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Homomorphism of Picture Fuzzy Subgroup of a Group

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Abstract. The notion of (r,s,t)-cut sets of a picture fuzzy set, $\mathcal{C}_{r,s,t}(Q)$, over the universe Y has been discussed to establish some characterisation of picture fuzzy subgroup of a picture fuzzy group. This paper introduces the concept of homomorphism of picture fuzzy subgroup of a group by delving into the relationship between the picture fuzzy set P of X and picture fuzzy set $f(Q)$ of Y, given a mapping $f: X \to Y$ via (r,s,t)-cut sets of picture fuzzy sets. Some results were also established regarding the picture fuzzy subgroup of G_1 and G_2 for f being a homomorphism.

Keywords: Picture Fuzzy Set, Picture Fuzzy subgroup, Cut set, Homomorphism

AMS Mathematics Subject Classification (2010): 03E72, 08A72

1. Introduction

Fuzzy set theory (FT) was pioneered by Zadeh [22], and since the inception it has gained a wide generalisations and extensions by several researchers. The extension of Zadeh's work was the introduction of intuitionistic fuzzy set theory (IFS) by Atanassov [1] by incorporating a non membership degree together with the membership degree in fuzzy sets. Cuong and Krinovich [7] extended both FS and IFS to picture fuzzy set (PFS) by incorporating a vital tool not taken into consideration by the previous researchers which is neutrality degree. Thus, PFS is made up of positive membership degree, neural membership degree and negative membership degree.

Rosenfield [19] introduced the notion of fuzzy group (FG) as a generalisation of classical group. Biswas [6] studied the Rosenfield's work and introduced the idea of intuitionistic fuzzy group (IFG). Sharma [21] contributed to the work of Biswas by studying some algebraic nature of intuitionistic fuzzy groups and obtained their properties via (α, β)-cut sets. Picture fuzzy subgroup (PFSG) was introduced by Dogra and Pal [15] in order to extend both FG and IFG. Sangodapo and Onasanya [20] contributed to the work of Dogra and Pal [15] to establish some characteristics of PFSG of a PFG via (r,s,t)-cut sets of a PFS.

Homomorphism is a structure preserving maps between two algebraic structures of the same type. The classical homomorphism was generalised by Rsenfield [19] to homomorphism of fuzzy subgroup of a group. Sharma [21] extended this to intuitionistic fuzzy subgroup of a group and by using the properties of cut sets, some results related to IFSG were obtained..

In this paper, we contributed to the work of Sangodapo and Onasanya [20] by introducing the concept of homomorphism of picture fuzzy subgroup of a group via (r, s, t)-cut sets. The relationship between the PFS P of X and PFS $f(Q)$ of Y, given a mapping $f: X \to Y$ was established, and some results regarding the picture fuzzy subgroup of G_1 and G_2 for f being a homomorphism were obtaind.

2. Preliminaries

This section gives the basic definitions and existing results relating to picture fuzzy subgroups.

Definition 2.1. *[7] A picture fuzzy set Q of Y is defined as*

$$Q = \{(y, \sigma_Q(y), \tau_Q(y), \gamma_Q(y)) | y \in Y\},$$

where the functions

$$\sigma_Q: Y \to [0,1], \tau_Q: Y \to [0,1] \text{ and } \gamma_Q: Y \to [0,1]$$

are called the positive, neutral and negative membership degrees of $y \in Q$, respectively, and $\sigma_Q, \tau_Q, \gamma_Q$ satisfy

$$0 \le \sigma_Q(y) + \tau_Q(y) + \gamma_Q(y) \le 1, \forall y \in Y.$$

For each $y \in Y$, $S_Q(y) = 1 - (\sigma_Q(y) - \tau_Q(y) - \gamma_Q(y))$ is called the refusal membership degree of $y \in Q$.

