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A Generalization of the Pigeonhole Principle

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Abstract. In this short note, we present a generalization of the pigeonhole principle.

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1. Introduction

In mathematics, the pigeonhole principle is also called Dirichlet drawer principle. Although it is intuitive, it can be used to obtain interesting results. For instance, the pigeonhole principle is used to find Ramsey number $R(3,3) = 6$ (see Example 13 on Page 352 in [1]). The pumping lemma for regular languages in computer science is obtained by using the pigeonhole principle. The elementary version of the pigeonhole principle can be stated as follows.

The Pigeonhole Principle. Suppose we place n pigeons into m pigeonholes, where $n > m$. Then there exists at least one pigeonhole contains at least two pigeons.

The pigeonhole principle can be generalized as follows (see Theorem 2 on Page 349 in [1]).

The Generalized Pigeonhole Principle. Suppose we place n pigeons into m pigeonholes, where $n > m$. Then there exists at least one pigeonhole contains at least $\lfloor \frac{n-1}{m} \rfloor + 1 = \lceil \frac{n}{m} \rceil$ pigeons.

The goal of this short note is to prove the following theorem which is a generalization of the generalized pigeonhole principle.

Theorem 1. Suppose we place n pigeons into m pigeonholes, where $n > m$. Assume that the probability that the jth pigeon is placed into ith pigeonhole is p_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq n$. Then there exists at least one pigeonhole contains at least $\lfloor \frac{n-1}{m} \rfloor + 1 = \lceil \frac{n}{m} \rceil$ pigeons which have positive placement probability for that pigeonhole.

Below are explanations for that Theorem 1 is a generalization of the generalized pigeonhole principle. In the generalized pigeonhole principle, our assignment of the j th, for each j with $1 \le j \le n$, pigeon into pigeonholes always satisfies that there exists an i, where $1 \le i \le m$, such that $p_{ij} = 1$ and $p_{kj} = 0$ for each $k \in \{1,2,\dots,m\} - \{i\}$. In this case, it is obvious that $p_{ij} > 0$ if and only if the i th pigeonhole contains the j th pigeon. Applying Theorem 1 to this scenario, we have that Theorem 1 is a generalization of the generalized pigeonhole principle.

2. Proofs

In order to prove Theorem 1, we need the following lemma.

Lemma 1. Let $A = (a_{ij})$ be an $m \times n$ real matrix such that $n > m$ and $a_{ij} \ge 0$. Suppose $\sum_{i=1}^{m} a_{ij} = u$ for each j with $1 \le j \le n$, where u is real and $u > 0$. Then there exists an i, $1 \le i \le m$, such that $|\{j : a_{ij} > 0\}| \ge \lfloor \frac{n-1}{m} \rfloor + 1$.

Proof of Theorem 1: Notice first that $a_{ij} \le u$ for each i and j with $1 \le i \le m$ and $1 \le j \le n$. Suppose, to the contrary, that for each i with $1 \le i \le m$, $|\{j : a_{ij} > 0\}| \le \lfloor \frac{n-1}{m} \rfloor$. Then

$$nu = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \le \sum_{i=1}^{m} \lfloor \frac{n-1}{m} \rfloor u \le mu \frac{n-1}{m} = nu - u,$$

a contradiction. This completes the proof of Lemma 1. □

Theorem 1 follows from by letting $u = 1$ and $a_{ij} = p_{ij}$, where $1 \le i \le m$ and $1 \le j \le n$, in Lemma 1.

3. Conclusions

In this short note, we provide a new generalization of the pigeonhole principle. It can be regarded as the probabilistic version of the pigeonhole principle. It is interesting to find applications of this probabilistic pigeonhole principle.

REFERENCES

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