ to $\mathcal{F}_p^{\gamma}(\mathcal{O},T)$ and the norm of its inverse is less than a constant depending only on d,p,δ,κ and K (thus independent of T).

Next, we prove a perturbation result because we do not allow ε depending on T.

Theorem 4.4. Take the operators L^0 and Λ^0 from Theorem 4.3, and let some operators L^1 and Λ^1 satisfy the requirements from Definition 4.1. We assert that there exists a constant $\varepsilon\in(0,1)$ if, for a constant K_1 and any $u,v\in H^{\gamma+2,\theta}_{p,d_1}(\mathcal{O})$, one gets

$$\begin{aligned}&\left\|\psi\left(L^1(u,t)-L^1(v,t)\right)\right\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O})}+\left\|\Lambda^1(u,t)-\Lambda^1(v,t)\right\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O})}\\&\le\varepsilon\left\|u_{xx}-v_{xx}\right\|_{H^{\gamma,\theta}_{p,d_1}(\mathcal{O})}+K_1\|u-v\|_{H^{\gamma+1,\theta}_{p,d_1}(\mathcal{O})},\end{aligned}\tag{30}$$

then, for any $(f,g)\in \mathbb{F}_p^{\gamma}(\mathcal{O},T)$, there admits a unique solution $u\in \mathcal{H}_{p,d_1}^{\gamma+2,\theta}(\mathcal{O},T)$ of the system

$$(L^0+L^1,\Lambda^0+\Lambda^1)u=-(f,g)\,, \tag{31}$$

where ε depending only on d,p,δ,κ and K. Furthermore, for this solution u, there exists a constant $\beta>0$ for any $\theta\in(d-\beta,d+\beta)$ satisfying

$$\|u\|_{\mathcal{H}_{p,d_1}^{\gamma+2,\theta}(\mathcal{O},T)}\le N\left\|(L^1(\cdot,0)+f,\Lambda^1(\cdot,0)+g)\right\|_{\mathbb{F}_p^{\gamma}(\mathcal{O},T)}, \tag{32}$$

where N depends only on d,p,δ,κ,K,K_1 and T (N is dependent of T if $K_1=0$).

Proof: Firstly, by Lemma 2.1 (v), one gets

$$\|u\|_{H_{p,d_1}^{\gamma+1,\theta}(\mathcal{O})}\le\varepsilon\|u_{xx}\|_{H_{p,d_1}^{\gamma,\theta}(\mathcal{O})}+N(\varepsilon,d,p)\|u\|_{H_{p,d_1}^{\gamma,\theta}(\mathcal{O})}.$$

Therefore without loss of generality we use the following inequality instead of (30)

$$\begin{aligned}&\left\|L^1(u,t)-L^1(v,t)\right\|_{H_{p,d_1}^{\gamma,\theta}(\mathcal{O})}+\left\|\Lambda^1(u,t)-\Lambda^1(v,t)\right\|_{H_{p,d_1}^{\gamma+1,\theta}(\mathcal{O})}\\&\le\varepsilon\|u_{xx}-v_{xx}\|_{H_{p,d_1}^{\gamma,\theta}(\mathcal{O})}+K_1\|u-v\|_{H_{p,d_1}^{\gamma,\theta}(\mathcal{O})}.\end{aligned}$$

Now fix $(f,g)\in \mathbb{F}_p^{\gamma}(\mathcal{O},T)$. Take $u\in \mathcal{H}_{p,d_1,0}^{\gamma+2,\theta}(\mathcal{O},T)$, note that $(L^1(u),\Lambda^1(u))\in \mathbb{F}_p^{\gamma}(\mathcal{O},T)$ and, by using Theorem 4.3, define $v\in \mathcal{H}_{p,d_1,0}^{\gamma+2,\theta}(\mathcal{O},T)$ as the unique solution of the equation

$$(L^0,\Lambda^0)v=-(f+L_1(u),g+\Lambda_1(u))\,.$$

By denoting $v=\mathcal{R}u$, we defining an operator $\mathcal{R}:\mathcal{H}_{p,d_1}^{\gamma+2}(\mathcal{O},T)\to \mathcal{H}_{p,d_1}^{\gamma+2}(\mathcal{O},T)$. System (31) is equivalent to the equation $u=\mathcal{R}u$. Therefore, to prove the existence and uniqueness of solutions to (31), we only need to show that, for an integer $n>0$, the operator $\mathcal{R}^n$ is a contraction in $\mathcal{H}_{p,d_1,0}^{\gamma+2}(\mathcal{O},T)$. By using the Theorem 4.3 and Minkowski inequality, for $t\le T$,