Definition 2.2. *[7] Let P and Q be two PFSs. Then, the inclusion, equality, union, intersection and complement are defined as follow:*

- $P \subseteq Q$ if and only if for all $y \in Y$, $\sigma_P(y) \le \sigma_Q(y)$, $\tau_P(y) \le \tau_Q(y)$ and $\gamma_P(y) \ge \gamma_Q(y)$.
- $P = Q$ if and only if $P \subseteq Q$ and $Q \subseteq P$.
- $P \cup Q = \{(y, \sigma_P(y) \vee \sigma_Q(y), \tau_P(y) \wedge \tau_Q(y)), \gamma_P(y) \wedge \gamma_Q(y)) | y \in Y\}$.
- $P \cap Q = \{(y, \sigma_P(y) \wedge \sigma_Q(y), \tau_P(y) \wedge \tau_Q(y)), \gamma_P(y) \vee \gamma_Q(y)) | y \in Y\}$.
- $\overline{P} = \{(y, \gamma_P(y), \tau_P(y), \sigma_P(y)) | y \in Y\}$.

Definition 2.3. *Let $(Y, .)$ be a groupoid and P, Q be two PFSs of Y. Then, the picture fuzzy product of P and Q, denoted by PoQ is defined as for any $y \in Y$,*

$$PoQ(y) = (\sigma_{PoQ}(y), \tau_{PoQ}(y), \eta_{PoQ}(y))$$

where

$$\sigma_{PoQ}(y) = \begin{cases} \vee_{xz=y} [\sigma_P(x) \wedge \sigma_Q(z)] \\ 0, \ if \ \ y \neq xz \end{cases},$$

$$\tau_{PoQ}(y) = \begin{cases} \vee_{xz=y} [\tau_P(x) \wedge \tau_Q(z)] \\ 0, \ if \ \ y \neq xz \end{cases},$$

$$\eta_{PoQ}(y) = \begin{cases} \wedge_{xz=y} [\eta_P(x) \vee \eta_Q(z)] \\ 1, \ if \ \ y \neq xz \end{cases}$$

Definition 2.4. *[15] Let* $Q = \{(y, \sigma_Q, \tau_Q, \eta_Q) | y \in Y\}$ *be PFS over the universe* Y. *Then,* (r, s, t)*-cut of* Q *is a crisp set of* Q, *denoted by* $C_{r,s,t}(Q)$ *and is defined by*

$$C_{r,s,t}(Q) = \{y \in Y | \sigma_Q(y) \geq r, \tau_Q(y) \geq s, \eta_Q(y) \leq t\}$$

$r, s, t \in [0,1]$ with the condition $0 < r + s + t \leq 1$.

Theorem 2.1. *[16] If* Q *and* R *are two PFSs of a universe* Y, *then the following holds*

- $C_{r,s,t}(Q) \subseteq C_{u,v,w}(Q)$ if $r \geq u, s \geq v, t \leq w$.
- $C_{1-s-t,s,t}(Q) \subseteq C_{r,s,t}(Q) \subseteq C_{r,1-r-t,t}(Q)$.
- $Q \subseteq R$ implies $C_{r,s,t}(Q) \subseteq C_{r,s,t}(R)$.
- $C_{r,s,t}(Q \cap R) = C_{r,s,t}(Q) \cap C_{r,s,t}(R)$.
- $C_{r,s,t}(Q \cup R) \supseteq C_{r,s,t}(Q) \cup C_{r,s,t}(R)$.
- $C_{r,s,t}(\cap Q_i) = \cap C_{r,s,t}(Q_i)$.
- $C_{1,0,0}(Q) = Y$.

Sangodapo and Onasanya [20] gave a counter example to show that Theorem 2.1 (ii) and (vii) were wrong, and the corrected version of the theorem was given in Theorem 3.1 of [20].

Theorem 2.2. *[20] Let* Q *and* R *be two PFSs of a universe* Y. *Then, the following assertions hold:*

- $C_{1-s-t,s,t}(Q) \subseteq C_{r,s,t}(Q) \subseteq C_{r,s,1-r-s}(Q)$,
- $C_{1-s-t,1-r-t,t}(Q) \subseteq C_{r,s,t}(Q) \subseteq C_{r,s,1-r-s}(Q)$,
- $C_{r,1-r-t,t}(Q) \subseteq C_{r,s,t}(Q) \subseteq C_{r,s,1-r-s}(Q)$,
- $C_{0,0,1}(Q) = Y$.

Thus, Z is represented by

$$Z_1 = \{\langle r, \sigma_{Z_1}^k(r), \tau_{Z_1}^k(r)), \eta_{Z_1}^k(r)\rangle | \, r \in Y\}$$

$k = 1, 2, \cdots, n$.

