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Max-norm and Square-max Norm of Fuzzy Matrices

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Abstract. Fuzzy matrices play an important role to model several uncertain systems. In this paper, two types of norms, viz. max norm and square-max norm of fuzzy matrices are introduced. Also, several properties are investigated.

Keywords: *Max norm, square-max norm, properties of max norm*

1. Introduction

The study of linear algebra has become more and more popular in the last few decades. People are attracted to this subject because of its beauty and its connection with many other pure and applied areas. In theoretical development of the subject as well as in many application, one often needs to measure the length of vectors. For this purpose, norm functions are consider on a vector space.

A norm on a real vector space V is a function $\|.\|: V \to R$ satisfying

1. $\|u\| > 0$ for any nonzero $u \in V$.
2. $\|ru\| = |r|\,\|u\|$ for any $r \in R$ and $u \in$ V.
3. $\|u+v\| \le \|u\| + \|v\|$ for any $u, v \in V$.

The norm is a measure of the size of the vector u where condition (1) requires the size to be positive, condition (2) requires the size to be scaled as the vector is scaled, and condition (3) is known as the triangle inequality and has its origin in the notion of distance in R^3. The condition (2) is called homogeneous condition and this condition ensure that the norm of the zero vector in V is 0; this condition is often included in the definition of a norm.

Common example of norms on R^n are the l_p norms,where $1 \le p \le \infty$, defined by

$$l_p(u) = \{\sum_{j=1}^{n} |u_j|^p\}^{\frac{1}{p}} \quad \text{if } 1 \le p < \infty \text{ and}$$

$$l_p(u) = \max_{1 \le j \le n} |u_j| \quad \text{if } p = \infty$$

for any $u = (u_1, u_2, ..., u_n)^t \in R^n$. Note that if one define an l_p function on R^n as define above with $0 < p < 1$, then it does not satisfy the triangle inequality, hence is not a norm.

Given a norm on a real vector space V, one can compare the norms of vectors, discuss convergence of sequence of vectors, study limits and continuity of transformations, and consider approximation problems such as finding the nearest element in a subset or a subspace of V to a given vector. These problems arise naturally in analysis, numerical analysis, differential equations, Markov chains, etc.

The norm of a matrix is a measure of how large its elements are. It is a way of determining the "size" of a matrix that is necessarily related to how many rows or columns the matrix has. The norm of a square matrix A is a non negative real number denoted by $\|A\|$. There are several different ways of defining a matrix norm but they all share the following properties:

1. $\|A\| \geq 0$ for any square matrix *A*.
2. $\|A\| = 0$ iff the matrix $A = 0$.
3. $\|KA\| = |K| \|A\|$ for any scaler K.
4. $\|A + B\| \leq \|A\| + \|B\|$ for any square matrix A, B.
5. $\|AB\| \leq \|A\|\|B\|$.

Different types of matrix norm:

The 1-norm

$$\|A\|_1 = \max_{1 \leq j \leq n} (\sum_{i=1}^{n} |a_{ij}|).$$

(the maximum absolute column sum). Simply we sum the absolute values down each column and then take the biggest answer (A useful reminder is that "1" is a tall, thin character and a column is a tall, thin quantity).

The infinity norm

$$\|A\|_\infty = \max_{1 \leq i \leq n} (\sum_{j=1}^{n} |a_{ij}|).$$

The infinity norm of a square matrix is the maximum of the absolute row sum. Simply we sum the absolute values along each row and then take the biggest answer.

Euclidean norm

$$\|A\|_E = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij})^2}.$$

The Euclidean norm of a square matrix is the square root of the sum of all the squares of the elements. This is similar to ordinary "Pythagorean" length where the size of a vector is found by taking the square root of the sum of the squares of all the elements.

Any definition you can define of which satisfies the five condition mentioned at the beginning of this section is a definition of a norm. There are many many possibilities, but the three given above are among the most commonly used.

Like vector norm and matrix norm, norm of a fuzzy matrix is also a function $\|.\| : M_n(F) \to [0,1]$ which satisfies the following properties

1. $\|A\| \geq 0$ for any fuzzy matrix A.
2. $\|A\| = 0$ iff the fuzzy matrix A =0.
3. $\|KA\| = |K| \|A\|$ for any scaler $K \in [0,1]$.
4. $\|A+B\| \leq \|A\| + \|B\|$ for any two fuzzy matrix A and B.
5. $\|AB\| \leq \|A\| \|B\|$ for any two fuzzy matrix A and B.

In this project paper we will define different type of norm on fuzzy matrices.

2. Why study different norms?

Different norm on a vector space can give rise to different geometrical and analytical structures. In an infinite dimensional vector space, the convergence of a sequence can vary depending on the choice of norm. This phenomena leads to many interesting questions and research in analysis and functional analysis.