$$\begin{aligned}\|\mathcal{R}u-\mathcal{R}v\|^p_{\mathcal{H}_{p,d_1}^{\gamma+2,\theta}(\mathcal{O},t)}&\le N\left\|L^1(u,t)-L^1(v,t),\Lambda^1(u,t)-\Lambda^1(v,t)\right\|^p_{\mathbb{F}_p^{\gamma}(\mathcal{O},T)}\\&\le N_0\varepsilon^p\|u-v\|^p_{\mathcal{H}_{p,d_1}^{\gamma+2,\theta}(\mathcal{O},t)}+N_0K_1^p\int_0^t E\|u(s)-v(s)\|^p_{H_{p,d_1}^{\gamma,\theta}(\mathcal{O})}\,\mathrm{d}s,\end{aligned}$$

where $N_0=N/p$. This gives the desired result if $K_1=0$. Also in this case estimate (32) follows obviously with N independent of T.

When $K_1 \neq 0$, by Lemma 2.9, one gets

$$E\|u(s)-v(s)\|^p_{H^{\gamma+2,\theta}_{p,d_1}(O)} \leq N_1\|u-v\|^p_{H^{\gamma+2,\theta}_{p,d_1}(O,s)},$$

where $s \leq T$ and N_1 depends only on d, p and T. Define $\mu := N_0\varepsilon^p$. Hence for all $t \leq T$, one gets

$$\|\mathcal{R}u-\mathcal{R}v\|^p_{H^{\gamma+2,\theta}_{p,d_1}(O,t)} \leq \mu\|u-v\|^p_{H^{\gamma+2,\theta}_{p,d_1}(O,t)} + N_2\int_0^t \|u_1-u_2\|^p_{H^{\gamma+2,\theta}_{p,d_1}(O,s)}\,\mathrm{d}s,$$

where N_2 depends only on $d, p, \delta, \kappa, K, K_1$ and T. Hence, by introduction,

$$\begin{aligned}
&\|\mathcal{R}^n u_1-\mathcal{R}^n u_2\|^p_{H^{\gamma+2,\theta}_{p,d_1}(O,t)} \leq \mu^n\|u_1-u_2\|^p_{H^{\gamma+2,\theta}_{p,d_1}(O,t)} \\
&+\sum_{k=1}^{n}\tbinom{n}{k}\mu^{n-k}N_2^k\int_0^t\frac{(t-s)^{k-1}}{(k-1)!}\|u_1-u_2\|_{H^{\gamma+2,\theta}_{p,d_1}(O,s)}\,\mathrm{d}s\,\|\mathscr{R}^n u_1-\mathscr{R}^n u_2\|^p_{H^{2,\theta}_{p,d_1}(O,T)} \\
&\leq \sum_{k=0}^{n}\tbinom{n}{k}\mu^{n-k}\,(TN_2)^k/k!\,\|u_1-u_2\|_{H^{2,\theta}_{p,d_1}(O,T)} \\
&\leq 2^n\mu^n\max_k\left\{(TN_2/\mu)^k/k!\right\}\|u_1-u_2\|_{H^{2,\theta}_{p,d_1}(O,T)}.
\end{aligned}$$

This allows us to find ε depending only on d, p, δ, κ and K, so that the operator $\mathcal{R}^n$ is a contraction in $H^{\gamma+2}_{p,d_1,0}(O,T)$ with coefficient $1/2$. Of course, this yields all our assertion. The theorem is proved. □

Lemma 4.5. Let Assumption 2.10 be satisfied. Then there exists $\varepsilon = \varepsilon(d, p, \gamma, \delta, \kappa, K) > 0$ and $\beta > 0$ such that system (1) admits a unique solution $u \in H^{\gamma+2,\theta}_{p,d_1,0}(O,T)$ of Furthermore, for this solution u, one gets

$$\|u\|^p_{H^{\gamma+2,\theta}_{p,d_1}(O,T)} \leq N\|(f,g)\|_{F^{\gamma}_p(O,T)},\ \|u\|^p_{H^{\gamma+2,\theta}_{p,d_1}(O,T)} \leq N\|(L,\Lambda)u\|_{F^{\gamma}_p(O,T)}, \tag{33}$$

for all $\theta \in (d-\beta, d+\beta)$, where N depends only on d, p, δ, κ, K and T.