Theorem 2.3. *Let* Q *be PFS of* G. *Then,* Q *is PFSG of* G *if and only if* $C_{r,s,t}(Q)$ *is a PFSG of* G *for all* $r, s, t \in [0,1]$ *with* $0 < r + s + t \leq 1$, *where* $\sigma_Q(e) \geq r, \tau_Q(e) \geq s, \eta_Q(e) \leq t$ *and* e *is the identity element of* G.

Proof: Suppose that Q is PFSG of G. Then, by Proposition 3.1 [20], $C_{r,s,t}(Q)$ is a PFSG of G.

Conversely, suppose that Q be PFS of G such that $C_{r,s,t}(Q)$ is a PFSG of G for all $r,s,t \in [0,1]$ with $0 < r+s+t \leq 1$. Let $a,b \in G$ and let $r = \sigma_Q(a) \wedge \sigma_Q(b)$, $s = \tau_Q(a) \wedge \tau_Q(b)$ and $t = \eta_Q(a) \vee \eta_Q(b)$. Then, $\sigma_Q(a) \geq r, \sigma_Q(b) \geq r, \tau_Q(a) \geq s, \tau_Q(b) \geq s$ and $\eta_Q(a) \leq t, \eta_Q(b) \leq t$. This implies that,

$$\sigma_Q(a) \geq r, \tau_Q(a) \geq s, \eta_Q(a) \leq t$$

and

$$\sigma_Q(b) \geq r, \tau_Q(b) \geq s, \eta_Q(b) \leq t.$$

Thus, $a \in C_{r,s,t}(Q)$ and $b \in C_{r,s,t}(Q)$ which means that $ab \in C_{r,s,t}(Q)$ since $C_{r,s,t}(Q)$ is a PFSG of G. Hence,

$$\sigma_Q(ab) \geq r = \sigma_Q(a) \wedge \sigma_Q(b),$$

$$\tau_Q(ab) \geq s = \tau_Q(a) \wedge \tau_Q(b)$$

and

$$\eta_Q(ab) \leq t = \eta_Q(a) \vee \eta_Q(b)$$

which implies that

$$\sigma_Q(ab) \geq \sigma_Q(a) \wedge \sigma_Q(b),$$

$$\tau_Q(ab) = \tau_Q(a) \wedge \tau_Q(b)$$

and

$$\eta_Q(ab)\eta_Q(a) \vee \eta_Q(b).$$

Also, let $a \in G$ and $\sigma_Q(a) = r$, $\tau_Q(a) = s$ and $\eta_Q(a) = t$. Thus, $\sigma_Q(a) \geq r$, $\tau_Q(a) \geq s$ and $\eta_Q(a) \leq t$ which implies that $a \in C_{r,s,t}(Q)$. Since, $a \in C_{r,s,t}(Q)$ is a PFSG of G, therefore there is $a^{-1} \in C_{r,s,t}(Q)$ which means that $\sigma_Q(a^{-1}) \geq r$, $\tau_Q(a^{-1}) \geq s$ and $\eta_Q(a^{-1}) \leq t$. So,

$$\sigma_Q(a^{-1}) \geq r = \sigma_Q(a), \tau_Q(a^{-1}) \geq s = \tau_Q(a) \text{ and } \eta_Q(a^{-1}) \leq t = \eta_Q(a).$$

Thus,

$$\sigma_Q(a) = \sigma_Q((a^{-1})^{-1}) \geq \sigma_Q(a^{-1}) \geq \sigma_Q(a) \Rightarrow \ \sigma_Q(a^{-1}) = \sigma_Q(a),$$
$$\tau_Q(a) = \tau_Q((a^{-1})^{-1}) \geq \tau_Q(a^{-1}) \geq \tau_Q(a) \Rightarrow \ \tau_Q(a^{-1}) = \tau_Q(a) \text{ and}$$
$$\eta_Q(a) = \eta_Q((a^{-1})^{-1}) \leq \eta_Q(a^{-1}) \leq \eta_Q(a) \Rightarrow, \eta_Q(a^{-1}) = \eta_Q(a).$$

Therefore, Q is PFSG of G.