In a finite dimensional vector space V, two norm $\|.\|_1$ and $\|.\|_2$ are said to be equivalent if there exist two positive constant such that

$$a\|v\|_1 \leq \|v\|_2 \leq \|v\|_1 \text{ for all } v \in V.$$

First, for a given sequence it may be easier to prove convergence with respect to one norm rather than another. In application such as numerical analysis one would like to use a norm that can determine convergence efficiently. Therefore, it is a good idea to have knowledge of different norms.

Second, sometimes a specific norm may be needed to deal with a certain problem. For instance, if one travels in Manhattan and wants to measure the distance from a location marked as the origin $(0,0)$ to a destination marked as (x, y) on the map, one may use the l_2 norm of (x, y), which measures the straight line distance between two points, or one may need to use the l_1 norm of v, which measures the distance for a taxi cab to drive from $(0,0)$ to (x, y). The l_1 norm is sometimes referred to as the taxi cab norm for this reason.

In approximation theory, solutions of a problem can vary with different problems. For example, if W is a subspace of R^n and v does not belongs to W, then for $1 < p < \infty$ there is a unique $u_0 \in W$ such that

$$\|v - u_0\| \leq \|v - u\| \qquad \text{for all } u \in W,$$

but the uniqueness condition may fail if $p = 1$ or ∞. To see a concrete example let $v = (1,0)$ and $W = \{(0, y) : y \in R\}$. Then for all $y \in [-1,1]$ we have $1 = \|v - (0, y)\| \leq \|v - w\|$ for all $w \in W$. For some problems, having a unique approximation is good, but for others it may be better to have many so that one of them can be chosen to satisfy additional conditions.

3. Fuzzy matrix

Fuzzy matrices were introduce for the first time by Thomason [42], who discussed the convergence of powers of fuzzy matrix. Ragab et al. [33,34] presented some properties of the min-max composition of fuzzy matrices. Hashimoto [18,19] studied the canonical form of a transitive fuzzy matrix. Hemashina et al. [20] Investigated iterates of fuzzy circulant matrices. Dererminant theory, powers and nilpotent conditions of matrices over a distributive lattice are consider by Zhang [43] and Tan [41]. After that Pal, Bhowmik, Adak, Shyamal, Mondal have done lot of works on fuzzy, intuitionistic fuzzy, interval-valued fuzzy, etc. matrices [1-12,24,25,27-32,35-39].

A Boolean matrix is a matrix with elements each has value 0 or 1. A fuzzy matrix is a matrix with elements having values in the closed interval [0,1]. We can still see that all fuzzy matrices are matrices but every matrix in general is not a fuzzy matrix. We see the fuzzy interval i.e., the unit interval is a subset of reals. Thus a matrix in general is not a fuzzy matrix since the unit interval [0,1] is contained in the set of reals. The big question is can we add two fuzzy matrices A and B and get the sum of them to be fuzzy matrix. The answer in general is not possible for the sum of two fuzzy matrices may turn out to be a matrix which is not a fuzzy matrix. If we add above two fuzzy matrix A and B then all entries in A+B will not lie in [0,1], hence A+B is only just a matrix and not a fuzzy matrix.

So only in case of fuzzy matrices the max or min operation are defined. Clearly under the max or min operations the resultant matrix is again a fuzzy matrix. In general to add two matrix we use max operation.

Now we wish to find the product of two fuzzy matrices X and Y where X and Y are compatible under multiplication. We see the product of two fuzzy matrices under usual matrix multiplication is not a fuzzy matrix. So we need to define a compatible operation analogous to product so that the product again happens to be a fuzzy matrix. However even for this new operation if the product XY is to be defined we need the number of columns of X is equal to the number of rows of Y. The two types of operations which we can have are max-min operation and min-max operation.

In fuzzy matrices only the elements are uncertain, while rows and columns are taken as certain. But in many real life situation we observe that rows and columns also be uncertain. For example, in afuzzy graph the vertices and edges both are uncertain. So, if we represent a fuzzy graph in matrix form where the membership values of vertices and edges represents the membership values of rows and columns and elements represent the membership values of the corresponding edge. That is, in this matrices rows and columns all are uncertain. We call this types of matrices are fuzzy matrices with fuzzy rows and columns. $A=[r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ be a fuzzy matrix with fuzzy rows and columns of order m×n. Here a_{ij}, $i=1,2,\ldots,m;$ $j=1,2,\ldots,n.$ represents the ij^{th} elements of A, $r_A(i)$ and $c_A(j)$ represents the membership values of ith row and jth column respectively for $i=1,2,\ldots,m;$ $j=1,2,\ldots,n.$

