Journal of Mathematics and Informatics
Vol. 30, 2026, 17-29
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 10 April 2026
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v30a03269

Journal of Mathematics and Informatics

On The Picture Fuzzy Abelian Subgroups of a Group

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Received 2 March 2026; accepted 8 April 2026

Abstract. This paper investigates the concept of picture fuzzy subgroups within the framework of group theory. A picture-fuzzy subgroup of a group is studied by extending classical subgroup properties to the picture-fuzzy environment using positive, neutral, and negative membership degrees. Furthermore, the notion of a picture fuzzy abelian subgroup and cyclic picture fuzzy subgroup of a group G is introduced as a special class of picture fuzzy subgroups. This picture-fuzzy abelian subgroup preserves the commutativity property in the fuzzy sense. Several characterisations and fundamental properties of both picture-fuzzy abelian subgroups and cyclic picture-fuzzy subgroups are established. The study contributes to the development of picture fuzzy algebra and provides a foundation for further investigations.

Keywords: Picture Fuzzy Subgroup, Picture Fuzzy Abelian Subgroup, Cut set, Picture Fuzzy Abelian Subgroup

AMS Mathematics Subject Classification (2010): 03E72, 08A72

1. Introduction

One important development in fuzzy set theory was the concept of fuzzy groups, introduced by Rosenfeld [18], which extends classical group theory to the fuzzy setting. This concept allows elements of the group to belong to the subgroup with varying degrees of membership, thereby providing a flexible tool for handling uncertainty in algebraic structures. Biswas [6] studied Rosenfield's work and introduced the idea of an intuitionistic fuzzy group (IFG). Sharma [25] contributed to the work of Biswas by studying some algebraic nature of intuitionistic fuzzy groups and obtained their properties via (α, β)-cut sets.

Cuong and Krinovich [8] introduced picture fuzzy set (PFS) theory by incorporating a vital tool not taken into consideration by the previous researchers, which is the neutrality degree. Thus, PFS is composed of positive, neutral, and negative membership degrees. This theory has been studied and applied extensively by several researchers, see [10, 11, 12, 14, 19, 21, 22, 23, 24, 25, 26]. One important direction in this development is the study of picture fuzzy subgroups, which generalise classical subgroups and fuzzy subgroups of a group introduced by Dogra and Pal [13], whereby extending both Rosenfield's and Biswas' works. Sangodapo and Onasanya [20] contributed to the work of Dogra and Pal [13] to establish some characteristics of PFSG via (r, s, t)-cut sets of a PFS.

In classical group theory, abelian groups play a fundamental role due to the commutativity of their binary operation. The concept of fuzzy abelian subgroup was first introduced in [17] by Mukherjee and Bhattacharya. Makamba in [16] identified a weakness in the definition of a fuzzy abelian group, which he corrected by giving a stronger definition. Sharma [27] studied mappings between intuitionistic fuzzy groups, which led him to introduce the notion of intuitionistic fuzzy group homomorphisms. Sharma [28] studied abelian groups and extended the concept to intuitionistic fuzzy abelian groups. In [24] Sangodapo, studied mappings between groups and introduced the notion of homomorphisms of picture fuzzy subgroups, where the image and inverse image of a picture fuzzy subgroup under a group homomorphism preserve the picture fuzzy subgroup structure.

In this paper, we have built on the work in [24] and extended intuitionistic fuzzy abelian subgroups by introducing the notions of picture fuzzy abelian subgroups and cyclic picture fuzzy subgroups of a group G, thereby extending abelian and cyclic groups to the picture fuzzy environment.

2. Preliminaries

This section provides the basic definitions and existing results on picture-fuzzy subgroups.

Definition 2.1. *[8] A picture fuzzy set Q of Y is defined as*

$$Q = \{(y, \sigma_Q(y), \tau_Q(y), \gamma_Q(y)) | y \in Y\},$$

where the functions

$$\sigma_Q: Y \to [0,1], \tau_Q: Y \to [0,1] \text{ and } \gamma_Q: Y \to [0,1]$$

are called the positive, neutral and negative membership degrees of $y \in Q$, respectively, and $\sigma_Q, \tau_Q, \gamma_Q$ satisfy

$$0 \le \sigma_Q(y) + \tau_Q(y) + \gamma_Q(y) \le 1, \forall y \in Y.$$

For each $y \in Y$, $S_Q(y) = 1 - (\sigma_Q(y) - \tau_Q(y) - \gamma_Q(y))$ is called the refusal membership degree of $y \in Q$.

Definition 2.2. *[13] Let $(G,*)$ be a crisp group and $Q = \{(y, \sigma_Q(y), \tau_Q(y), \eta_Q(y)) \mid y \in G\}$ be a PFS in G. Then, Q is called picture fuzzy subgroup of G (PFSG) if*

(i) $\sigma_Q(a*b) \ge \sigma_Q(a) \wedge \sigma_Q(b), \tau_Q(a*b) \ge \tau_Q(a) \wedge \tau_Q(b), \eta_Q(a*b) \le \eta_Q(a) \vee \eta_Q(b)$

(ii) $\sigma_Q(a^{-1}) \ge \sigma_Q(a), \tau_Q(a^{-1}) \ge \tau_Q(a), \eta_Q(a^{-1}) \le \eta_Q(a)$ for all $a, b \in G$.

Notice that a^{-1} is the inverse of $a \in G$,

or equivalently, Q is a PFSG of G if and only if

$$\sigma_Q(a*b^{-1}) \ge \sigma_Q(a) \wedge \sigma_Q(b), \tau_Q(a*b^{-1}) \ge \tau_Q(a) \wedge \tau_Q(b), \eta_Q(a*b^{-1}) \le \eta_Q(a) \vee \eta_Q(b).$$

Definition 2.3. *[13] Let $(G,*)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, Q is called picture fuzzy normal subgroup of G, denoted by PFNSG if*

$$\sigma_{Qa}(b) = \sigma_{aQ}(b), \tau_{Qa}(b) = \tau_{aQ}(b), \eta_{Qa}(b) = \eta_{aQ}(b)$$

for all $a, b \in G$.

