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Approximated Technique to Solve a Time-Fractional Diffusion-Convection Reaction Equation Using Natural Transform

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Abstract. In this study, we investigate the time-fractional diffusion–convection-reaction equation and develop an approximate numerical solution using the Natural Decomposition Method (NDM). The proposed approach employs the Natural transform in conjunction with the Caputo fractional derivative operator, followed by the application of the Riemann–Liouville fractional integral. Nonlinear terms are efficiently treated through Adomian polynomials, which facilitate decomposition and enhance computational tractability. The findings demonstrate that the Natural approach provides high accuracy with comparatively fast convergence. All symbolic and numerical computations, as well as graphical illustrations, are performed using MATLAB. To validate the effectiveness and applicability of the method, several illustrative examples are presented, confirming its reliability in solving fractional partial differential equations. The results underscore the potential of the NDM as a powerful tool for addressing a broad class of fractional diffusion–convection reaction problems.

Keywords: Caputo derivative, Diffusion Convection equation, Riemann-Liouville derivative, Natural Transform, Time-fractional derivatives.

AMS Mathematics Subject Classification (2020): 34C20

1. Introduction

The mathematical equation describing diffusion difficulties has attracted the attention of numerous academics over the years. Cherniha and Serov [1] provided a fresh analysis and accurate solutions for non-linear diffusion equations. New modification equations were derived by Kuske and Mileniski [2] for the hexagon-style in reaction-diffusion systems. These systems exhibit more non-linearities than Smith-Hohenberg models or Rayleigh-Bernard convection. Matano et al. [3] investigated the interaction and diffusion equations using the spatially heterogeneous interaction term. If this reaction's term coefficient is far higher than the dispersion coefficient, the strong interface between two separate phases will be visible. They demonstrated that the motion equation for this interface includes a drift term even though drift was absent from the original diffusion equations. The researchers in [4] investigated the uniqueness and existence of the solution to the self-similarity of diffusion equation. A study was performed to examine fast gas flow models by heating various materials using a microwave and by porous media. A nonlinear diffusion-advection-reaction model is being investigated in[5]and[6-7]. According to[8][9][10], the general theory of this model focused on existence, uniqueness and regularity. In particularly, we consider the nonlinear fractional PDE that describes an arrangement that can be better understood by looking at the solutions of fractional-order DEs. Fractional derivatives offer more accurate representations of real-world issues than integer-order derivatives. Because of their numerous uses in science, fractional PDEs have proven to be a useful tool for describing the diffusion processes[11], viscoelasticity and electrical phenomena [12]. The Fractional PDE was discovered that, when taken along the time scaling limit, fractional time derivatives typically appear as infinitesimal generators of the time evolution. Therefore, the need to clarify the ideas of equilibrium, stability states, and temporal evolution at the long-term limit justifies the significance of looking into fractional equations. To find the approximate solutions, a variety of effective approaches have been presented for solving fractional partial differential equations. The double Laplace formulas for partial fractional derivatives were developed by the authors in[13], and they can be used to solve a fractional heat equation under specific initial and boundary conditions. They use the fractional complex transform approach in[14] to study the transport equations in fractal porous media. Numerous authors have studied fractional order PDEs in recent years using a variety of techniques, including the variational iteration method(VIM) in [15-16], the homotopy perturbation method in [17-18] and [19], the Laplace transform(LT) and Laplace homotopy perturbation method in [20], and the homotopy analysis technique in [21].

We apply the Natural decomposition method (NDM) to fractional derivatives, and use it to solve initial value fractional differential equations. The authors of [26],[28],[29],[30]and [31] examined a number of Natural transform (NT) properties, using this knowledge to create simple and effective methods for handling ordinary and partial differential equations. There is clearly a desire to learn more about this change and apply it to a variety of mathematical and physical research challenges. The NT, which is linear and bilateral with scale and unit preserving properties[26], can be utilized to solve a wide range of difference and differential equation problems without the need for a new frequency domain. We use the NT to solve the generic form. of nonlinear time-Fractional diffusion equation

$$\partial_\tau^\lambda \omega(x,t) = \omega_{xx}(x,t) - \alpha\omega_x(x,t) - \beta\omega(x,t), \qquad \text{for } 0 < \lambda \le 1 \tag{1}$$