Theorem 2.4. *Let $(Y,.)$ be a groupoid and P,Q be two PFSs of Y. Then,*

$$C_{r,s,t}(PoQ) = C_{r,s,t}(P)C_{r,s,t}(Q).$$

Proof: By the cut set definition, we have that

$$C_{r,s,t}(PoQ) = \{y \in Y | \sigma_{PoQ}(y) \geq r, \tau_{PoQ}(y) \geq s, \eta_{PoQ}(y) \leq t\}.$$

Let $h \in C_{r,s,t}(PoQ) \Leftrightarrow \sigma_{PoQ}(h) \geq r, \tau_{PoQ}(h) \geq s, \eta_{PoQ}(h) \leq t$.

$$\sigma_{PoQ}(h) = \begin{cases} \vee_{xz=h} [\sigma_P(x) \wedge \sigma_Q(z)] \\ \\ 0, \ \ if \ \ \ h \neq xz \end{cases},$$

$$\tau_{PoQ}(h) = \begin{cases} \vee_{xz=h}\,[\tau_P(x) \wedge \tau_Q(z)] \\ \\ 0, \;\; if \;\;\; h \neq xz \end{cases},$$

and

$$\eta_{PoQ}(h) = \begin{cases} \wedge_{xz=h}\,[\eta_P(x) \vee \eta_Q(z)] \\ \\ 1, \;\; if \;\;\; h \neq xz \end{cases}$$

Therefore,

$\vee_{xz=h}\,[\sigma_P(x) \wedge \sigma_Q(z)] \geq r, \vee_{xz=h}\,[\tau_P(x) \wedge \tau_Q(z)] \geq s$ and $\wedge_{xz=h}\,[\eta_P(x) \vee \eta_Q(z)] \leq t$.

There exist x_1, z_1, x_2, z_2 and x_3, z_3 in Y such that $h = x_1 z_1$, $h = x_2 z_2$ and $h = x_3 z_3$, and that $\sigma_P(x_1) \wedge \sigma_Q(z_1) \geq r$, $\tau_P(x_2) \wedge \tau_Q(z_2) \geq s$ and $\eta_P(x_3) \vee \eta_Q(z_3) \leq t$

$\Leftrightarrow \sigma_P(x_1) \geq r, \sigma_Q(z_1) \geq r, \tau_P(x_2) \geq s, \tau_Q(z_2) \geq s$ and $\eta_P(x_3) \leq t, \eta_Q(z_3) \leq t$.

Let $\eta_P(x_1) \leq 1 - r - s$, $\eta_Q(z_1) \leq 1 - r - s$, $\tau_P(x_2) \geq 1 - r - t$, $\tau_Q(z_2) \geq 1 - r - t$ and $\sigma_P(x_3) \geq 1 - s - t, \sigma_Q(z_3) \geq 1 - s - t$

$\Leftrightarrow x_1 \in C_{r,s,1-r-s}(P), z_1 \in C_{r,s,1-r-s}(Q)$, $x_2 \in C_{r,1-r-t,t}(P), z_2 \in C_{r,1-r-t,t}(Q)$ and $x_3 \in C_{1-s-t,s,t}(P), z_3 \in C_{1-s-t,s,t}(Q)$.

By Theorem 2.2, we have for every PFS P of Y,

$$C_{1-s-t,s,t}(P) \subseteq C_{r,s,t}(P) \subseteq C_{r,s,1-r-s}(P)$$

$\Rightarrow$

$$h = x_3 z_3 \in C_{1-s-t,s,t}(P)C_{1-s-t,s,t}(Q) \subseteq C_{r,s,t}(P)C_{r,s,t}(Q).$$

Thus, $h \in C_{r,s,t}(P)C_{r,s,t}(Q)$.

Therefore, $C_{r,s,t}(PoQ) = C_{r,s,t}(P)C_{r,s,t}(Q)$.

Theorem 2.5. *Let P and Q be two PFSG of group G. Then, PoQ is a PFSG of G if and only if $PoQ = QoP$.*

Proof: PoQ is a PFSG of $G \Rightarrow C_{r,s,t}(PoQ)$ is a PFSG of G for all $r, s, t \in [0,1]$ with $r + s + t \leq 1$. The P and Q are PFSG of group $G \Leftrightarrow C_{r,s,t}(P)$ and $C_{r,s,t}(Q)$ are PFSG of $G, \forall\, r, s, t \in [0,1]$ with $r + s + t \leq 1$. Thus, $C_{r,s,t}(P)C_{r,s,t}(Q)$ is a PFSG of $G, \Leftrightarrow C_{r,s,t}(P)C_{r,s,t}(Q) = C_{r,s,t}(Q)C_{r,s,t}(P) \Leftrightarrow C_{r,s,t}(PoQ) = C_{r,s,t}(QoP)$ for all $r, s, t \in [0,1]$ with $r + s + t \leq 1, \Leftrightarrow PoQ = QoP$.