The above definition can be redefined as;

Definition 2.4. *A PFSG $Q = (\sigma_Q, \tau_Q, \eta_Q)$ of a group G is said to be a PFNSG of G if $\sigma_Q(ab) = \sigma_Q(ba), \tau_Q(ab) = \tau_Q(ba)$ and $\eta_Q(ab) = \eta_Q(ba)\ \forall\ a, b \in G$, or equivalently, Q a PFSG of G is said to be normal if and only if*

$$\sigma_Q(b^{-1}ab) = \sigma_Q(a), \sigma_Q(b^{-1}ab) = \sigma_Q(a) \text{ and } \eta_Q(b^{-1}ab) = \eta_Q(a)\ \forall\ a, b \in G.$$

Definition 2.5. *[13] Let $Q = \{(x, \sigma_Q, \tau_Q, \eta_Q) | a \in X\}$ be PFS over the universe X. Then, (r, s, t)-cut set of Q is the crisp set in Q, denoted by $C_{\sigma,\tau,\gamma}(Q)$ and is defined by*

$$C_{r,s,t}(Q) = \{a \in X | \sigma_Q(a) \geq r, \tau_Q(a) \geq s, \eta_Q(a) \leq t\}$$

$r, s, t \in [0,1]$ with the condition $0 \leq r + s + t \leq 1$

Theorem 2.1. *[13] Let $(G,*)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, Q is a PFSG (PFNSG) if and only if $C_{r,s,t}(Q)$ is a crisp subgroup (normal) of G.*

Definition 2.6. *Let $Q = \{(a, \sigma_Q(a), \tau_Q(a), \eta_Q(a)) \mid a \in X\}$ and $R = \{(b, \sigma_R(b), \tau_R(b), \eta_R(b)) \mid b \in Y\}$ be two PFSs. Then the Cartesian product of Q and R is the PFS*

$$P \times Q = \{(a, b), \sigma_{Q\times R}(a, b), \tau_{Q\times R}(a, b), \eta_{Q\times R}(a, b) \mid (a, b) \in X \times Y\},$$

where $\sigma_{Q\times R}(a, b) = \sigma_Q(a) \wedge \sigma_R(b)$, $\tau_{Q\times R}(a, b) = \tau_Q(a) \wedge \tau_R(b)$ and $\eta_{Q\times R}(a, b) = \eta_Q(a) \vee \eta_R(b)$ for all $(a, b) \in X \times Y$.

Definition 2.7. *[24] Let Y_1 and Y_2 be two nonempty sets and $f: Y_1 \to Y_2$ be a mapping. Let P and Q be two PFSs of Y_1 and Y_2, respectively. Then, the image of P under f denoted by $f(P)$ is defined as*

$$f(P)(y_2) = (\sigma_{f(P)}(y_2), \tau_{f(P)}(y_2), \eta_{f(P)}(y_2)),$$

where

$$\sigma_{f(P)}(y_2) = \begin{cases} \vee\{\sigma_P(y_1): y_1 \in f^{-1}(y_2)\} \\ 0, \ otherwise \end{cases},$$

$$\tau_{f(P)}(y_2) = \begin{cases} \vee\{\tau_P(y_1): y_1 \in f^{-1}(y_2)\} \\ 0, \ otherwise \end{cases},$$

and

$$\eta_{f(P)}(y_2) = \begin{cases} \wedge\{\eta_P(y_1): y_1 \in f^{-1}(y_2)\} \\ 1, \ otherwise \end{cases}$$

Thus,

$$f(P)(y_2) = \begin{cases} (\vee\{\sigma_P(y_1): y_1 \in f^{-1}(y_2)\}, \vee\{\tau_P(y_1): y_1 \in f^{-1}(y_2)\}, \wedge\{\eta_P(y_1): y_1 \in f^{-1}(y_2)\}) \\ (0,0,1), \quad otherwise \end{cases}$$

The pre-image of Q under f, denoted by $f^{-1}(Q)$ is also defined as

$$f^{-1}(Q)(y_1) = (\sigma_{f^{-1}(Q)}(y_1), \tau_{f^{-1}(Q)}(y_1), \eta_{f^{-1}(Q)}(y_1))$$

where

$$\sigma_{f^{-1}(Q)}(y_1) = \sigma_Q(f(y_1)), \tau_{f^{-1}(Q)}(y_1) = \tau_Q(f(y_1)), \eta_{f^{-1}(Q)}(y_1) = \eta_Q(f(y_1)).$$

Thus,

$$f^{-1}(Q)(y_1) = (\sigma_Q(f(y_1)), \tau_Q(f(y_1)), \eta_Q(f(y_1))).$$

Remark 2.1. *[24] For any $y_1 \in Y_1$, we have $\sigma_{f(P)}(f(y_1)) \geq \sigma_P(y_1), \tau_{f(P)}(f(y_1)) \geq \tau_P(y_1)$ and $\eta_{f(P)}(f(y_1)) \leq \eta_P(y_1)$.*

Theorem 2.2. *[24] Let $f: Y_1 \to Y_2$ be a mapping. Then,*

- $f(C_{r,s,t}(P)) \subseteq C_{r,s,t}(f(P))$, for all $P \in PFS(Y_1)$
- $f^{-1}(C_{r,s,t}(Q)) = C_{r,s,t}(Q)(f^{-1}(Q))$, for all $Q \in PFS(Y_2)$.

3. Picture fuzzy abelian subgroups

This section introduces the concept of picture fuzzy abelian subgroup of a group and establishes the properties associated to it.