Here ∂_τ^λ is the fractional Caputo derivative (FCD) of order α respect on the time variable. and $\boldsymbol{I_0^{1-\lambda}}$is the fractional integral of Riemann-Liouville in the time variable of order $1-\lambda$, defined for every summable function ω as

$$I_0^\lambda \omega(x,t) = \frac{1}{\Gamma(\lambda)}\int_0^t (t-\tau)^{\lambda-1}\omega(x,\tau)d\tau. \tag{2}$$

Then the FCD of order λ respect on the time variable is given as follows

$$\partial_\tau^\lambda \omega(x,t) = I_0^{1-\lambda}\omega_t(x,t) = \frac{1}{\Gamma(1-\lambda)}\int_0^t (t-\tau)^{-\lambda}\omega_\tau(x,\tau)d\tau. \tag{3}$$

The Mittag-Leffler function(MLF) [27]is an essential function that is widely used in fractional calculus. The MLF solves non-integer order DEs in the same way as the exponential does for integer order DEs. The exponential function is actually a very particular variation—among an infinite number—of this function that appears to be used by everyone. Eq.(1) provides the conventional definition of Mittag-Leffler.

$$\mathbb{E}_{\lambda}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\lambda k+1)},\quad \lambda>0. \tag{4}$$

The exponential function represents the case $\lambda=1$,

$$\mathbb{E}_{1}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(k+1)}=e^{t}. \tag{5}$$

In addition, it is usual to represent the MLF using two arguments, λ and μ, so that

$$\mathbb{E}_{\lambda,\mu}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\lambda k+\mu)},\quad \lambda>0,\mu>0. \tag{6}$$

The Natural transform (NT) is defined over the set of function

$$\mathbb{A}=\left\{\omega\,(x,t)|\exists C,\epsilon_{1},\epsilon_{2}>0,|\omega(x,t)|<Ce^{\frac{|t|}{\mu_{j}}},if\ t\in(-1)^{j}\times[0,\infty)\right\}$$

as given by the following formula

$$\mathbb{N}[\omega(x,t)]=\frac{1}{p}\int_{0}^{\infty}\omega(x,t)e^{\frac{-st}{p}}\,dt,\qquad s,p\in(0,\infty) \tag{7}$$

In Belgacem et al. [26], demonstrated that the NT represents the theoretical dual to the LT. As a result, one should be able to compete with it on a large scale for solving of the problem. Many of the NT's unique qualities are discussed and summarized in [26][28]. The NT of the fractional derivative introduced by Caputo is stated by

$$\mathbb{N}\left[\partial_{\tau}^{\lambda}\omega(x,t)\right]=\frac{s^{\alpha}}{p^{\alpha}}\mathbb{N}[\omega(x,t)]-\sum_{k=0}^{n-1}\frac{s^{\alpha-1-k}\omega^{(\mathrm{k})}(x,0)}{p^{\alpha-k}},\qquad(n-1<\lambda\leq n). \tag{8}$$

2. Analysis of Natural Adomian Decomposition Method (NADM)

The nonlinear The time-Fractional diffusion-convection equation (1) is defined together with an initial"

$$\omega(x,0)=\phi(x). \tag{9}$$

use the **NADM** to Eq.(1), then it transforms into

$$\mathbb{N}\left[\partial_{\tau}^{\lambda}\omega(x,t)\right]=\mathbb{N}[\omega_{xx}(x,t)-\alpha\omega_{x}(x,t)-\mu\omega(x,t)+\Psi(\omega)],\quad t\in(0,t^{*})$$

$$\frac{s^{\lambda}}{p^{\lambda}}\mathbb{N}[\omega(x,t)]-\sum_{k=0}^{n-1}\frac{s^{\lambda-1-k}\omega^{(\mathrm{k})}(x,0)}{p^{\lambda-k}}=\mathbb{N}[\omega_{xx}(x,t)-\alpha\omega_{x}(x,t)-\mu\omega(x,t)+\Psi(\omega)],$$

$$\frac{s^{\lambda}}{p^{\lambda}}\mathbb{N}[\omega(x,t)]=\sum_{k=0}^{n-1}\frac{s^{\lambda-1-k}\omega^{(\mathrm{k})}(x,0)}{p^{\lambda-k}}+\mathbb{N}[\omega_{xx}(x,t)-\alpha\omega_{x}(x,t)-\mu\omega(x,t)+\Psi(\omega)]$$