Definition 1. [41] *A **fuzzy matrix** (FM) of order \$m\times n\$ is defined as* $A=< a_{ij}, a_{ij\mu} >$ *where* $a_{ij\mu}$ *is the membership value of the ij-th element* a_{ij} *in A.*

An $n\times n$ *fuzzy matrix R is called reflexive iff* $r_{ii}=1$ *for all i=1,2,...,n. It is called* α *-reflexive iff* $r_{ii}\geq\alpha$ *for all i=1,2,...,n where* $\alpha\in[0,1]$. *It is called weakly reflexive iff* $r_{ii}\geq r_{ij}$ *for all i,j=1,2,...,n. An* $n\times n$ *fuzzy matrix R is called irreflexive iff* $r_{ii}=0$ *for all i=1,2,...,n.*

Definition 2. *An $n \times n$ fuzzy matrix S is called symmetric iff $s_{ij} = s_{ji}$ for all i,j=1,2,...,n. It is called antisymmetric iff $S \wedge S' \le I_n$ where I_n is the usual unit matrix.*

Note that the condition $S \wedge S' \le I_n$, means that $s_{ij} \wedge s_{ji} = 0$ for all $i \ne j$ and $s_{ii} \le 1$ for all i. So if $S_{ij} = 1$ then $s_{ji} = 0$, which the crisp case.

Definition 3. *An $n \times n$ fuzzy matrix N is called nilpotent iff $N^n = 0$ (the zero matrix). If $N^m = 0$ and $N^{m-1} \ne 0$; $1 \le m \le n$ then N is called nilpotent of degree m. An $n \times n$ fuzzy matrix E is called idempotent iff $E^2 = E$. It is called transitive iff $E^2 \le E$. It is called compact iff $E^2 \ge E$.*

Definition 4. *A triangular fuzzy matrix of order $m \times n$ is defined as $A = (a_{ij})_{m \times n}$ where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ is the ij^{th} element of A, m_{ij} is the mean value of a_{ij} and α_{ij}, β_{ij} are left and right spread of a_{ij} respectively.*

4. Max norm

We already know that norm of a fuzzy matrix can be define in several ways. It is also known to us that every norm must be satisfied the five condition which already we discuss in introduction. Now we define a new type of norm called max norm which gives the maximum element of the fuzzy matrix.

Max norm of a fuzzy matrix $A \in M_n(F)$ is denoted by $\|A\|_M$ and defined by

$$\|A\|_M = \bigvee_{i,j=1}^{n} a_{ij}$$

Lemma 1. *All the condition of norm are satisfied by $\|A\|_M = \bigvee_{i,j=1}^{n} a_{ij}$.*

Proof: Let us consider

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & .... & b_{1n} \\ b_{21} & b_{22} & .... & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & .... & b_{nn} \end{bmatrix}$$

We also consider $\bigvee_{i,j=1}^{n} a_{ij} = a_{pq}$ and $\bigvee_{i,j=1}^{n} b_{ij} = b_{kl}$.

So, $\|A\|_M = a_{pq}$ and $\|B\|_M = b_{kl}$.

(i) Clearly $\|A\|_M \ge 0$ and $\|A\|_M = 0$ iff $\bigvee_{i,j=1}^{n} a_{ij} = 0$

i.e. iff $a_{ij} = 0$, for all, i.e. iff $A = 0$

(ii) Let $\alpha \in [0,1]$ then $\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & .... & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & .... & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & .... & \alpha a_{nn} \end{bmatrix}$

if $\alpha > a_{pq}$ then $\bigvee_{i,j=1}^{n} \alpha a_{ij} = a_{pq}$. So, $\|\alpha A\|_M = \|A\|_M = |\alpha| \|A\|_M$

If $\alpha < a_{pq}$ then $\bigvee_{i,j=1}^{n} \alpha a_{ij} = \alpha$. So, $\|\alpha A\|_M = |\alpha| = |\alpha| \|A\|_M$.

If $\alpha = a_{pq}$ then obviously $\|\alpha A\|_M = |\alpha| \|A\|_M$.

(iii) Now

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & .... & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & .... & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & .... & a_{nn}+b_{nn} \end{bmatrix}$$

if $a_{pq} \le b_{kl}$ then $\bigvee_{i,j=1}^{n} (a_{ij}+b_{ij}) = b_{kl}$. So, $\|A+B\|_M = \|B\|_M = \|A\|_M + \|B\|_M$.

If $a_{pq} \ge b_{kl}$ then $\bigvee_{i,j=1}^{n} (a_{ij}+b_{ij}) = a_{pq}$. So , $\|A+B\|_M = \|A\|_M = \|A\|_M + \|B\|_M$.