Definition 3.1. *Let G be a group and Q a PFSG Then, the picture fuzzy normalizer of Q in G, denoted by $\mathcal{N}$, is defined as*

$$\mathcal{N}(Q) = \{g \in G: \sigma_Q(g^{-1}yg) = \sigma_Q(y), \tau_Q(g^{-1}yg) = \tau_Q(y) \text{ and } \eta_Q(g^{-1}yg) = \eta_Q(y), \forall\, y \in G\}.$$

Theorem 3.1. *Let Q be a PFSG of a group G. Then,*

- $\mathcal{N}(Q)$ is a subgroup of G.
- Q is a PFNSG of G if and only if $\mathcal{N}(Q) = G$.
- Q is a PFNSG of the group $\mathcal{N}(Q)$.

Proof:

- Let $g, h \in \mathcal{N}(Q), y, z \in G$. Then,

$$\sigma_Q(g^{-1}yg) = \sigma_Q(y), \tau_Q(g^{-1}yg) = \tau_Q(y) \text{ and } \eta_Q(g^{-1}yg) = \eta_Q(y) \qquad (i$$

$$\sigma_Q(h^{-1}zh) = \sigma_Q(z), \tau_Q(h^{-1}zh) = \tau_Q(z) \text{ and } \eta_Q(h^{-1}zh) = \eta_Q(z) \qquad (ii)$$

Put $z = g^{-1}yg$ in (ii) we get,

$$\sigma_Q(h^{-1}g^{-1}ygh) = \sigma_Q(g^{-1}yg),$$

$$\tau_Q(h^{-1}g^{-1}ygh) = \tau_Q(g^{-1}yg)$$

and

$$\eta_Q(h^{-1}g^{-1}ygh) = \eta_Q(g^{-1}yg).$$

Using (i) we get,

$$\sigma_Q(h^{-1}g^{-1}ygh) = \sigma_Q(g^{-1}yg) = \sigma_Q(y),$$

$$\tau_Q(h^{-1}g^{-1}ygh) = \tau_Q(g^{-1}yg) = \tau_Q(y)$$

and

$$\eta_Q(h^{-1}g^{-1}ygh) = \eta_Q(g^{-1}yg) = \eta_Q(y).$$

which imply that

$$\sigma_Q((gh)^{-1}y(gh)) = \sigma_Q(y),$$

$$\tau_Q((gh)^{-1}y(gh)) = \tau_Q(y)$$

and

$$\eta_Q((gh)^{-1}y(gh)) = \eta_Q(y).$$

This means that, $gh \in \mathcal{N}(Q)$. Next, replace y in (i) with y^{-1}, we get,

$$\sigma_Q(g^{-1}y^{-1}g) = \sigma_Q(y^{-1}) = \sigma_Q(y),$$

$$\tau_Q(g^{-1}y^{-1}g) = \tau_Q(y^{-1}) = \tau_Q(y)$$

and

$$\eta_Q(g^{-1}y^{-1}g) = \eta_Q(y^{-1}) = \eta_Q(y)$$

That is;

$$\sigma_Q((gyg^{-1})^{-1}) = \sigma_Q(gyg^{-1}) = \sigma_Q(y),$$

$$\tau_Q((gyg^{-1})^{-1}) = \tau_Q(gyg^{-1}) = \tau_Q(y)$$

and

$$\eta_Q((gyg^{-1})^{-1}) = \eta_Q(gyg^{-1}) = \eta_Q(y),$$

which imply that

$$\sigma_Q((g^{-1})^{-1}y(g^{-1})) = \sigma_Q(y),$$

$$\tau_Q((g^{-1})^{-1}y(g^{-1})) = \tau_Q(y)$$

and

$$\eta_Q((g^{-1})^{-1}y(g^{-1})) = \eta_Q(y)$$

Thus, $g^{-1} \in \mathcal{N}(Q)$. Therefore, $\mathcal{N}(Q)$ is a subgroup of G.

• Suppose that $\mathcal{N}(Q) = G$. Then, $\sigma_Q(g^{-1}yg) = \sigma_Q(y), \tau_Q(g^{-1}yg) = \tau_Q(y)$ and $\eta_Q(g^{-1}yg) = \eta_Q(y)$ for all $g, y \in G$. Hence, Q is a PFNSG of the group G.

Conversely, suppose that Q is a PFNSG of the group G. Then,

$$\sigma_Q(g^{-1}yg) = \sigma_Q(y), \tau_Q(g^{-1}yg) = \tau_Q(y) \text{ and } \eta_Q(g^{-1}yg) = \eta_Q(y).$$

This means that, the set

$$\{g \in G: \sigma_Q(g^{-1}yg) = \sigma_Q(y), \tau_Q(g^{-1}yg) = \tau_Q(y) \text{ and } \eta_Q(g^{-1}yg) = \eta_Q(y), \forall\, y \in G\} = G.$$

Therefore, $\mathcal{N}(Q) = G$.

• Let $g, h \in \mathcal{N}(Q)$. Then,

$$\sigma_Q(g^{-1}yg) = \sigma_Q(y), \tau_Q(g^{-1}yg) = \tau_Q(y) \text{ and } \eta_Q(g^{-1}yg) = \eta_Q(y), \forall\, y \in G.$$

Replace y with gh, we get,

$$\sigma_Q(gh) = \sigma_Q(g^{-1}ghg) = \sigma_Q(hg),$$

$$\tau_Q(gh) = \tau_Q(g^{-1}ghg) = \tau_Q(hg)$$

and

$$\eta_Q(gh) = \eta_Q(g^{-1}ghg) = \eta_Q(hg)$$

Hence, Q is a PFNSG of the group $\mathcal{N}(Q)$.