Then we obtain

$$\mathbb{N}[\omega(x,t)]=\frac{p^{\lambda}}{s^{\lambda}}\sum_{k=0}^{n-1}\frac{s^{\lambda-1-k}\omega^{(\mathrm{k})}(x,0)}{p^{\lambda-k}}+\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}[\omega_{xx}(x,t)-\alpha\omega_{x}(x,t)-\mu\omega(x,t)+\Psi(\omega)] \tag{10}$$

By applying the convolution theorem and the inverse Natural transform to both sides of equation (10), we can now obtain

$$\omega(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\sum_{k=0}^{n-1}\frac{s^{\lambda-1-k}\omega^{(\mathrm{k})}(x,0)}{p^{\lambda-k}}+\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}[\omega_{xx}(x,t)-\alpha\omega_{x}(x,t)+\mu\omega(x,t)+\Psi(\omega)]\right),$$

$$\omega(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\sum_{k=0}^{n-1}\frac{s^{\lambda-1-k}\omega^{(\mathrm{k})}(x,0)}{p^{\lambda-k}}\right)+\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}[\omega_{xx}(x,t)-\alpha\omega_{x}(x,t)+\mu\omega(x,t)+\Psi(\omega)]\right). \tag{11}$$

It may be expressed in the form, and $\alpha=1$,$\mu=1$

$$\omega(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\sum_{k=0}^{n-1}\frac{s^{\lambda-1-k}\omega^{(\mathrm{k})}(x,0)}{p^{\lambda-k}}\right)+\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}[\omega_{xx}(x,t)-\omega_{x}(x,t)+\omega(x,t)+\Psi(\omega)]\right).$$

We now represent the solution as the infinite series given below

$$\omega(x,t)=\sum_{n=0}^{\infty}\omega_{n}(x,t), \tag{12}$$

where the nonlinear term $\Psi(\omega)$is decomposed as the following

$$\Psi(\omega(\mathrm{x},\mathrm{t}))=\sum_{\mathrm{n}=0}^{\infty}\mathrm{B}_{\mathrm{n}}(\omega_{0},\omega_{1},\ldots,\omega_{\mathrm{n}}), \tag{13}$$

where B_{n} can be calculated

$$\mathrm{B}_{\mathrm{n}}=\frac{1}{\mathrm{n}!}\left[\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{d}^{\lambda}}\Psi\left(\sum_{\mathrm{n}=0}^{\infty}\lambda^{\mathrm{k}}\omega_{\mathrm{n}}\right)\right]. \tag{14}$$

Substitution of Eq. (12) and (13) in Eq.(11) leads to

$$\sum_{n=0}^{\infty}\omega_{n}(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\sum_{k=0}^{n-1}\frac{s^{\lambda-1-k}\phi^{(\mathrm{k})}(x,0)}{p^{\lambda-k}}\right) \tag{15 }$$

$$+\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}\left[\sum_{\mathrm{n=0}}^{\infty}(\omega_{\mathrm{n}})_{\mathrm{xx}}-\sum_{\mathrm{n=0}}^{\infty}(\omega_{\mathrm{n}})_{\mathrm{x}}+\sum_{\mathrm{n=0}}^{\infty}\omega_{\mathrm{n}}+\sum_{\mathrm{n=0}}^{\infty}\mathrm{B}_{\mathrm{n}}(\omega_0,\omega_1,\dots,\omega_{\mathrm{n}})\right]\right)$$

By comparing both sides of the Eq. (15), we get

$$\omega_0(x,t)=\phi(x),$$

$$\omega_1(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}[\phi_{xx}-\phi_x+\phi+\mathrm{B}_0(\phi)]\right),$$

$$\omega_2(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}[(\omega_1)_{xx}-(\omega_1)_x+\omega_1+\mathrm{B}_1(\mathrm{u}_0,\mathrm{u}_1)]\right)$$

$$\vdots$$

$$\omega_n(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}\big((\omega_{n-1})_{xx}-(\omega_{n-1})_x+\omega_{n-1}+\mathrm{B}_{\mathrm{n}}(\mathrm{u}_0,\mathrm{u}_1,\dots,\mathrm{u}_{\mathrm{n}})\big)\right).$$

Finally, the analytical solution $\omega(x,t)$ is approximated using a truncated series:

$$\omega(x,t)=\lim_{n\to\infty}\varphi_n(x,t), \qquad (16)$$

where $\varphi_n(x,t)=\sum_{k=0}^{n}\omega_k(x,t)$ is the sequence of partial sums to the series Eq.(15).