Therefore $\|A+B\|_M = \|A\|_M + \|B\|_M$, for all $A, B \in M_n(F)$.

(iv) Now

$$AB = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & \sum_{i=1}^{n} a_{1i}b_{i2} & \cdots & \sum_{i=1}^{n} a_{1i}b_{in} \\ \sum_{i=1}^{n} a_{2i}b_{i1} & \sum_{i=1}^{n} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{n} a_{2i}b_{in} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{n} a_{ni}b_{i1} & \sum_{i=1}^{n} a_{ni}b_{i2} & \cdots & \sum_{i=1}^{n} a_{ni}b_{in} \end{bmatrix}$$

As $a_{ij} + b_{ij} = \bigvee\{a_{ij}, b_{ij}\}$ and $a_{ij}.b_{ij} = \bigwedge\{a_{ij}, b_{ij}\}$ therefore maximum element of AB is less than or equal to minimum of a_{pq} and b_{kl}. Here the symbol $\bigvee$ used to indicate maximum element and the symbol $\bigwedge$ used to indicate minimum element.

Thus, $\|AB\|_M \le \|A\|_M \|B\|_M$. Hence all the condions of norm are proved.

5. Properties of max norm

Properties 1. *For every fuzzy matrix* A, $\|A\|_M = \|A^T\|_M$ *always hold.*

Proof: As maximum element of A and A^T are equal so the above property hold trivially.

Properties 2. *For any two fuzzy matrix A and B in $M_n(F)$,*
$\|(A+B)^T\|_M = \|A^T\|_M + \|B^T\|_M$ always hold.

Proof: $\|(A+B)^T\|_M = \|A+B\|_M$ [using first property of max-norm.]

$= \|A\|_M + \|B\|_M$ [from the definition of max-norm.]

$= \|A^T\|_M + \|B^T\|_M$ [using first property of max-norm.]

Properties 3. *If A and B are two fuzzy matrices and $A \le B$ then $\|A\|_M \le \|B\|_M$.*

Proof: As $A \le B$ therefore $a_{ij} \le b_{ij}$ for all i, j.

$$\Rightarrow \bigvee_{i,j=1}^{n} a_{ij} \le \bigvee_{i,j=1}^{n} b_{ij}. \Rightarrow \|A\|_M \le \|B\|_M$$

Example 1. *Let* $A = \begin{bmatrix} 0.5 & 0.3 & 0.1 \\ 0.4 & 0.6 & 0.4 \\ 0.3 & 0.2 & 0.1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0.6 & 0.4 & 0.2 \\ 0.5 & 0.7 & 0.5 \\ 0.4 & 0.3 & 0.2 \end{bmatrix}$

$\therefore \|A\|_M = 0.6$ and $\|B\|_M = 0.7$. So, $\|A\|_M < \|B\|_M$.

Properties 4. *If A and B are two fuzzy matrices and $A \le B$ then $\|AC\|_M \le \|BC\|_M$ for all $C \in M_n(F)$.*

Proof: As $A \le B$ therefore $a_{ij} \le b_{ij}$ for all i, j.

This implies that $a_{ij}c_{ij} \le b_{ij}c_{ij}$ for all values of $c_{ij}; 1 \le i \le n; 1 \le j \le n$.

So, $\bigvee_{i,j=1}^{n} a_{ij}c_{ij} \le \bigvee_{i,j=1}^{n} b_{ij}c_{ij} \Rightarrow \|AC\|_M \le \|BC\|_M$

Properties 5. *For any two fuzzy matrix A and $B(\ne A)$ AB and BA may or may not be equal but $\|AB\|_M = \|BA\|_M$ always hold.*

Example 2. *Let* $A = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.6 & 0.2 & 0.4 \\ 0.7 & 0.4 & 0.3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0.4 & 0.6 & 0.2 \\ 0.7 & 0.3 & 0.5 \\ 0.8 & 0.5 & 0.4 \end{bmatrix}$

$$\therefore \; AB = \begin{bmatrix} 0.5 & 0.3 & 0.5 \\ 0.4 & 0.6 & 0.4 \\ 0.4 & 0.6 & 0.4 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0.6 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0.3 \\ 0.5 & 0.5 & 0.4 \end{bmatrix}$$

$\therefore \; \|AB\|_M = 0.6$ and $\|BA\|_M = 0.6$. So, $\|AB\|_M = \|BA\|_M$.

Properties 6. *If* $\|A_1\|_M \le \|A_2\|_M \le ... \le \|A_n\|_M$ *then*

(i) $\|A_1 + A_2 + ... + A_n\|_M = \|A_n\|_M$

(ii) $\|A_1 A_2 ... A_n\|_M \le \|A_1\|_M$.