Definition 3.2. *Let G be a group and Q a PFSG of G. Then, the picture fuzzy centralizer of Q in G, denoted by $\mathcal{C}(Q)$ is defined as*

$$\mathcal{C}(Q) = \{g \in G: \sigma_Q([g, y]) = \sigma_Q(e), \tau_Q([g, y]) = \tau_Q(e) \text{ and } \eta_Q([g, y]) = \eta_Q(e), \forall\, y \in G\}$$

where $[y, h] = (y^{-1}h^{-1}yh)$ is called the commutator of $y, h \in G$.

Theorem 3.2. *Let Q be a PFSG of a group G. Then,*

- $\mathcal{C}(Q)$ is a subgroup of G.
- $\mathcal{C}(Q)$ is a normal subgroup of $\mathcal{N}(Q)$.

Proof:

- $\mathcal{C}(Q) \neq \emptyset$ since $e \in \mathcal{C}(Q)$. Let $g, h \in \mathcal{C}(Q)$. Then,

$$\sigma_Q([g, y_1]) = \sigma_Q(e), \tau_Q([g, y_1]) = \tau_Q(e) \text{ and } \eta_Q([g, y_1]) = \eta_Q(e)$$

and

$$\sigma_Q([h, y_2]) = \sigma_Q(e), \tau_Q([h, y_2]) = \tau_Q(e) \text{ and } \eta_Q([h, y_2]) = \eta_Q(e)$$

for all $y_1, y_2 \in G$. This imply that

$$\sigma_Q([g, y_1]) = \sigma_Q(g^{-1}y_1^{-1}gy_1) = \sigma_Q(e),$$

$$\tau_Q([g, y_1]) = \tau_Q(g^{-1}y_1^{-1}gy_1) = \tau_Q(e),$$

$$\eta_Q([g, y_1]) = \eta_Q(g^{-1}y_1^{-1}gy_1) = \eta_Q(e) \qquad (i)$$

$$\sigma_Q([h, y_2]) = \sigma_Q(h^{-1}y_2^{-1}hy_2) = \sigma_Q(e),$$

$$\tau_Q([h, y_2]) = \tau_Q(h^{-1}y_2^{-1}hy_2) = \tau_Q(e),$$

$$\eta_Q([h, y_2]) = \eta_Q(h^{-1}y_2^{-1}hy_2) = \eta_Q(e) \qquad (ii)$$

In (ii), replace y_2 with $g^{-1}kg$, thus,

$$\sigma_Q(h^{-1}g^{-1}k^{-1}ghg^{-1}kg) = \sigma_Q(e),$$

$$\tau_Q(h^{-1}g^{-1}k^{-1}ghg^{-1}kg) = \tau_Q(e)$$

and

$$\eta_Q(h^{-1}g^{-1}k^{-1}ghg^{-1}kg) = \eta_Q(e)$$

this imply that,

$$\sigma_Q((gh)^{-1}k^{-1}(gh)k)(k^{-1}g^{-1}kg) = \sigma_Q(e),$$

$$\tau_Q((gh)^{-1}k^{-1}(gh)k)(k^{-1}g^{-1}kg) = \tau_Q(e)$$

and

$$\eta_Q((gh)^{-1}k^{-1}(gh)k)(k^{-1}g^{-1}kg) = \eta_Q(e)$$

So by (i) we get,

$$\sigma_Q((gh)^{-1}k^{-1}(gh)k) = \sigma_Q(e),$$

$$\tau_Q((gh)^{-1}k^{-1}(gh)k) = \tau_Q(e)$$

and

$$\eta_Q((gh)^{-1}k^{-1}(gh)k) = \eta_Q(e)$$

which means that, $gh \in \mathcal{C}(Q)$.

Also, from (i), we get,

$$\sigma_Q(e) = \sigma_Q(g^{-1}y_1^{-1}gy_1) = \sigma_Q((g^{-1}y_1^{-1}gy_1))^{-1} = \sigma_Q(y_1^{-1}g^{-1}y_1g)$$

$$\tau_Q(e) = \tau_Q(g^{-1}y_1^{-1}gy_1) = \tau_Q((g^{-1}y_1^{-1}gy_1))^{-1} = \tau_Q(y_1^{-1}g^{-1}y_1g)$$

and

$$\eta_Q(e) = \eta_Q(g^{-1}y_1^{-1}gy_1) = \eta_Q((g^{-1}y_1^{-1}gy_1))^{-1} = \eta_Q(y_1^{-1}g^{-1}y_1g)$$

This means that

$$\sigma_Q(e) = \sigma_Q(y_1^{-1}g^{-1}y_1g)$$

$$\tau_Q(e) = \tau_Q(y_1^{-1}g^{-1}y_1g),$$

$$\eta_Q(e) = \eta_Q(y_1^{-1}g^{-1}y_1g) \qquad (iii)$$

Replace y_1 with sg^{-1} in (iii), we get,

$$\sigma_Q(e) = \sigma_Q(gs^{-1}g^{-1}sg^{-1}g) = \sigma_Q(gs^{-1}g^{-1}s),$$

$$\tau_Q(e) = \tau_Q(gs^{-1}g^{-1}sg^{-1}g) = \tau_Q(gs^{-1}g^{-1}s)$$

$$\eta_Q(e) = \eta_Q(gs^{-1}g^{-1}sg^{-1}g) = \eta_Q(gs^{-1}g^{-1}s)$$

Hence, $g^{-1} \in \mathcal{C}(Q)$. Therefore, $\mathcal{C}(Q)$ is a subgroup of G.