Remark 2.1. *If Q is a PFSG of G, then $PoP = P$.*

Proof: Since Q is a PFSG of G, it means that $C_{r,s,t}(P)$ is a PFSG of G, for all $r, s, t \in [0,1]$ with $r + s + t \leq 1$. Thus, $C_{r,s,t}(P)C_{r,s,t}(P) = C_{r,s,t}(P)$ this implies that $C_{r,s,t}(PoP) = C_{r,s,t}(P)$ for all $r, s, t \in [0,1]$ with $r + s + t \leq 1$.

Hence, $PoP = P$.

Definition 2.5. *Let Y_1 and Y_2 be two nonempty sets and $f: Y_1 \to Y_2$ be a mapping. Let P and Q be two PFSs of Y_1 and Y_2, respectively. Then, the image of P under f denoted by $f(P)$ is defined as*

$$f(P)(y_2) = (\sigma_{f(P)}(y_2), \tau_{f(P)}(y_2), \eta_{f(P)}(y_2)),$$

where

$$\sigma_{f(P)}(y_2) = \begin{cases} \vee \{\sigma_P(y_1): y_1 \in f^{-1}(y_2)\} \\ 0, \ otherwise \end{cases},$$

$$\tau_{f(P)}(y_2) = \begin{cases} \vee \{\tau_P(y_1): y_1 \in f^{-1}(y_2)\} \\ 0, \ otherwise \end{cases},$$

and

$$\eta_{f(P)}(y_2) = \begin{cases} \wedge \{\eta_P(y_1): y_1 \in f^{-1}(y_2)\} \\ 1, \ otherwise \end{cases}$$

Thus,

$$f(P)(y_2) = \begin{cases} (\vee \{\sigma_P(y_1): y_1 \in f^{-1}(y_2)\}, \vee \{\tau_P(y_1): y_1 \in f^{-1}(y_2)\}, \wedge \{\eta_P(y_1): y_1 \in f^{-1}(y_2)\}) \\ (0,0,1), \qquad otherwise \end{cases}$$

The pre-image of Q under f, denoted by $f^{-1}(Q)$ is also defined as

$$f^{-1}(Q)(y_1) = (\sigma_{f^{-1}(Q)}(y_1), \tau_{f^{-1}(Q)}(y_1), \eta_{f^{-1}(Q)}(y_1))$$

where

$$\sigma_{f^{-1}(Q)}(y_1) = \sigma_Q(f(y_1)), \tau_{f^{-1}(Q)}(y_1) = \tau_Q(f(y_1)), \eta_{f^{-1}(Q)}(y_1) = \eta_Q(f(y_1)).$$

Thus,

$$f^{-1}(Q)(y_1) = (\sigma_Q(f(y_1)), \tau_Q(f(y_1)), \eta_Q(f(y_1))).$$

Theorem 2.6. *Let $f: Y_1 \to Y_2$ be a mapping. Then,*

- $f(C_{r,s,t}(P)) \subseteq C_{r,s,t}(f(P))$, for all $P \in PFS(Y_1)$
- $f^{-1}(C_{r,s,t}(Q)) = C_{r,s,t}(Q)(f^{-1}(Q))$, for all $Q \in PFS(Y_2)$

Proof:

- Let $y_2 \in f(C_{r,s,t}(P))$, there exists $y_1 \in C_{r,s,t}(P)$ such that $f(y_1) = y_2$ and $\sigma_P(y_1) \geq r, \tau_P(y_1) \geq s$, and $\eta_P(y_1) \leq t$.

This means that $\vee \{\sigma_P(y_1): y_1 \in f^{-1}(y_2)\} \geq r, \vee \{\tau_P(y_1): y_1 \in f^{-1}(y_2)\} \geq s$ and $\wedge \{\sigma_P(y_1): y_1 \in f^{-1}(y_2)\} \leq t$.