Properties 7. *Max norm follows Laplace transform.*

Proof: Let $A, B \in M_n(F)$ and $\alpha, \beta \in [0,1]$ then

$\|\alpha A + \beta B\|_M = \|\alpha A\|_M + \|\beta B\|_M = |\alpha| \, \|A\|_M + |\beta| \, \|B\|_M$.

Definition 5. *Define a mapping* $d : M_n(F) \times M_n(F) \to [0,1]$ *as* $d(A,B) = \|A + B\|_M$ for all A, B in $M_n(F)$.

Proposition 1. *The above mapping d satisfies the following condition for all* A, B, C *in* $M_n(F)$

(i) $d(A,B) \ge 0$ and $d(A,B) = 0$ iff $A = B = 0$.

(ii) $d(A,B) = d(B,A)$

(iii) $d(A,B) \le d(A,C) + d(B,C)$ for all A, B, C in $M_n(F)$.

Proof: (i) $d(A,B) = \|A + B\|_M \ge 0$ [by first condition of norm.]

Again $d(A,B) = 0 \Leftrightarrow \|A + B\|_M = 0$

$\Leftrightarrow A + B = 0$

$\Leftrightarrow A = 0$ and $B = 0$

(ii) $d(A,B) = \|A + B\|_M = \|B + A\|_M = d(B,A)$

(iii) $d(A,B) = \|A + B\|_M \le \|A + B\|_M + \|C\|_M = \|A + B + C\|_M = \|A + B + C + C\|_M$

$= \|(A + C) + (B + C)\|_M = \|A + C\|_M + \|B + C\|_M = d(A,C) + d(B,C)$

So, $d(A,B) \le d(A,C) + d(B,C)$ for all A, B, C in $M_n(F)$.

Example 3. *Let* $A = \begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.6 & 0.9 & 0.6 \\ 0.1 & 0.7 & 0.7 \end{bmatrix}$, $B = \begin{bmatrix} 0.6 & 0.2 & 0.1 \\ 0.4 & 0.3 & 0.7 \\ 0.6 & 0.7 & 0.4 \end{bmatrix}$ and

$C = \begin{bmatrix} 0.6 & 0.4 & 0.6 \\ 0.2 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.4 \end{bmatrix}$. $\therefore A+B = \begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.6 & 0.9 & 0.7 \\ 0.6 & 0.7 & 0.7 \end{bmatrix}$

(i) $d(A,B) = \|A+B\|_M = 0.9 > 0$

(ii) $B+A = \begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.6 & 0.9 & 0.7 \\ 0.6 & 0.7 & 0.7 \end{bmatrix}$, then $\|B+A\|_M = 0.9$.

So, $d(A,B) = d(B,A)$.

(iii) Now $A+C = \begin{bmatrix} 0.8 & 0.4 & 0.6 \\ 0.6 & 0.9 & 0.6 \\ 0.1 & 0.7 & 0.7 \end{bmatrix}$ and $B+C = \begin{bmatrix} 0.6 & 0.4 & 0.6 \\ 0.4 & 0.5 & 0.7 \\ 0.6 & 0.7 & 0.4 \end{bmatrix}$.

$\therefore \|A+C\|_M = 0.9$ and $\|B+C\|_M = 0.7$.

Then $d(A,C) + d(B,C) = 0.9 + 0.7 = 0.9$.

So, $d(A,B) = d(A,C) + d(B,C)$.

Theorem 1. *If* $A, A', B, B' \in M_n(F)$ *then* $d(A,B) + d(A'+B') = d(A,A') + d(B,B')$.

Proof: $d(A,B) + d(A',B')$

$= \|A+B\|_M + \|A'+B'\|_M = \|A\|_M + \|B\|_M + \|A'\|_M + \|B'\|_M$

$= (\|A\|_M + \|A'\|_M) + (\|B\|_M + \|B'\|_M) = \|A+A'\|_M + \|B+B'\|_M = d(A,A') + d(B,B')$

Theorem 2. *If* $A, B \in M_n(F)$ *and* $A \le B$ *then* $d(A,C) \le d(B,C)$ *for all* $C \in M_n(F)$.

Proof: As $A \le B$.

So, $\|A\|_M \le \|B\|_M$

$\Rightarrow \|A\|_M + \|C\|_M \le \|B\|_M + \|C\|_M$

$\Rightarrow \|A+C\|_M \le \|B+C\|_M$

$\Rightarrow d(A,C) \le d(B,C)$ for all $C \in M_n(F)$.

Definition 6. *Define a mapping* $d' : M_n(F) \times M_n(F) \to [0,1]$ *as* $d'(A,B) = \min\{\|A\|_M, \|B\|_M\}$ for all A, B in $M_n(F)$.