$\sigma_Q(e) = \sigma_Q(gs^{-1}g^{-1}sg^{-1}g)$
$= \sigma_Q(gs^{-1}g^{-1}s),$
$\tau_Q(e) = \tau_Q(gs^{-1}g^{-1}sg^{-1}g)$
$= \tau_Q(gs^{-1}g^{-1}s)$ and
$\eta_Q(e) = \eta_Q(gs^{-1}g^{-1}sg^{-1}g)$
$= \eta_Q(gs^{-1}g^{-1}s)(i)$

- Let $g \in \mathcal{C}(Q)$ and $h \in \mathcal{N}(Q)$. From equations (i) and (ii),

$$\sigma_Q(g^{-1}y_1^{-1}gy_1) = \sigma_Q(e),\ \tau_Q(g^{-1}y_1^{-1}gy_1) = \tau_Q(e),\ \eta_Q(g^{-1}y_1^{-1}gy_1) = \eta_Q(e)$$

and

$$\sigma_Q(h^{-1}y_2^{-1}hy_2) = \sigma_Q(e),\ \tau_Q(h^{-1}y_2^{-1}hy_2) = \tau_Q(e),\ \eta_Q(h^{-1}y_2^{-1}hy_2) = \eta_Q(e),$$

for all $y_1\ y_2 \in G$.

Let $y_2 = g^{-1}y_1^{-1}gy_1$ in equation (ii) and using equation (i), we get,

$$\sigma_Q(h^{-1}g^{-1}y_1^{-1}gy_1h) = \sigma_Q(g^{-1}y_1gy_1) = \sigma_Q(e),$$

$$\tau_Q(h^{-1}g^{-1}y_1^{-1}gy_1h) = \tau_Q(g^{-1}y_1gy_1) = \tau_Q(e),$$

$$\eta_Q(h^{-1}g^{-1}y_1^{-1}gy_1h) = \eta_Q(g^{-1}y_1gy_1) = \eta_Q(e) \qquad (iv)$$

Also, let $y_1 = hkh^{-1}$ which means that

$$\sigma_Q(h^{-1}g^{-1}hk^{-1}h^{-1}ghkh^{-1}h) = \sigma_Q(e),$$

$$\tau_Q(h^{-1}g^{-1}hk^{-1}h^{-1}ghkh^{-1}h) = \tau_Q(e),$$

$$\eta_Q(h^{-1}g^{-1}hk^{-1}h^{-1}ghkh^{-1}h) = \eta_Q(e)$$

i.e;

$$\sigma_Q(h^{-1}g^{-1}hk^{-1}h^{-1}ghk) = \sigma_Q(e),$$

$$\tau_Q(h^{-1}g^{-1}hk^{-1}h^{-1}ghkh) = \tau_Q(e),$$

$$\eta_Q(h^{-1}g^{-1}hk^{-1}h^{-1}ghkh) = \eta_Q(e)$$

This imply that

$$\sigma_Q((h^{-1}gh)^{-1}k^{-1}(h^{-1}gh)k) = \sigma_Q(e),$$

$$\tau_Q((h^{-1}gh)^{-1}k^{-1}(h^{-1}gh)k) = \tau_Q(e),$$

$$\eta_Q((h^{-1}gh)^{-1}k^{-1}(h^{-1}gh)k) = \eta_Q(e)$$

which means that $h^{-1}gh \in \mathcal{C}(Q)$.
Therefore, $\mathcal{C}(Q)$ is a normal subgroup of $\mathcal{N}(Q)$.

Theorem 3.3. *Let Q be a PFNSG of a group G.*
Let $\mathcal{N} = \{g \in G: \sigma_Q(g) = \sigma_Q(e), \tau_Q(g) = \tau_Q(e)$ and $\eta_Q(g) = \eta_Q(e)\}$. Then, $\mathcal{N} \subseteq \mathcal{C}(Q)$.

Proof: Let Q be a PFNSG of a group G. This implies that

$$\sigma_Q(y_2^{-1}y_1y_2) = \sigma_Q(y_1), \tau_Q(y_2^{-1}y_1y_2) = \tau_Q(y_1) \text{ and } \eta_Q(y_2^{-1}y_1y_2) = \eta_Q(y_1)$$

for all $y_1, y_2 \in G$.

Let $g \in \mathcal{N}$. Then, $\sigma_Q(g) = \sigma_Q(e), \tau_Q(g) = \tau_Q(e)$ and $\eta_Q(g) = \eta_Q(e)$.

Now,

$$\begin{aligned}
&\sigma_Q([g, y_1]) = \sigma_Q(g^{-1}y_1^{-1}gy_1) \\
&\geq \sigma_Q(g^{-1}) \wedge \sigma_Q(y_1^{-1}gy_1) \\
&= \sigma_Q(g) \wedge \sigma_Q(g) \\
&= \sigma_Q(e) \wedge \sigma_Q(e) \\
&= \sigma_Q(e)
\end{aligned}$$

$$\begin{aligned}
&\tau_Q([g, y_1]) = \tau_Q(g^{-1}y_1^{-1}gy_1) \\
&\geq \tau_Q(g^{-1}) \wedge \tau_Q(y_1^{-1}gy_1) \\
&= \tau_Q(g) \wedge \tau_Q(g) \\
&= \tau_Q(e) \wedge \tau_Q(e) \\
&= \tau_Q(e)
\end{aligned}$$

$$\begin{aligned}
&\eta_Q([g, y_1]) = \eta_Q(g^{-1}y_1^{-1}gy_1) \\
&\leq \eta_Q(g^{-1}) \vee \eta_Q(y_1^{-1}gy_1) \\
&= \eta_Q(g) \vee \eta_Q(g)
\end{aligned}$$

$= eta_Q(e) \vee \eta_Q(e)$
$= \eta_Q(e)$
So, $g \in \mathcal{C}(Q)$. Hence, $\mathcal{N} \subseteq \mathcal{C}(Q)$.

