Journal of Mathematics and Informatics
Vol. 6, 2016, 15-19
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 13 September 2016
www.researchmathsci.org

Journal of
Mathematics and Informatics

Weak Contraction Principle in b-Metric Spaces

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Received 31 July 2016; accepted 28 August 2016

Abstract. In this paper, we establish the existence of a unique fixed point for weak contractions in the context of b-metric spaces. The main theorem is supported with an illustrative example. The line of research is the study of generalizations of Banach's contraction mapping principle in spaces which are more general than metric spaces.

Keywords: b-metric space, fixed point, weak contraction

AMS Mathematics Subject Classification (2010): 54H25, 47H10

1. Introduction and mathematical preliminaries

The concept of b-metric space was introduced by Bakhtin in [4] which was further used by Czerwik in [8]. It is one of the several generalizations of metric spaces which have appeared in recent literatures as, for instances, partial metric spaces [2,6,12], G-metric spaces [14,16], fuzzy metric spaces [10], etc. A few more works on b-metric spaces can be foundin [3,11,13]. In the following we describe the essential features of the spaces which are relevant to our studies in this paper.

Definition 1.1. Let X be a non-empty set and let $s \geq 1$ be a given real number. A function
$d: X \times X \to R_+$, is called a b-metric provided that, for all$x, y, z \in X$,

1) $d(x,y) = 0$ iff $x = y$,
2) $d(x,y) = d(y,x)$,
3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

A pair (X, d) is called a b-metric space. If $s = 1$, then it reduces to the usual metric space.

Example 1.2. The spacel_p $(0 < p < 1)$,
$l_p = \{(x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$,
Together with the function $d: l_p \times l_p \to \mathbb{R}_+$

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}},$$

where$x = x_n$, $y = y_n \in l_p$is a b-metric space.It can be shown that

$$d(x,y) \leq \frac{1}{2^p}[d(x,y)+d(y,z)]$$

Example 1.3. The $L_p\,(0<p<1)$ of all real functions $x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p\,dt < \infty$, is a b-metric space if we take $d(x,y) = \left[\int_0^1 |x(t)-y(t)|^p\,dt\right]^{\frac{1}{p}}$ for each$x, y \in L_p$.

Definition 1.4. Let (X,d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $m \geq n(\varepsilon)$, $d(x_n, y_n) < \varepsilon$.

Definition 1.5. Let (X,d) be a b-metric space. Then a sequence $\{x_n\}$ in $X$is convergent if and only if for all $\varepsilon > 0$, there exists $x \in X$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq n(\varepsilon)$ where $n(\varepsilon) \in N$.

In this case we write
$\lim_{n\to\infty} x_n = x$.

In this paper, we study weak contraction which is intermediate between a contraction and a non-expansion. It was detained first in Hilbert spaces by Alber et al [1] and then was adopted to metric spaces by Rhoades [15]. Further the concept was elaborated in a good number of papers [5,7,9] all of which are metric spaces.

Our result is a unique fixed point result in b-metric spaces for weak contractions. The definition of weak contraction is suitably modified for the b-metric spaces. The main result is supported with an example.

Definition 1.6. The b-metric space is complete if every Cauchy sequence in it is convergent.

2. Main results

Theorem 2.1. Let (X,d,s) be a complete b-metric space and $s \geq 1$ be a given real number. Let $T: X \to X$ be a mapping such that

$$d(Tx,Ty) \leq d(x,y) - \Phi(d(x,y)) \ldots\ldots\ldots\ldots\ldots\ldots(2.1)$$

where $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\lim_{n\to\infty} \inf \Phi(t_n) > (s-1)l$ whenever $\lim\sup t_n \geq l > 0$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be any element. We constructthe sequence$\{x_n\}$ by$x_n = Tx_{n-1}$, $n \geq 1$.

$$\text{Then } d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq d(x_{n-1}, x_n) - \Phi(d(x_{n-1}, x_n)) \ldots\ldots\ldots\ldots(2.2)$$

which implies $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ (using property of Φ)

It follows that $\{d(x_{n-1}, x_n)\}$ is a monotone decreasing sequence of non-negative real numbers and hence $d(x_{n-1}, x_n) \to l$ as $n \to \infty$.

Since $l > 0$ we have

$$\lim_{n\to\infty} \inf \Phi(d(x_{n-1}, x_n)) > 0 \ldots\ldots\ldots\ldots\ldots\ldots(2.3)$$

Taking limit in (2.2) we get

$$l \leq l - \lim_{n\to\infty} \inf \left(\Phi(x_{n-1}, x_n)\right)$$

which is a contradiction by (2.3).

Therefore, $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$ ....……………………………………………..…(2.4)

We next prove that $\{x_n\}$is a Cauchy sequence.