3. Numerical experiments

Example 3.1. Let the one-dimensional parabolic fractional diffusion equation:

$$\begin{matrix}\partial_\tau^{\lambda}\omega(x,t)=\omega_{xx}(x,t)\\ \omega(x,0)=\sin(x)\end{matrix}, \quad 0<\lambda\le 1 \qquad (17)$$

From Eq.(15), we using NDM

$$\omega_0(x,t)=\sin(x)$$

$$\omega_n(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}\{(\omega_{n-1})_{xx}\}\right),$$

$$\omega_1(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}\{(\omega_0)_{xx}\}\right)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}\{-\sin(x)\}\right)$$

$$=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}(-\sin(x))\right)=-\frac{\sin(x)t^{\lambda}}{\Gamma(\lambda+1)}.$$

$$\omega_2(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}\{(\omega_1)_{xx}\}\right)=\frac{\sin(x)t^{2\lambda}}{\Gamma(2\lambda+1)}.$$

$$\omega_3(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}\{(\omega_2)_{xx}\}\right)=-\frac{\sin(x)t^{3\lambda}}{\Gamma(3\lambda+1)}.$$

$$\omega_4(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}\{(\omega_3)_{xx}\}\right)=\frac{\sin(x)t^{4\lambda}}{\Gamma(4\lambda+1)}.$$

In the same way

$$\omega_5(x,t)=\mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}\{(\omega_4)_{xx}\}\right)=-\frac{\sin(x)t^{5\lambda}}{\Gamma(5\lambda+1)}$$

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$$\omega_n(x,t) = \mathbb{N}^{-1}\left(\frac{p^\lambda}{s^\lambda}\mathbb{N}\{(\omega_{n-1})_{xx}\}\right) = \frac{\sin(x)(-t^\lambda)^n}{\Gamma(n\lambda+1)}.$$

Using the sequence of partial sums

$$\varphi_n(x,t) = \sum_{k=0}^{n}\omega_k(x,t) = \omega_0(x,t) + \omega_1(x,t) + \cdots + \omega_n(x,t)$$

By substituting the above solutions, it is transformed into

$$\varphi_n(x,t) = \sin(x)\left[1 + \frac{t^\lambda}{\Gamma(\lambda+1)} + \frac{t^{2\lambda}}{\Gamma(2\lambda+1)} \cdots \frac{(-t^\lambda)^n}{\Gamma(n\lambda+1)}\right] \tag{18}$$

Using the one-parameter MLF, the problem's solution

$$\lim_{n\to\infty}\varphi_n(x,t) = \sin(x)\sum_{k=0}^{\infty}\frac{(-t^\lambda)^k}{\Gamma(k\lambda+1)}.$$

Then from Eq.(16)

$$\omega(x,t) = \sin(x)\, E_{\lambda,1}(-t^\lambda). \tag{19}$$

The Eq. (19) for $\lambda = 1$ is approximate to the form $\omega(x,t) = \sin(x)\, e^{-t}$ which is the exact solution of Eq.(17) for $\lambda = 1$.

Example 3.2. Let the one-dimensional parabolic fractional diffusion equation:

$$\begin{aligned} \partial_\tau^\lambda \omega(x,t) &= \omega_{xx}(x,t) + 2\omega_x(x,t) \\ \omega(x,0) &= e^{-x} \end{aligned} \quad . \qquad 0 < \lambda \le 1 \tag{20}$$

From Eq. (15), we using NDM

$$\omega_0(x,t) = e^{-x}$$

$$\omega_n(x,t) = \mathbb{N}^{-1}\left(\frac{p^\lambda}{s^\lambda}\mathbb{N}\{(\omega_{n-1})_{xx} + 2(\omega_{n-1})_x\}\right),$$

$$\omega_1(x,t) = \mathbb{N}^{-1}\left(\frac{p^\lambda}{s^\lambda}\mathbb{N}\{(\omega_0)_{xx} + 2(\omega_0)_x\}\right) = -\frac{e^{-x}t^\lambda}{\Gamma(\lambda+1)}.$$