Definition 3.3. *Let Q be a PFSG of a group G. Then, Q is called a picture fuzzy abelian subgroup (PFASG) if and only if $\mathcal{C}_{r,s,t}(Q)$ is an abelian subgroup of G, for all $r, s, t \in [0,1]$ with 0 < $r + s + t \leq 1$.*

Remark 3.1. *If G is an abelian group, then every PFSG of G is a PFASG of G but the converse need not hold. See the example below.*

Example 3.1. *Let*

$$G = \mathbb{D}_4 = \langle a, b \mid a^4 = e, b^2 = e, bab = a^{-1} \rangle$$

be the dihedral group of order 8. Define a PFS Q on G by assigning membership elements in the abelian subgroup $Q = \{e, a^2\}$ and zero otherwise. Define the membership functions as:

For $g \in G$,
- If $g \in G$: $\sigma_Q(g) = 0.7, \tau_Q(g) = 0.15, \eta_Q(g) = 0.05$
- If $g \in G \backslash Q$: $\sigma_Q(g) = 0.1, \tau_Q(g) = 0.05, \eta_Q(g) = 0.7$.

Clearly, $0 \leq \sigma_Q(g) + \tau_Q(g) + \eta_Q(g) \leq 1, \ \forall \ g \in G$.

Note:
(i) Q ia s PFSG of G because; e has maximal membership, the inverse of a^2 is itself, membership degrees are preserved and the closure property holds.
(ii) Q is a PFASG of G because; within the support of Q, $\{e, a^2\}$ we have

$$ea^2 = a^2 e, a^2 a^2 = e.$$

Hence, for all g_1, g_2 with significant membership

$$\sigma_Q(g_1 g_2) = \sigma_Q(g_2 g_1), \tau_Q(g_1 g_2) = \tau_Q(g_2 g_1) \text{ and } \eta_Q(g_1 g_2) = \eta_Q(g_2 g_1).$$

This shows that, $G = \mathbb{D}_4$ is a non-abelian group but picture fuzzy subgroup Q behaves abelian within its support which means that, Q is a PFASG of a non-abelian group G.

Theorem 3.4. *Let Q be a PFASG of G. Then, the set*
$G^* = \{g \in G: \sigma_Q(gh) = \sigma_Q(hg), \tau_Q(gh) = \tau_Q(hg) \text{ and } \eta_Q(gh) = \eta_Q(hg) \ \forall \ g, h \in G\}$
is a PFASG of G.

Proof: Since G is a PFASG of group G, $\mathcal{C}_{r,s,t}(Q)$ is a PFASG, for all $r, s, t \in [0,1]$ with $0 \leq r + s + t \leq 1$.

Next is to show that G^* is a PFASG of G. Since G^* has an identity, i.e; $e \in G^*$, it means that $G^* \neq \emptyset$.

Let $g, h \in G^*$. Then,

$$\sigma_Q(gy_1) = \sigma_Q(y_1 g), \sigma_Q(hy_1) = \sigma_Q(y_1 h),$$

$$\tau_Q(gy_1) = \tau_Q(y_1 g), \tau_Q(hy_1) = \tau_Q(y_1 h) \text{ and}$$

$$\eta_Q(gy_1) = \eta_Q(y_1 g), \eta_Q(hy_1) = \eta_Q(y_1 h) \ \forall \ g, h \in G$$

Thus,

$$\sigma_Q((gh)y_1) = \sigma_Q(g(hy_1)) = \sigma_Q((hy_1)g) = \sigma_Q(h(y_1g)) = \sigma_Q((y_1g)h) = \sigma_Q(y_1(gh)),$$

$$\tau_Q((gh)y_1) = \tau_Q(g(hy_1)) = \tau_Q((hy_1)g) = \tau_Q(h(y_1g)) = \tau_Q((y_1g)h) = \tau_Q(y_1(gh))$$

and

$$\eta_Q((gh)y_1) = \eta_Q(g(hy_1)) = \eta_Q((hy_1)g) = \eta_Q(h(y_1g)) = \eta_Q((y_1g)h) = \eta_Q(y_1(gh))$$

hold for all $y_1 \in G$. Hence, $gh \in G^*$.

Also, let $g \in G^*$ this means that

$$\sigma_Q(gy_1) = \sigma_Q(y_1g), \tau_Q(gy_1) = \tau_Q(y_1g) \text{ and } \eta_Q(gy_1) = \eta_Q(y_1g)$$

hold for all $y_1 \in G$.

Let $y_1 = y_1^{-1}$, then

$$\sigma_Q(gy_1^{-1}) = \sigma_Q(y_1^{-1}g), \tau_Q(gy_1^{-1}) = \tau_Q(y_1^{-1}g) \text{ and } \eta_Q(gy_1^{-1}) = \eta_Q(y_1^{-1}g)$$

Thus,

$$\sigma_Q(g^{-1}y_1) = \sigma_Q((g^{-1}y_1)^{-1}) = \sigma_Q(y_1^{-1}g) = \sigma_Q(gy_1^{-1}) = \sigma_Q((gy_1^{-1})^{-1}) = \sigma_Q(y_1g^{-1})$$

$$\tau_Q(g^{-1}y_1) = \tau_Q((g^{-1}y_1)^{-1}) = \tau_Q(y_1^{-1}g) = \tau_Q(gy_1^{-1}) = \tau_Q((gy_1^{-1})^{-1}) = \tau_Q(y_1g^{-1})$$

$$\eta_Q(g^{-1}y_1) = \eta_Q((g^{-1}y_1)^{-1}) = \eta_Q(y_1^{-1}g) = \eta_Q(gy_1^{-1}) = \eta_Q((gy_1^{-1})^{-1}) = \eta_Q(y_1g^{-1})$$

for all $y_1 \in G$. So, $g^{-1} \in G^*$. Therefore, G^* is a PFASG of G.

Next, to show that G^* is a PFASG of group G.