Proof: We have already proved the Theorem 4.4 for the system with coefficients independent of x. The only proof we need is to extend the Theorem 4.4 to the version that the condition have already mentioned in Assumption 2.10. This is very similar with the proof of Theorem 6.6 in [7] where the theorem is proved when $O = \mathbb{R}^d$, $d_1 = 1$. The only difference is that one needs to use Theorem 4.4 in this article in place of Theorem 6.4 in [1]. Hence the theorem is proved. □

Proof of Theorem 2.11: For $\lambda \in [0,1]$ we consider the operation

$$du^k = (L_k^\lambda u + f)dt + (\Lambda_k^{m,\lambda} u + g^m)dw_t^m \tag{34}$$

with zero initial condition, where

$$L_k^\lambda = \lambda L_k^0 + (1-\lambda)L_k \text{ , } L^\lambda = (L_1^\lambda, L_2^\lambda, \cdots, L_{d_1}^\lambda) \text{ ,}$$

$$\Lambda_{k,m}^\lambda = \lambda \Lambda_{k,m}^0 + (1-\lambda)\Lambda_{k,m} \text{ , } \quad \Lambda^\lambda = (\Lambda_{1,m}^\lambda, \Lambda_{2,m}^\lambda, \cdots, \Lambda_{d_1,m}^\lambda)$$

and (f,g) is an arbitrary element in $\mathcal{F}_p^\gamma(\mathcal{O},T)$. Take a $\lambda_0 \in [0,1]$ and assume that for $\lambda = \lambda_0$ system (34) with zero initial data admits a unique solution $u \in H_{p,d_1,0}^{\gamma+2,\theta}(\mathcal{O},T)$. Actually, according to Theorem 3.4 this assumption is satisfied for $\lambda_0 = 1$. Then the following operator

$$\mathcal{R}_{\lambda_0} : \mathcal{F}_p^\gamma(\mathcal{O},T) \to H_{p,d_1,0}^{\gamma+2,\theta}(\mathcal{O},T)$$

can be hold, such that $\mathcal{R}_{\lambda_0}(f,g) = u$. From (33) one can found that

$$\left\| \mathcal{R}_{\lambda_0}(f,g) \right\|_{H_{p,d_1}^{\gamma+2,\theta}(\mathcal{O},T)} \le N \left\| (f,g) \right\|_{\mathcal{F}_p^\gamma(\mathcal{O},T)} . \tag{35}$$

When $\lambda \ne \lambda_0$, the system (24) can be changed as follow

$$du^k = \left(L_k^{\lambda_0} u + (\lambda - \lambda_0)(L_k u - L_k^0 u) + D_i f_k^i + f_k^0 \right) dt + \left(\Lambda_k^{m,\lambda_0} u + (\lambda - \lambda_0)(\Lambda_k^m u - \Lambda_k^{m,0} u) + g_m^k \right) dw_t^m$$

or

$$du = \left(L^{\lambda_0} u + (\lambda - \lambda_0)(Lu - L^0 u) + D_i f^i + f^0 \right) dt + \left(\Lambda^{m,\lambda_0} u + (\lambda - \lambda_0)(\Lambda u - \Lambda^0 u) + g^k \right) dw_t \text{ .} \tag{36}$$

Next, we solve the system (36) by iterations. Define $u_0 = 0$ and

$$u_{j-1} = \mathcal{R}_{\lambda_0} \left(L_{\lambda_0} u_j + (\lambda - \lambda_0)(Lu_j - L_0 u_j) + D_i f^i + f^0, (\lambda - \lambda_0)(\Lambda u_j - \Lambda_0 u_j) + g^k \right).$$