This implies that $\sigma_P(y_1) \geq r, \tau_P(y_1) \geq s$ and $\eta_P(y_1) \leq t. \Rightarrow \ y_2 \in C_{r,s,t}(f(P))$.

Therefore, $f(C_{r,s,t}(P)) \subseteq C_{r,s,t}(f(P))$, for all $P \in PFS(Y_1)$.

- $C_{r,s,t}(Q)(f^{-1}(Q)) = \{y_1 \in Y_1: \sigma_Q(f(y_1) \geq r, \tau_Q(f(y_1) \geq s, \eta_Q(f(y_1) \leq t\}$

$$= \{y_1 \in Y_1: \sigma_Q(f(y_1)) \geq r, \tau_Q(f(y_1)) \geq s, \eta_Q(f(y_1)) \leq t\}$$
$$= \{y_1 \in Y_1: f(y_1) \in C_{r,s,t}(Q)\}$$
$$= \{y_1 \in Y_1: y_1 \in f^{-1}(C_{r,s,t}(Q))\}$$
$$= f^{-1}(C_{r,s,t}(Q))$$

Hence, $f^{-1}(C_{r,s,t}(Q)) = C_{r,s,t}(Q)(f^{-1}(Q))$, for all $Q \in PFS(Y_2)$.

3. Homomorphism of picture fuzzy group

This section gives some properties of homomorphism of a picture fuzzy subgroup of a group using a simpler approach called cut set of picture fuzzy sets.

Theorem 3.1. *Let $f: G \to G^*$ be a surjective homomorphism and Q a PFSG of G. Then, $f(Q)$ is a PFSG of G^*.*

Proof: By Theorem 2.3, it is suffices to show that $C_{r,s,t}(f(Q))$ is a PFSG of G^* for all $r, s, t \in [0,1]$ with $r + s + t \leq 1$.

Let $q_1, q_2 \in C_{r,s,t}(f(Q))$, then

$$\sigma_{f(Q)}(q_1) \geq r, \tau_{f(Q)}(q_1) \geq s, \eta_{f(Q)}(q_1) \leq t$$

and

$$\sigma_{f(Q)}(q_2) \geq r, \tau_{f(Q)}(q_2) \geq s, \eta_{f(Q)}(q_2) \leq t.$$

By Theorem 2.6 (i), we have $f(C_{r,s,t}(Q)) \subseteq C_{r,s,t}(f(Q))$, for all $Q \in PFS(G)$.

Therefore, there exists $p_1, p_2 \in G$ such that $\sigma_Q(p_1) \geq \sigma_{f(Q)}(p_1) \geq r$,

$\tau_Q(p_1) \geq \tau_{f(Q)}(p_1) \geq s, \eta_Q(p_1) \leq \eta_{f(Q)}(p_1) \leq t$

and $\sigma_Q(p_2) \geq \sigma_{f(Q)}(p_2) \geq r$, $\tau_Q(p_2) \geq \tau_{f(Q)}(p_2) \geq s$, $\eta_Q(p_2) \leq \eta_{f(Q)}(p_2) \leq t$.

This implies that

$$\sigma_Q(p_1) \geq r, \tau_Q(p_1) \geq s, \eta_Q(p_1) \leq t$$

and

$$\sigma_Q(p_2) \geq r, \tau_Q(p_2) \geq s, \eta_Q(p_2) \leq t.$$

So, $\sigma_Q(p_1) \wedge \sigma_Q(p_2) \geq r$, $\tau_Q(p_1) \wedge \tau_Q(p_2) \geq s$ and $\eta_Q(p_1) \leq \eta_Q(p_2) \leq t$.

Since Q is a PFSG of G, we have

$\sigma_Q(p_1 p_2^{-1}) \geq \sigma_Q(p_1) \wedge \sigma_Q(p_2) \geq r$, $\tau_Q(p_1 p_2^{-1}) \geq \tau_Q(p_1) \wedge \tau_Q(p_2) \geq s$

and $\eta_Q(p_1 p_2^{-1}) \leq \eta_Q(p_1) \vee \eta_Q(p_2) \leq t$.

This implies that $\sigma_Q(p_1 p_2^{-1}) \geq r$, $\tau_Q(p_1 p_2^{-1}) \geq s$ and $\eta_Q(p_1 p_2^{-1}) \leq t$.