Let $g, h \in G^*$, assume without loss of generality; let $\sigma_Q(g) = r, \tau_Q(g) \leq 1 - r, \eta_Q(g) \leq 1 - r - s$ and $\sigma_Q(h) = u, \tau_Q(h) \leq 1 - u, \eta_Q(h) \leq 1 - u - v$ where $r, s, t, u, v, w \in [0,1]$ with $0 \leq r + s + t \leq 1$ and $0 \leq u + v + w \leq 1$. Then, $g \in \mathcal{C}_{r,1-r,1-r-s}(Q)$, $h \in \mathcal{C}_{u,1-u,1-u-v}(Q)$.

Let $r < u$. Then, $\sigma_Q(h) = u > r, \tau_Q(h) \leq 1 - u < 1 - r$ and $\eta_Q(h) \leq 1 - u - v < 1 - r - s$, which imply that $h \in \mathcal{C}_{r,1-r,1-r-s}(Q)$. Thus, $g, h \in \mathcal{C}_{r,1-r,1-r-s}(Q)$. Therefore, $gh = hg$. Hence, G^* is a PFASG of G.

Corollary 3.1.

(i) If Q is a PFASG of G, then Q is also a PFNSG of G.

(ii) $\mathcal{C}(Q) = G^*$.

Theorem 3.5. *Let Q be a PFASG of G. Then, $\mathcal{C}(Q)$ is a PFASG of G.*

Theorem 3.6. *Let P and Q be two PFSGs of groups G_1 and G_2, respectively. Then, $P \times Q$ is a PFASG of $G_1 \times G_2$ if and only if P and Q are PFASGs of G_1 and G_2, respectively.*

Proof: Suppose that $P \times Q$ is a PFASG of $G_1 \times G_2$. Then, $\mathcal{C}_{r,s,t}(P \times Q)$ is a PFASG of $G_1 \times G_2$, i.e; $\mathcal{C}_{r,s,t}(P) \times \mathcal{C}_{r,s,t}(Q)$ is a PFSG of $G_1 \times G_2$. This implies that $\mathcal{C}_{r,s,t}(P)$ and $\mathcal{C}_{r,s,t}(Q)$ are PFASG of G_1 and G_2, respectively. Hence, P and Q are PFASG of G_1 and G_2, respectively.

Conversely, suppose that P and Q are PFASG of G_1 and G_2, respectively. Then, $\mathcal{C}_{r,s,t}(P)$ and $\mathcal{C}_{r,s,t}(Q)$ are PFASG of G_1 and G_2, respectively for all $r, s, t \in [0,1]$ with $0 \leq r + s + t \leq 1$. Thus, $\mathcal{C}_{r,s,t}(P) \times \mathcal{C}_{r,s,t}(Q)$ is a PFSG of $G_1 \times G_2$. But $\mathcal{C}_{r,s,t}(P \times Q) = \mathcal{C}_{r,s,t}(P) \times \mathcal{C}_{r,s,t}(Q)$.
Therefore, $\mathcal{C}_{r,s,t}(P) \times \mathcal{C}_{r,s,t}(Q)$ is a PFASG of $G_1 \times G_2$, for all $r, s, t \in [0,1]$ with $0 \leq r + s + t \leq 1$.

Definition 3.4. *Let Q be a PFSG of a group G. Then, Q is called cyclic picture fuzzy subgroup of G, denoted by CPFSG if $\mathcal{C}_{r,s,t}(Q)$ is a CPFSG of G, for all $r, s, t \in [0,1]$ with* $0 \leq r + s + t \leq 1$.

Remark 3.2.
• If G is a cyclic group, then every PFSG of G is a CPFSG of G, but converse need not be true.
Proof: Let $G = \langle a \rangle$ be cyclic group and let Q be any PFSG of G. Then, it is true that

$$\sigma_Q(a^p) \geq \sigma_Q(a^{p-1}) \geq \sigma_Q(a^{p-2}) \geq \cdots \geq \sigma_Q(a^2) \geq \sigma_Q(a),$$

$$\tau_Q(a^p) \geq \tau_Q(a^{p-1}) \geq \tau_Q(a^{p-2}) \geq \cdots \geq \tau_Q(a^2) \geq \tau_Q(a)$$

and

$$\eta_Q(a^p) \leq \eta_Q(a^{p-1}) \leq \eta_Q(a^{p-2}) \leq \cdots \leq \eta_Q(a^2) \leq \eta_Q(a).$$

hold for all $p \in \mathbb{N}$. Therefore, if $a^q \in \mathcal{C}_{r,s,t}(Q)$, for some $q \in \mathbb{N}$, then $a^q, a^{q+1}, a^{q+2}, \cdots \in \mathcal{C}_{r,s,t}(Q)$, i.e; $\mathcal{C}_{r,s,t}(Q) = \langle a^{-1} \rangle$, which is a CPFSG, for all $r, s, t \in [0,1]$ with $0 \leq r + s + t \leq 1$. Hence, Q is a CPFSG of G.
Converse need not be true, see Example 3.1.

• Every CPFSG of a group G is a PFSG, but the converse need not be true.
Proof: It is obvious.

4. Homomorphism of picture fuzzy abelian groups

Theorem 4.1. *Let $f: G \to G^*$ be homomorphism of a group G into a group G^*. Let P be a PFASG of group G^*. Then, $f^{-1}(P)$ is a PFASG of group G.*
Proof: Let P be a PFASG of group G^*. Therefore, $\mathcal{C}_{r,s,t}(P)$ is a PFASG of G^*, for all $r, s, t \in [0,1]$ with $0 \leq r + s + t \leq 1$. By Theorem 2.6 [24], we have

$$\mathcal{C}_{r,s,t}(f^{-1}(P)) = f^{-1}(\mathcal{C}_{r,s,t}(P))$$
$$= \{a \in G_1 \mid f(a) \in \mathcal{C}_{r,s,t}(P)\}.$$