$$\omega_2(x,t) = \mathbb{N}^{-1}\left(\frac{p^\lambda}{s^\lambda}\mathbb{N}\{(\omega_1)_{xx} + 2(\omega_1)_x\}\right) = \frac{e^{-x}t^{2\lambda}}{\Gamma(2\lambda+1)}.$$

$$\omega_3(x,t) = \mathbb{N}^{-1}\left(\frac{p^\lambda}{s^\lambda}\mathbb{N}\{(\omega_2)_{xx} + 2(\omega_2)_x\}\right) = -\frac{e^{-x}t^{3\lambda}}{\Gamma(3\lambda+1)}.$$

$$\omega_4(x,t) = \mathbb{N}^{-1}\left(\frac{p^\lambda}{s^\lambda}\mathbb{N}\{(\omega_3)_{xx} + 2(\omega_3)_x\}\right) = \frac{e^{-x}t^{4\lambda}}{\Gamma(4\lambda+1)}.$$

In the same way

$$\omega_5(x,t) = \mathbb{N}^{-1}\left(\frac{p^\lambda}{s^\lambda}\mathbb{N}\{(\omega_4)_{xx} + 2(\omega_4)_x\}\right) = -\frac{e^{-x}t^{5\lambda}}{\Gamma(5\lambda+1)}.$$

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$$\omega_n(x,t) = \mathbb{N}^{-1}\left(\frac{p^{\lambda}}{s^{\lambda}}\mathbb{N}\{(\omega_{n-1})_{xx} + 2(\omega_{n-1})_x\}\right) = \frac{e^{-x}(-t^{\lambda})^n}{\Gamma(n\lambda+1)}.$$

From the sequence of partial sums

$$\varphi_n(x,t) = \sum_{k=0}^{n}\omega_k(x,t) = \omega_0(x,t) + \omega_1(x,t) + \cdots + \omega_n(x,t)$$

By substituting the above solutions, it is transformed into

$$\varphi_n(x,t) = e^{-x}\left[1 - \frac{t^{\lambda}}{\Gamma(\lambda+1)} + \frac{t^{2\lambda}}{\Gamma(2\lambda+1)} \pm \cdots \frac{(-t^{\lambda})^n}{\Gamma(n\lambda+1)}\right] \tag{21}$$

Using the one-parameter MLF, the problem's solution"

$$\lim_{n\to\infty}\varphi_n(x,t) = e^{-x}\sum_{k=0}^{\infty}\frac{(-t^{\lambda})^k}{\Gamma(k\lambda+1)}.$$

Then from Eq.(16)

$$\omega(x,t) = e^{-x}E_{\lambda,1}(-t^{\lambda}). \tag{22}$$

The Eq.(22) for $\lambda = 1$ is approximate to the form $\omega(x,t) = e^{-(x+t)}$ which is the exact solution of Eq.(20) for $\lambda = 1$.

4. Numerical results and discussion

The numerical solution of the Cauchy problem(CP) represented the time-fractional diffusion-advection equation (1) with the initial condition $\omega_0(x) = \phi(x)$, can be described and illustrated in above three application examples. The numerical values in Figure 1 represents the approximate solution $\omega(x,t)$ of the CP(17) in Example 3.1. The numerical solution approximated to the exact solution ω_{Exact} given by equation (19) through different values of t and $x = 1, 0.9, 0.8$ and 0.7; with values $\lambda = 0.7, 0.8, 0.9$ and 1 for Example 3.1. In Figure 1, we observe that the approximate solution for Example 3.1 decreases when t increases and for fixed values of x. In Figure 2, the approximate solution $\omega(x,t)$ of the CP(17) in Example 3.1 show the graph through different values of x, t when $\lambda = 1, 0.9, 0.8$ and 0.7; respectively. Therefore, both graphs in Figure 1 and Figure 2 comes close to the exact solution ω_{Exact} for $0 < \lambda \leq 1$. Now, the approximate solution $\omega(x,t)$ of the CP(20) to Example 3.2 is described in Figure 3. The numerical values approach to the exact solution ω_{Exact} given by the equation (22) through different values of t and $x = 0.75, 1.5, 2.25, 3.00$; with values $\lambda = 1.0, 0.9, 0.8$ and 0.7; for Example 3.2. Figure 3, we observe that the approximate solution for Example 3.2 decreases when t increases for fixed values of x, for $0 < \lambda \leq 1$. In Figure 4, the approximate solution $\omega(x,t)$ of the CP(20) in Example 3.2 show the graph of the approximate solution among different values of x, t when $\lambda = 1, 0.9, 0.8$ and 0.7; respectively. We can determine from the preceding reasoning and the numerical answers that the absolute error is extremely tiny, indicating that the suggested NDM is very successful in providing the analytical solutions for the time-fractional diffusion-convection problem with ease and without the need for any assumptions.