Meaning that $p_1 p_2^{-1} \in C_{r,s,t}(Q) \;\Rightarrow\; f(p_1 p_2^{-1}) \in f(C_{r,s,t}(Q)) \subseteq C_{r,s,t}(f(Q))$,

$$\Rightarrow\; f(p_1) f(p_2^{-1}) \in C_{r,s,t}(f(Q)) \;\Rightarrow\; q_1 q_2^{-1} \in C_{r,s,t}(f(Q)).$$

Therefore, $C_{r,s,t}(f(Q))$ is a PFSG of G^*.

Theorem 3.2. *Let $f: G \to G^*$ be homomorphism of G into another group G^* and P a PFSG of G^*. Then, $f^{-1}(P)$ is a PFSG of G.*

Proof: By Theorem 2.3, it is suffices to show that $C_{r,s,t}(f^{-1}(P))$ is a PFSG of G for all $r, s, t \in [0,1]$ with $r + s + t \leq 1$.

Let $p_1, p_2 \in C_{r,s,t}(f^{-1}(Q))$, then

$$\sigma_{f^{-1}(P)}(p_1) \geq r, \tau_{f^{-1}(P)}(p_1) \geq s, \eta_{f^{-1}(P)}(p_1) \leq t$$

and

$$\sigma_{f^{-1}(P)}(p_2) \geq r, \tau_{f^{-1}(P)}(p_2) \geq s, \eta_{f^{-1}(P)}(p_2) \leq t$$

that is;

$$\sigma_P(f(p_1)) \geq r, \tau_P(f(p_1)) \geq s, \eta_P(f(p_1)) \leq t$$

and

$$\sigma_P(f(p_2)) \geq r, \tau_P(f(p_2)) \geq s, \eta_P(f(p_2)) \leq t.$$

Meaning that

$\sigma_P(f(p_1)) \wedge \sigma_P(f(p_2)) \geq r, \tau_P(f(p_1)) \wedge \tau_P(f(p_2)) \geq s$ and $\eta_P(f(p_1)) \vee \eta_P(f(p_2)) \leq t$.

Since P is a PFSG of G^*, we have

$\sigma_P(f(p_1) f(p_2)^{-1}) \geq \sigma_P(f(p_1)) \wedge \sigma_P(f(p_2)) \geq r$

$\tau_P(f(p_1)f(p_2)^{-1}) \geq \tau_P(f(p_1)) \wedge \tau_P(f(p_2)) \geq s$
and $\eta_P(f(p_1)f(p_2)^{-1}) \leq \eta_p(f(p_1)) \vee \eta_p(f(p_2)) \leq t$.
Thus, $\sigma_P(f(p_1)f(p_2)^{-1}) \geq r, \tau_P(f(p_1)f(p_2)^{-1}) \geq s$ and $\eta_P(f(p_1)f(p_2)^{-1}) \leq t$, which means $f(p_1)f(p_2)^{-1} \in C_{r,s,t}(P) \Rightarrow f(p_1(p_2)^{-1}) \in C_{r,s,t}(P)$.
By Theorem 2.6 (ii), we have $p_1(p_2)^{-1} \in f^{-1}(C_{r,s,t}(P)) = C_{r,s,t}(f^{-1}(P))$.
So, $p_1(p_2)^{-1} \in C_{r,s,t}(f^{-1}(P))$.
Therefore, $C_{r,s,t}(f^{-1}(P))$ is a PFSG of G.

Theorem 3.3. *Let $f: G \to G^*$ be a surjective homomorphism and Q a PFNSG of G. Then, $f(Q)$ is a PFNSG of G^*.*

Proof: Let $g^* \in G^*$ and $q \in f(Q)$. Then, there exists $g \in G$ and $p \in Q$ such that $f(p) = q$ and $f(g) = g^*$. Since Q is a PFNSG of G, therefore, $\sigma_Q(g^{-1}pg) = \sigma_Q(p), \tau_Q(g^{-1}pg) = \tau_Q(p)$ and $\eta_Q(g^{-1}pg) = \eta_Q(p)$ for all $p \in Q$ and $g \in G$. So,