If possible, let$\{x_n\}$ be not a Cauchy sequence. Then there exists $\varepsilon > 0, k > 0$, for which there exists two sequences $\{m(k)\}$and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$$
$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon$$

Then, for all > 0 ,

$$\begin{aligned}\varepsilon &\leq d(x_{m(k)}, x_{n(k)})\\ &\leq s[d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})]\\ &\leq s\varepsilon + d(x_{n(k)-1}, x_{n(k)})\end{aligned}$$

Taking limit infimum of the above inequality, we obtain

$$\begin{aligned}\varepsilon &\leq \lim_{k\to\infty} \inf d(x_{m(k)}, x_{n(k)})\\ &\leq \lim_{k\to\infty} \sup d(x_{m(k)}, x_{n(k)}) \leq s\varepsilon \quad \text{……………………..(2.5)}\end{aligned}$$

Again, for all $k > 0$,

$$\begin{aligned}d(x_{m(k)-1}, x_{n(k)-1}) &\leq s[d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)}, x_{n(k)-1})]\\ &\leq sd(x_{m(k)-1}, x_{m(k)}) + s^2[d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1})]\end{aligned}$$

And

$$\begin{aligned}d(x_{m(k)}, x_{n(k)}) &\leq s[d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})]\\ &\leq s\, d(x_{m(k)}, x_{m(k)-1}) + s^2[d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})]\end{aligned}$$

Taking infimum of the above inequalities and from (2.4) and (2.5) we get,

$$\begin{aligned}\frac{\varepsilon}{s^2} &\leq \lim_{k\to\infty} \inf d(x_{m(k)-1}, x_{n(k)-1})\\ &\leq \lim_{k\to\infty} \sup d(x_{m(k)-1}, x_{n(k)-1})\\ &\leq s\varepsilon\end{aligned}$$

Now from (1.1) putting $x = x_{m(k)-1}$and$y = x_{n(k)-1}$, we obtain

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)-1}, x_{n(k)-1}) - \Phi(d(x_{m(k)-1}, x_{n(k)-1}))$$

Consequently,

$$\Phi(d(x_{m(k)-1}, x_{n(k)-1})) \leq d(x_{m(k)-1}, x_{n(k)-1}) - d(x_{m(k)}, x_{n(k)})$$

Taking limit supremum on both sides

$$\begin{aligned}&\lim_{k\to\infty} \sup \Phi(d(x_{m(k)-1}, x_{n(k)-1}))\\ &\leq \lim_{k\to\infty} \sup d(x_{m(k)-1}, x_{n(k)-1}) - \lim_{k\to\infty} \inf d(x_{m(k)}, x_{n(k)})\\ &\leq s\varepsilon - \varepsilon = (s-1)\varepsilon\end{aligned}$$

But by (2.5) and property of Φ,

$\lim_{k\to\infty} \sup \Phi(d(x_{m(k)}, x_{n(k)})) > (s-1)\varepsilon$, which is a contradiction.

Therefore $\{x_n\}$is a Cauchy sequence and hence $x_n \to x \in X$ since (X, d) is complete.

Then

$$\begin{aligned}d(x, Tx) &\leq s[d(x, x_{n+1}) + d(x_{n+1}, Tx)]\\ &= s[d(x, x_{n+1}) + d(Tx_n, Tx)]\\ &= sd(x, x_{n+1}) + s^2[d(x_n, x) - \Phi(d(x_n, x))]\\ &\leq sd(x, x_{n+1}) + s^2 d(x_n, x)\end{aligned}$$

Taking $n \to \infty$ in the above inequality we obtain

$$d(x, Tx) = 0 \text{ that is } x = Tx.$$

If x and y are two fixed points of T, then $d(x,y) > 0$ and

$$d(x,y) = d(Tx,Ty)$$
$$= d(x,y) - \Phi(d(x,y))$$
$$< d(x,y) \text{ , (by property of } \Phi)$$

which is a contradiction. Therefore$x = y$,which shows that the fixed point is unique.

Example 2.2. Let $X = [0,1]$ be equipped with the b-metric $d(x,y) = |x - y|^2$ for all $x, y \in X$. It can be checked that for all $x, y, z \in X$ we have

$$d(x,y) \leq 2[d(x,z) + d(z,y)].$$

Then (X, d) is a b-metric space with parameter $s = 2$ and it is complete.
Let $T: X \to X$ be defined as

$$Tx = x - \frac{x^2}{2} \ \ , x \in [0,1]$$

And $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ be defined as

$$\Phi(\mathrm{t}) = \frac{t^2}{2} \ \ , t \in [0,1].$$

Then for $x, y \in X$,

$$d(Tx,Ty) = \left|\left(x - \frac{x^2}{2}\right) - \left(y - \frac{y^2}{2}\right)\right|^2$$
$$= \left|(x - y) - \left(\frac{x^2}{2} - \frac{y^2}{2}\right)\right|^2$$
$$\leq |x - y|^2 - \frac{1}{2}\left|\frac{x^2}{2} - \frac{y^2}{2}\right|^4$$
$$= d(x,y) - \Phi(d(x,y))$$

We conclude that inequality (2.1) remains valid for Φ and consequently by an application of theorem (2.1), T has a unique fixed point. It is seen that 0 is the unique fixed point of T.

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