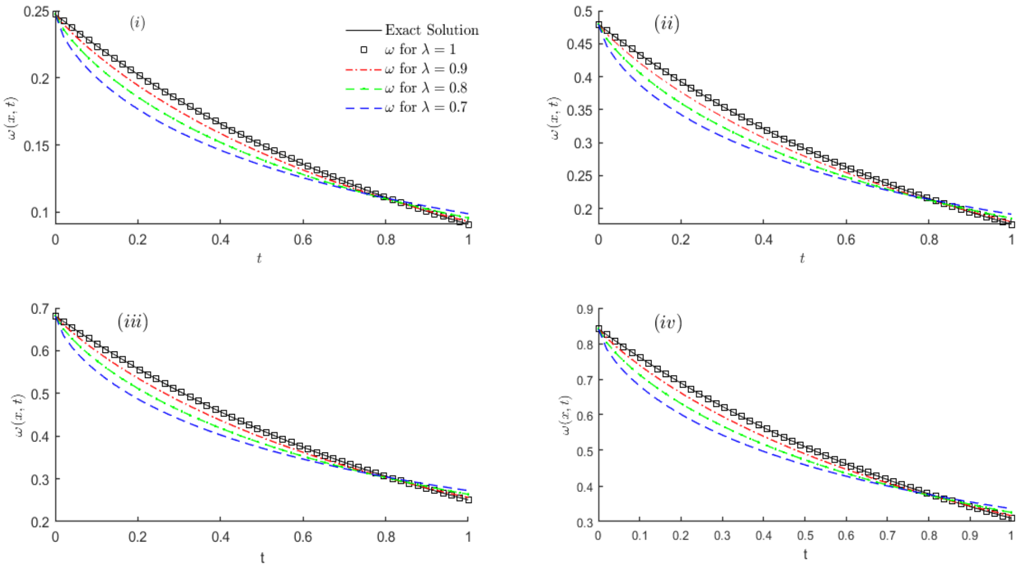

Figure 1. For Example 3.1, (i) x = 0.25; (ii) x = 0.50; (iii) x = 0.75; (iv) x = 1.0; and among different values of t when $\lambda = 0.7, 0.8, 0.9, 1.0$; it shows the graphs of the approximated solutions.