Let $a_1, a_2 \in \mathcal{C}_{r,s,t}(f^{-1}(P))$ be any two points. Then, $f(a_1), f(a_2) \in \mathcal{C}_{r,s,t}(P)$ since $\mathcal{C}_{r,s,t}(P)$ is a PFASG of G^*. Therefore, we have

$$f(a_1)f(a_2) = f(a_2)f(a_1) \text{ implies that}$$
$$f(a_1a_2) = f(a_2a_1).$$

So,

$$\sigma_P(f(a_1a_2)) = \sigma_P(f(a_2a_1)),$$
$$\tau_P(f(a_1a_2)) = \tau_P(f(a_2a_1)) \text{ and}$$
$$\eta_P(f(a_1a_2)) = \eta_P(f(a_2a_1))$$

This means that

$\sigma_{f^{-1}(P)}(a_1a_2) = \sigma_{f^{-1}(P)}(a_2a_1)$,
$\tau_{f^{-1}(P)}(a_1a_2) = \tau_{f^{-1}(P)}(a_2a_1)$ and
$\eta_{f^{-1}(P)}(a_1a_2) = \eta_{f^{-1}(P)}(a_2a_1)$

Thus, $a_1a_2 = a_2a_1$. Therefore, $\mathcal{C}_{r,s,t}(f^{-1}(P))$ is a PFASG of G, for all $r,s,t \in [0,1]$ with $0 \leq r+s+t \leq 1$.
Hence, $f^{-1}(P)$ is a PFASG of G.

Theorem 4.2. *Let $f: G \to G^*$ be surjective homomorphism and Q be a PFASG of group G1. Then, $f(Q)$ is a PFASG of group G^*.*
Proof: Let Q be a PFASG of group G, then $\mathcal{C}_{r,s,t}(Q)$ is a PFASG of G, for all $r,s,t \in [0,1]$ with $0 \leq r+s+t \leq 1$.
Let $b_1, b_2 \in \mathcal{C}_{r,s,t}(f(Q))$. Then, there exists $a_1, a_2 \in G$ such that $f(a_1) = b_1, f(a_2) = b_2$. Therefore, $f(a_1), f(a_2) \in \mathcal{C}_{r,s,t}(f(Q))$ as $\mathcal{C}_{r,s,t}(Q)$ is a PFASG of G. Thus, there exists $\mathcal{C}_{u,v,w}(Q)$ such that $a_1, a_2 \in \mathcal{C}_{u,v,w}(Q)$ for all $u,v,w \in [0,1]$ with $0 \leq u+v+w \leq 1$. But $\mathcal{C}_{u,v,w}(Q)$ is a PFASG of G. Therefore, $a_1a_2 = a_2a_1 \Rightarrow f(a_1a_2) = f(a_2a_1) \Rightarrow f(a_1)f(a_2) = f(a_2)f(a_1)$ i.e; $b_1b_2 = b_2b_1$. So, $\mathcal{C}_{r,s,t}(f(Q))$ is a PFASG of G^*.
Hence, $f(Q)$ is a PFASG of group G^*.

Theorem 4.3. *Let $f: G \to G^*$ be homomorphism of a group G into a group G^*. Let P be a CPFSG of group G^*. Then, $f^{-1}(P)$ is a CPFSG of group G.*
Proof: Since P is a CPFSG of group G^*, then $\mathcal{C}_{r,s,t}(P)$ is a CPFSG of G^*, for all $r,s,t \in [0,1]$ with $0 \leq r+s+t \leq 1$.
Let $\mathcal{C}_{r,s,t}(P) = \langle g_2 \rangle$, for some $g_2 \in G^*$. Now, $g_2 \in G^*$, there exists $g_1 \in G$ such that $f(g_1) = g_2$. So, $\mathcal{C}_{r,s,t}(P) = \langle f(g_2) \rangle$. Thus,

$$f^{-1}(\mathcal{C}_{r,s,t}(P)) = \mathcal{C}_{r,s,t}(f^{-1}P) = \langle g_1 \rangle.$$

Therefore, $f^{-1}(P)$ is a CPFSG of group G.

Theorem 4.4. *Let $f: G_1 \to G^*$ be surjective homomorphism and Q be a CPFSG of group G. Then, $f(Q)$ is a CPFSG of group G^*.*
Proof: Let Q be a CPFSG of group G, then $\mathcal{C}_{r,s,t}(Q)$ is a CPFSG of G, for all $r,s,t \in [0,1]$ with $0 \leq r+s+t \leq 1$.
Let $g \in \mathcal{C}_{r,s,t}(f(Q))$ be any element. Since f is surjective, let $g = f(g^*)$, for some $g^* \in G$. Thus, for all $g^* \in G$ (also for all $g \in \mathcal{C}_{r,s,t}(f(Q))$) there exists $\mathcal{C}_{u,v,w}(Q)$ such that $g^* \in \mathcal{C}_{u,v,w}(Q)$. But $\mathcal{C}_{u,v,w}(Q)$ is a CPFSG of G. Let $\mathcal{C}_{u,v,w}(Q) = \langle g_1 \rangle$. Thus, $g^* = (g_1)^{**}$. So,

$$g = f(g^*) = f((g_1)^{**}) = (f(g_1))^{**}$$

i.e; $\mathcal{C}_{r,s,t}(f(Q))$ is a CPFSG of G^*. Therefore, $f(Q)$ is a CPFSG of group G^*.

5. Conclusion

In the development of group theory, abelian groups play a vital role which has led to the interest in studying picture fuzzy abelian subgroup of a group G. Thus, in this paper, PFASG and CPFSG have been introduced after studying the PFSG of a group G as special cases of PFSGs and established several characterisations and fundamental properties. So, this study has contributed to the development of picture fuzzy algebra.

Acknowledgement: The author would like to express sincere appreciation to the esteemed reviewers for their valuable comments and suggestions, which significantly contributed to the quality of the paper.

Conflict of interest: The author declares no conflicts of interest.

Authors' Contributions: This is the sole work of the author.

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