$\sigma_{f(Q)}((g^*)^{-1}pg^*) = \sigma_{f(Q)}(f(g^{-1}pg))$ (since f is homomorphism)
$= \sigma_{f(Q)}(q')$, where $q' = f(g^{-1}pg) = (g^*)^{-1}qg^*$
$= \vee \{\sigma_Q(p'): f(p') = q', p' \in G\}$
$= \vee \{\sigma_Q(p'): f(p') = f(g^{-1}pg), p' \in G\}$
$= \vee \{\sigma_Q(g^{-1}pg): f(g^{-1}pg) = q' = (g^*)^{-1}qg^*, p \in Q, g \in G\}$
$= \vee \{\sigma_Q(g^{-1}pg): f(g^{-1}pg) = (g^*)^{-1}qg^*, p \in Q, g \in G\}$
$= \vee \{\sigma_Q(g^{-1}pg): f(g^{-1})f(p)f(g) = (g^*)^{-1}qg^*, p \in Q, g \in G\}$
$= \vee \{\sigma_Q(p): (g^*)^{-1}f(p)g^* = (g^*)^{-1}qg^*, p \in G\}$
$= \vee \{\sigma_Q(p): f(p) = q, p \in G\}$
$= \sigma_{f(Q)}(q)$,

$\tau_{f(Q)}((g^*)^{-1}pg^*) = \tau_{f(Q)}(f(g^{-1}pg))$ (since f is homomorphism)
$= \sigma_{f(Q)}(q')$, where $q' = f(g^{-1}pg) = (g^*)^{-1}qg^*$
$= \vee \{\tau_Q(p'): f(p') = q', p' \in G\}$
$= \vee \{\tau_Q(p'): f(p') = f(g^{-1}pg), p' \in G\}$
$= \vee \{\tau_Q(g^{-1}pg): f(g^{-1}pg) = q' = (g^*)^{-1}qg^*, p \in Q, g \in G\}$
$= \vee \{\tau_Q(g^{-1}pg): f(g^{-1}pg) = (g^*)^{-1}qg^*, p \in Q, g \in G\}$
$= \vee \{\tau_Q(g^{-1}pg): f(g^{-1})f(p)f(g) = (g^*)^{-1}qg^*, p \in Q, g \in G\}$
$= \vee \{\tau_Q(p): (g^*)^{-1}f(p)g^* = (g^*)^{-1}qg^*, p \in G\}$
$= \vee \{\tau_Q(p): f(p) = q, p \in G\}$
$= \tau_{f(Q)}(q)$

and

$\eta_{f(Q)}((g^*)^{-1}pg^*) = \eta_{f(Q)}(f(g^{-1}pg))$ (since f is homomorphism)
$= \eta_{f(Q)}(q')$, where $q' = f(g^{-1}pg) = (g^*)^{-1}qg^*$
$= \wedge \{\eta_Q(p'): f(p') = q', p' \in G\}$
$= \wedge \{\eta_Q(p'): f(p') = f(g^{-1}pg), p' \in G\}$
$= \wedge \{\eta_Q(g^{-1}pg): f(g^{-1}pg) = q' = (g^*)^{-1}qg^*, p \in Q, g \in G\}$
$= \wedge \{\eta_Q(g^{-1}pg): f(g^{-1}pg) = (g^*)^{-1}qg^*, p \in Q, g \in G\}$
$= \wedge \{\eta_Q(g^{-1}pg): f(g^{-1})f(p)f(g) = (g^*)^{-1}qg^*, p \in Q, g \in G\}$

$=\wedge\{\eta_Q(p):(g^*)^{-1}f(p)g^*=(g^*)^{-1}qg^*,p\in G\}$
$=\wedge\{\eta_Q(p):f(p)=q,p\in G\}$
$=\eta_{f(Q)}(q)$.

Hence, $fQ)$ is a PFNSG of group G^*.

5. Conclusion

In this paper, it has been established that the Picture Fuzzy MultiRelation (PFMR) is an extension of the Picture Fuzzy Relation (PFR). Some operations (union, intersection and complement) and some operators (Arithmetic mean operator, Geometric mean operator and Harmonic mean operator) have been studied with examples. Finally, the composition of PFMRs was introduced, and some of its properties were obtained. For future work, some applications of PFMR in decision-making, medical diagnosis, electoral systems, appointment procedures and pattern recognition will be explored.

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