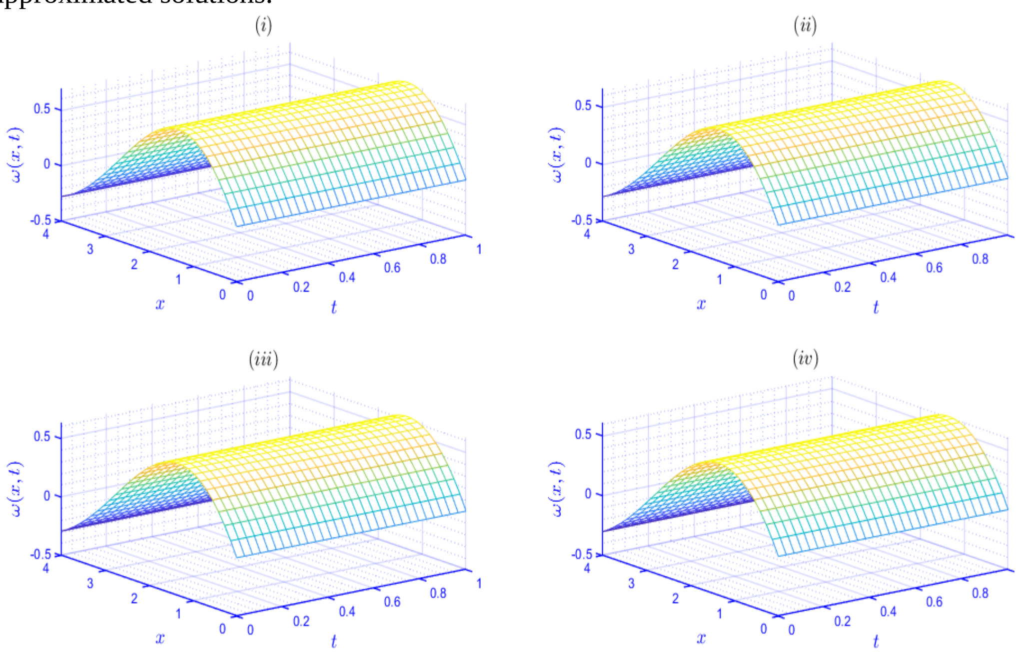

Figure 2. For Example 3.1, (i) $\alpha = 0.7$; (ii) $\alpha = 0.8$; (iii) $\alpha = 0.9$; (iv) $\alpha = 1.0$; and for various values of x and t; it shows the graphs of the approximated solutions.

x	t	$\lambda = 0.7$ ω	$\lambda = 0.8$ ω	$\lambda = 0.9$ ω	ω_{Exact}	Absolute Error $\lvert\omega - \omega_{Exact}\rvert$
0.25	0.25	0.16764	0.17597	0.18441	0.19267	0.00826
	0.50	0.13503	0.13912	0.14414	0.15005	0.00591
	0.75	0.11396	0.11404	0.11494	0.11686	0.00192
	1.00	0.09886	0.09573	0.09304	0.09101	0.00203
0.50	0.25	0.32486	0.34100	0.35736	0.37337	0.01601
	0.50	0.26168	0.26959	0.27932	0.29078	0.01146
	0.75	0.22084	0.22099	0.22274	0.22646	0.00372
	1.00	0.19158	0.18551	0.18029	0.17637	0.00323
1.00	0.25	0.57018	0.59851	0.627228	0.65533	0.02811
	0.50	0.45929	0.47317	0.490252	0.51037	0.02012
	0.75	0.38762	0.38786	0.390952	0.39748	0.00653
	1.00	0.33626	0.32561	0.316449	0.30956	0.00689

Table 1. For Example 3.1, $\lambda = 0.7, 0.8, 0.9$; and the exact solution ($\lambda = 1.0$) and for specific values of x.

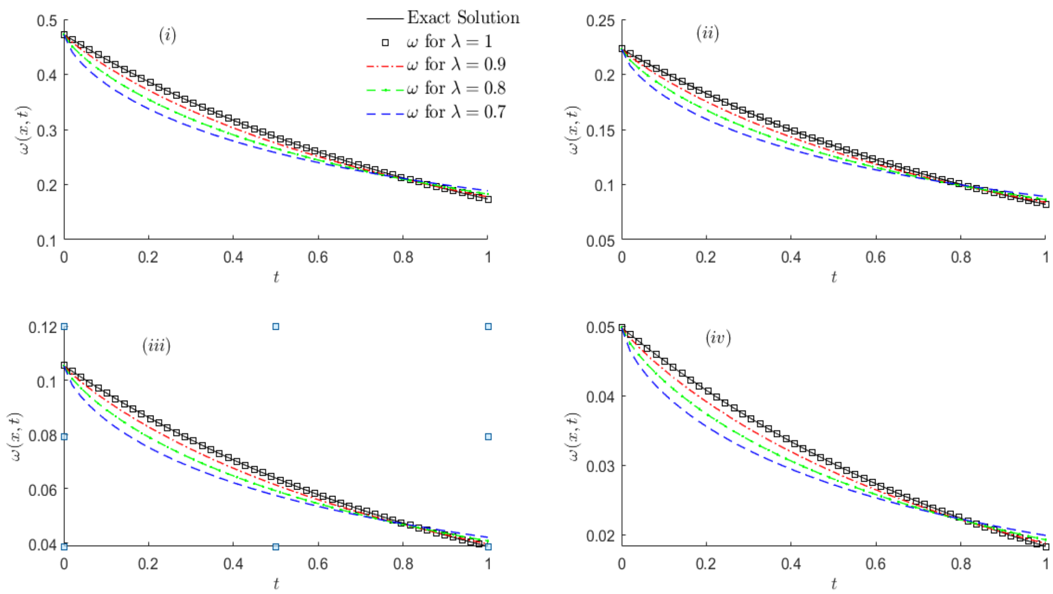

Figure 3. For Example 3.1, (i) x = 0.25; (ii) x = 0.50; (iii) x = 0.75; (iv) x = 1.0; and among different values of t when $\lambda = 0.7, 0.8, 0.9, 1.0$; it shows the graphs of the approximated solutions.