Proposition 2. *The above mapping* d' *satisfies the following condition for all* A, B, C *in* $M_n(F)$.

(i) $d'(A,B) \geq 0$ and $d'(A,B) = 0$ iff $A = 0$ or $B = 0$ or both $A = B = 0$
(ii) $d'(A,B) = d'(B,A)$.

Proof: (i) $d'(A,B) = min\{\|A\|_M, \|B\|_M\} \geq 0$ as $\|A\|_M \geq 0$ and $\|B\|_M \geq 0$
Now $d'(A,B)$ =min$\{\|A\|_M, \|B\|_M\} = 0$
$\Rightarrow \|A\|_M = 0$ or $\|B\|_M = 0$ or both $\|A\|_M = \|B\|_M = 0$
$\Rightarrow$ either $A = 0$ or $B = 0$ or both $A = B = 0$
(ii) $d'(A,B)$ =min$\{\|A\|_M, \|B\|_M\}$ =min$\{\|B\|_M, \|A\|_M\} = d'(B,A)$.

Proposition 3. *If* $A, B \in M_n(F)$ *and* $A \leq B$ *then* $d'(A,C) \leq d'(B,C)$ *for all* $C \in M_n(F)$.

Proof: Since $A \leq B$, so $\|A\|_M \leq \|B\|_M$.
Now $d'(A,C)$ =min$\{\|A\|_M, \|C\|_M\}$ and $d'(B,C)$ =min$\{\|B\|_M, \|C\|_M\}$.

Case-1:
If $\|A\|_M \leq \|B\|_M \leq \|C\|_M$ then $d'(A,C) = \|A\|_M \leq \|B\|_M = d'(B,C)$
i.e $d'(A,C) \leq d'(B,C)$.

Case-2:
If $\|C\|_M \leq \|A\|_M \leq \|B\|_M$ then $d'(A,C) = \|C\|_M = d'(B,C)$.

Case-3:
If $\|A\|_M \leq \|C\|_M \leq \|B\|_M$ then $d'(A,C) = \|A\|_M$ and $d'(B,C) = \|C\|_M$.
So, $d'(A,C) \leq d'(B,C)$.
Therefore $d'(A,C) \leq d'(B,C)$ for all $C \in M_n(F)$.

Definition 7. *For all* A *in* $M_n(F)$ *we define*

$$A_{sup} = \{x \in M_n(F) : \|x\|_M > \|A\|_M\}$$
$$A_{inf} = \{x \in M_n(F) : \|x\|_M < \|A\|_M\}$$
$$A_{equ} = \{x \in M_n(F) : \|x\|_M = \|A\|_M\}.$$

Clearly $M_n(F) = A_{sup} \cup A_{inf} \cup A_{equ}$.
The set A_{sup} is called *max-superior* to A, A_{inf} is called *max-inferior* to A and A_{equ} is called *max-equivalent* to A.

Theorem 3. *For each* A *in* $M_n(F)$ *the following results hold true*
(i) If $X \in A_{sup}$ (or A_{inf} or A_{equ}) then X^T is also in A_{sup} (or A_{inf} or A_{equ}) where X^T is the transpose of X.
(ii) If $A_1 \in A_{sup}$, $A_2 \in A_{inf}$ and $A_3 \in A_{equ}$ then $\|A_1 + A_2 + A_3\|_M = \|A_1\|_M$.

(iii) $\|A_1A_2A_3\|_M \le \|A_2\|_M$

(iv) $A^T \in A_{equ}$ for all A in $M_n(F)$.

Proof: (i) Since maximum element of A and A^T are equal therefore $\|A\|_M = \|A^T\|_M$. So, if $X \in A_{sup}$ then $\|X\|_M > \|A\|_M \Rightarrow \|X^T\|_M > \|A\|_M$.

Therefore, $X^T \in A_{sup}$. Similarly other two cases also hold.

(ii) $A_1 \in A_{sup} \Rightarrow \|A_1\|_M > \|A\|_M$

$A_2 \in A_{inf} \Rightarrow \|A_2\|_M < \|A\|_M$

$A_3 \in A_{equ} \Rightarrow \|A_3\|_M = \|A\|_M$

So, we can write $\|A_2\|_M < \|A\|_M = \|A_3\|_M < \|A_1\|_M$ (i)

From above relation it is clear that maximum element of A_1 is grater than maximum element of A_2 and maximum element of A_3.

Therefore, maximum element of $A_1 + A_2 + A_3$ = maximum element of A_1.

This implies $\|A_1 + A_2 + A_3\|_M = \|A_1\|_M$.

(iii) We know that $\|AB\|_M \le \|A\|_M \|B\|_M$.