According to (35), one gets

$$\begin{aligned} \left\| u_{j+1} - u_j \right\|_{H_{p,d_1}^{\gamma+2,\theta}(\mathcal{O},T)} &\le N \left| \lambda - \lambda_0 \right| \left\| (Lu - L_0 u) + D_i f^i + f^0, (\Lambda u - \Lambda_0 u) + g^k \right\|_{\mathcal{F}_p^\gamma(\mathcal{O},T)} \\ &\le N_1 \left| \lambda - \lambda_0 \right| \left\| u_{j+1} - u_j \right\|_{H_{p,d_1}^{\gamma+2,\theta}(\mathcal{O},T)} , \end{aligned}$$

where N_1 is independent of j, λ and λ_0. If $N_1 \left| \lambda - \lambda_0 \right| \le 1/2$, then u_j is a Cauchy sequence in $H_{p,d_1}^{\gamma+2,\theta}(\mathcal{O},T)$, which converges by Lemma 2.9. Its limit satisfies

$$u = R_{\lambda_0}\left(L_{\lambda_0}u + (\lambda-\lambda_0)(Lu - L_0 u) + D_i f^i + f^0, (\lambda-\lambda_0)(\Lambda u - \Lambda_0 u) + g^k\right),$$

which is equivalent to (34). In this way we show that if (34) is solvable for λ_0, then it is solvable for λ satisfying $N_1|\lambda-\lambda_0| \le 1/2$. In finite mumble of steps starting with $\lambda = 1$, we get to $\lambda = 0$. Hence the theorem is proved.□

Remark 4.6. Using Corollary 3.5 instead of Theorem 3.4 and following the proof of Theorem 2.11, we can extend Theorem 2.11 to the case when $\mathscr{M}^i$s are diagonalizable by an orthogonal matrix $\mathscr{T}(\omega, x)$, if $\mathscr{T}^* \mathscr{M}^i \mathscr{T}$ is diagonal for each i which is $F_0 \times B(\mathbb{R}_d)$-measurable $d_1 \times d_1$ orthogonal matrix $\mathscr{T}(\omega, x)$.

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Appendix

In the appendix, we present the Ito-Wentzell formula to the following stochastic differential equation

$$\mathrm{d}u_t(x) = f_t(x)\mathrm{d}t + g_t^k(x)\mathrm{d}w_t^k, t \le T \ , \tag{37}$$

where $f, u \in D$, $g \in \ell_2$.

Definition A1 We say that the equality (37) holds in the sense of distributions if only if for any $\phi \in C_0^\infty$, with probability one we have

$$(u_t(x), \phi) = (u_0(x), \phi) + \int_0^t (f_s, \phi)\mathrm{d}t + \int_0^t (g_s^k, \phi)\mathrm{d}w_t^k, \ t \le T \ .$$

Let x_t be an $\mathbb{R}^d$-valued stochastic process given by

$$x_t^i = \int_0^t b_s^i \mathrm{d}s + \sum_{k=1}^{\infty} \int_0^t \sigma_s^{ik} \mathrm{d}w_s^k ,$$

where $b_t = (b_t^i), \sigma_t^k = (\sigma_t^{ik})$ are predictable $\mathbb{R}^d$-valued processes such that for all ω and $s, T \in \mathbb{R}_+$ we have $\operatorname{trace}(a_s) < \infty$ and

$$\int_0^t \left(|b_t| + \operatorname{trace}(a_s) \right) \mathrm{d}t < \infty,$$

where $a_t = (a_t^{ij})$ and $2a_t^{ij} = (\sigma_t^{i\cdot}, \sigma_t^{j\cdot})_{\ell_2}$, so that

$$2 \operatorname{trace}(a_s) = \sum_{i=1}^{d} \sum_{k=1}^{\infty} \left| \sigma_t^{ik} \right|^2$$

Here is the Ito-Wentzell formula taken from Reference [7] (also see [20]).

Theorem A2 (Ito-Wentzell formula) Let $f, u \in D$, $g \in \ell_2$. Introduce

$$v_t(x) = u_t(x + x_t)$$

and assume that (1.2) holds (in the sense of distributions). Then

$$\begin{aligned} \mathrm{d}v_t(x) = & \left[f_t(x + x_t) + a_t^{ij} D_{ij} v_t(x) + b_t^i D_i v_t(x) + \left(D_i g_t(x + x_t), \sigma_t^{i\cdot} \right)_{\ell_2} \right] \mathrm{d}t \\ & + \left[g_t^k (x + x_t) + D_i v_t(x) \sigma_t^{ik} \right] \mathrm{d}w_t^k . \end{aligned} \tag{a1}$$

Here, the summation convention over the repeated indices $i, j = 1, \cdots, d$ (and $k = 1, 2, \cdots$) is enforced.