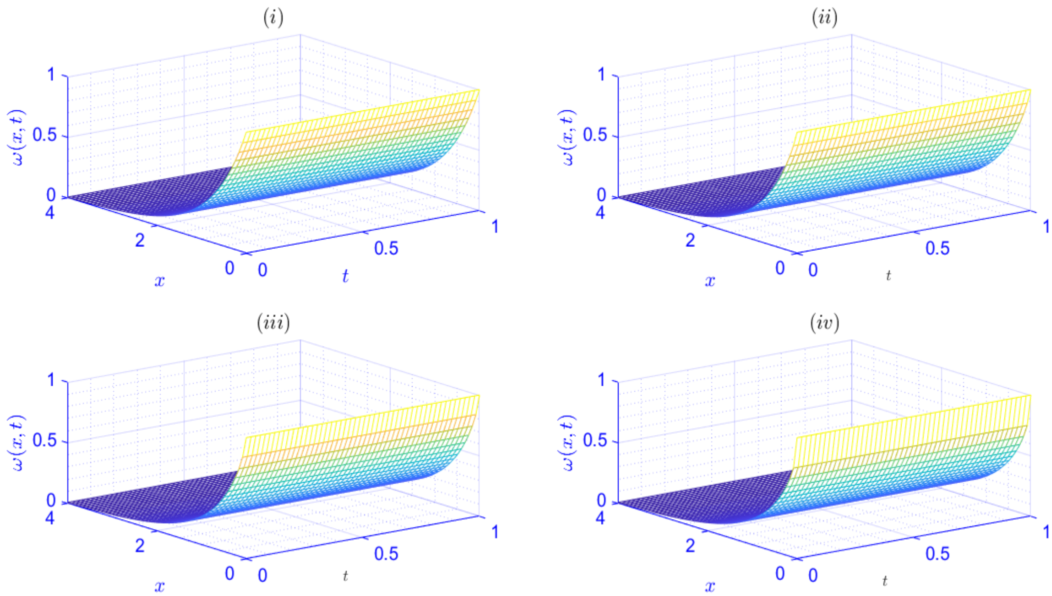

Figure 4. For Example 3.1, (i) $\alpha = 0.7$; (ii) $\alpha = 0.8$; (iii) $\alpha = 0.9$; (iv) $\alpha = 1.0$; and for various values of x and t; it shows the graphs of the approximated solutions.

x	t	$\lambda = 0.7$ ω	$\lambda = 0.8$ ω	$\lambda = 0.9$ ω	ω_{Exact}	Absolute Error $\lvert\omega - \omega_{Exact}\rvert$
0.25	0.25	0.038774	0.082085	0.173774	0.03878	0.00826
	0.50	0.030197	0.063928	0.135335	0.03020	0.00591
	0.75	0.023518	0.049787	0.105399	0.02352	0.00192
	1.00	0.018316	0.038774	0.082085	0.01832	0.00203
0.50	0.25	0.32486	0.34100	0.35736	0.37337	0.01601
	0.50	0.26168	0.26959	0.27932	0.29078	0.01146
	0.75	0.22084	0.22099	0.22274	0.22646	0.00372
	1.00	0.19158	0.18551	0.18029	0.17637	0.00323
1.00	0.25	0.57018	0.59851	0.627228	0.65533	0.02811
	0.50	0.45929	0.47317	0.490252	0.51037	0.02012
	0.75	0.38762	0.38786	0.390952	0.39748	0.00653
	1.00	0.33626	0.32561	0.316449	0.30956	0.00689

Table 2. For Example 3.1, $\lambda = 0.7, 0.8, 0.9$; and the exact solution ($\lambda = 1.0$) and for specific values of x.

5. Conclusion

The approximate numerical solution for the time-fractional diffusion-convection equation considered by using the (NDM). The fractional integral of Riemann-Liouville(RL) used in conjunction with the NT of the Caputo fractional derivative operator in the suggested method. Adomian polynomials are particularly handled for easily managing the nonlinear term. The fractional derivatives are characterized by the Caputo(C) and Riemann-Liouville(RL) sense. It is found that the NDM is rapid and accurate. All computations and graphics were performed in MATLAB. To prove the usefulness and validity of the proposed method, illustrative examples are provided.

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