Then we have $\|A_1A_2A_3\|_M \le \|A_1\|_M \|A_2A_3\|_M \le \|A_1\|_M \|A_2\|_M \|A_3\|_M = \|A_2\|_M$ [using(i)]

So, $\|A_1A_2A_3\|_M \le \|A_2\|_M$.

(iv) As $\|A\|_M = \|A^T\|_M$ therefore $A^T \in A_{equ}$ for all A in $M_n(F)$.

6. Square-max norm

Here we will define another new norm of fuzzy matrix named Square-Max norm. In this norm at first we will find the maximum element of the fuzzy matrix and then square it. Square-max norm of a fuzzy matrix A is denoted by $\|A\|_{SM}$ and define by

$$\|A\|_{SM} = (\bigvee_{i,j=1}^{n} a_{ij})^2 = (\|A\|_M)^2.$$

Lemma 2. *All the conditions of norm are satisfied by* $\|A\|_{SM} = (\bigvee_{i,j=1}^{n} a_{ij})^2 = (\|A\|_M)^2$.

Proof: Let us consider

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & .... & b_{1n} \\ b_{21} & b_{22} & .... & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & .... & b_{nn} \end{bmatrix}$$

Let maximum element of A be a_{pq} i.e. $\bigvee_{i,j=1}^{n} a_{ij} = a_{pq}$

and maximum element of B be b_{kl} i.e. $\bigvee_{i,j=1}^{n} b_{ij} = a_{kl}$.

$\|A\|_{SM} = (a_{pq})^2$ and $\|B\|_{SM} = (b_{kl})^2$

(i) Obviously $\|A\|_{SM} \geq 0$ and $\|A\|_{SM} = 0$ iff $a_{pq} = 0$ i.e. iff $a_{ij} = 0$ for all $i, j,$ i.e. iff A=0

(ii) Now we define scalar multiplication of a matrix as follows

$$\alpha a_{ij} = \begin{cases} \sqrt{\|A\|_{SM}} & if \quad |\alpha| > \|A\|_{SM} \\ \sqrt{|\alpha|} & if \quad |\alpha| \leq \|A\|_{SM} \end{cases} \text{ for all } i, j.$$

So, if $|\alpha| > \|A\|_{SM}$ then $\|\alpha A\|_{SM} = \|A\|_{SM} = |\alpha| \, \|A\|_{SM}$

and if $|\alpha| < \|A\|_{SM}$ then $\|\alpha A\|_{SM} = |\alpha| = |\alpha| \, \|A\|_{SM}$.

Therefore $\|\alpha A\|_{SM} = |\alpha| \, \|A\|_{SM}$ for all $\alpha \in [0,1]$.

(ii)

Now $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & .... & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & .... & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & .... & a_{nn}+b_{nn} \end{bmatrix}$

If $a_{pq} < b_{kl}$ then $\bigvee(a_{ij} + b_{ij}) = b_{kl}$ and then $\|A+B\|_{SM} = \|B\|_{SM} = \|A\|_{SM} + \|B\|_{SM}$.

Again if $a_{pq} > b_{kl}$ then $\bigvee(a_{ij} + b_{ij}) = a_{pq}$ and then

$\|A+B\|_{SM} = \|A\|_{SM} = \|A\|_{SM} + \|B\|_{SM}$.

(iii)

Now $AB = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & \sum_{i=1}^{n} a_{1i}b_{i2} & .... & \sum_{i=1}^{n} a_{1i}b_{in} \\ \sum_{i=1}^{n} a_{2i}b_{i1} & \sum_{i=1}^{n} a_{2i}b_{i2} & .... & \sum_{i=1}^{n} a_{2i}b_{in} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{n} a_{ni}b_{i1} & \sum_{i=1}^{n} a_{ni}b_{i2} & .... & \sum_{i=1}^{n} a_{ni}b_{in} \end{bmatrix}$

As $a_{ij} + b_{ij} = \bigvee\{a_{ij}, b_{ij}\}$ and $a_{ij}.b_{ij} = \bigwedge\{a_{ij}, b_{ij}\}$ therefore maximum element of AB is less than or equal to minimum of a_{pq} and b_{kl}. Here the symbol $\bigvee$ used to indicate maximum element and the symbol $\bigwedge$ used to indicate minimum element.

So, $\|AB\|_{SM} \leq \|A\|_{SM} \|B\|_{SM}$.

Hence all the conditions of norm are satiesfied by Square-max norm.

Theorem 4. *Addition of two norm of fuzzy matrices is also a norm.*

Proof: Let $\|.\|_1$ and $\|.\|_2$ be two norm on $M_n(F)$.

Now let us consider a function $\|.\|: M_n(F) \to [0,1]$ define by $\|A\| = \|A\|_1 + \|A\|_2$ for all $A \in M_n(F)$.

(i) Since $\|A\|_1 \geq 0$ and $\|A\|_2 \geq 0$ so, $\|A\| \geq 0$

Again $\|A\|_1 = 0$ and $\|A\|_2 = 0$ iff $A = 0$. So, $\|A\| = 0$ iff $A = 0$.

(ii) Let $\alpha \in F$ then $\|\alpha A\| = \|\alpha A\|_1 + \|\alpha A\|_2$

$= |\alpha| \|A\|_1 + |\alpha| \|A\|_2$ [as $\|.\|_1$ and $\|.\|_2$ be two norm]

$= |\alpha| (\|A\|_1 + \|A\|_2) = |\alpha| \|A\|$

(iii) Let $A, B \in M_n(F)$

$\|A+B\| = \|A+B\|_1 + \|A+B\|_2$

$\leq \|A\|_1 + \|B\|_1 + \|A\|_2 + \|B\|_2 = (\|A\|_1 + \|A\|_2) + (\|B\|_1 + \|B\|_2) = \|A\| + \|B\|$

So, $\|A+B\| \leq \|A\| + \|B\|$ for all $A, B \in M_n(F)$.

(iv) $\|AB\| = \|AB\|_1 + \|AB\|_2$

$\leq \|A\|_1 \|B\|_1 + \|A\|_2 \|B\|_2 \leq \|A\|_1 \|B\|_1 + \|A\|_1 \|B\|_2 + \|A\|_2 \|B\|_1 + \|A\|_2 \|B\|_2$

$= (\|A\|_1 + \|A\|_2)(\|B\|_1 + \|B\|_2) = \|A\| \|B\|$

So, $\|AB\| \leq \|A\| \|B\|$ for all $A, B \in M_n(F)$.

Therefore $\|.\|$ fulfill all the conditions of norm and hence $\|.\|$ is a norm on $M_n(F)$.

Theorem 5. *Scalar multiplication of a norm of fuzzy matrices is not a norm.*

Proof : Let $\|.\|_1$ be a norm on $M_n(F)$.

Case-1*:*

Now let $\|.\|: M_n(F) \to [0,1]$ be a function define by $\|A\| = C\|A\|_1$ for all $A \in M_n(F)$ and $C \in (0,1)$.

(i) As $\|A\|_1 \geq 0$ and $C > 0$, so $\|A\| \geq 0$ and $\|A\| = 0$ iff $A = 0$.

(ii) Let $\alpha \in F$ then $\|\alpha A\| = C\|\alpha A\|_1 = C|\alpha| \|A\|_1$ [as $\|.\|_1$ is a norm]

$= |\alpha| (C\|A\|_1) = |\alpha| \|A\|$ for all $\alpha \in F$.

(iii) $\|A+B\| = C\|A+B\|_1 \leq C(\|A\|_1 + \|B\|_1) = C\|A\|_1 + C\|B\|_1$

$= \|A\| + \|B\|$ for all $A, B \in M_n(F)$.

(iv) $\|AB\| = C\|AB\|_1 \leq C\|A\|_1 \|B\|_1 \not\leq C\|A\|_1 C\|B\|_1$ $[\because C \in (0,1)]$

So, $\|.\|$ dose not fulfill all the condions of a norm .

Therefore $\|.\|$ is not a norm on $M_n(F)$.

Case-2:

As $\|A\| \geq 0$ for all $A \in M_n(F)$ so, C never be negative.

Case-3:

Let C=0, then $\|A\| = C\|A\|_1 = 0$ always hold.

So, $\|A\| = 0$ and it does not imply $A = 0$.Therefore $C \neq 0$

Case-4:

Let $C > 1$, then $\|A\| = C\|A\|_1$ does not belongs to $[0,1]$ for all $C > 1$.

From above fore cases it is clear that scalar multiplication of a norm never become a norm.

7. Conclusion

In this paper, we define max-norm and square-max norm of fuzzy matrices. In different situation we use different norm. Somewhere max norm is suitable to use than square-max norm, somewhere square-max norm is suitable than max norm. We already prove that max norm satisfied Laplace transformation. So max norm is very important things in application area. Using these norm we can define conditional number to check whether a system of linear equation is ill posed or well posed. Norm of fuzzy matrices can take a effective contribution to solve a fuzzy system of linear equation. Similarly we can define norm on triangular fuzzy matrix, circulant triangular fuzzy matrix, fuzzy membership matrix etc. fuzzy membership matrix is used in medical diagnosis and decision making. So, if we define norm on fuzzy membership matrix then it will take a effective contribution on medical science.

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