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The Algebraic Connectivity, the Laplacian Spectral Radius and Some Hamiltonian Properties of Graphs

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Abstract. Let G = (V, E) be a graph. The second smallest eigenvalue and the largest eigenvalue of the Laplacian matrix of G are called, respectively, the algebraic connectivity and the Laplacian spectral radius of G. In this note, we present sufficient conditions based on the algebraic connectivity or the Laplacian spectral radius for some Hamiltonian properties of graphs.

Keywords: The algebraic connectivity, the Laplacian spectral radius, Hamiltonian graph, traceable graph.

AMS Mathematics Subject Classification (2020): 05C45, 05C50

1. Introduction

In this section, we first introduce the definitions, terminologies, and notations which will be used in this paper. They have appeared in the author's previous papers, such as [5]. We will repeat them here. We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let G = (V(G), E(G)) be a graph. The number of vertices and the number of edges in G are denoted by n and e, respectively. The degree of a vertex v is denoted by $d_G(v)$. We use $\delta = d_1 \leq d_2 \leq ... \leq d_n = \Delta$ to denote the degree sequence of a graph G. A subset of V(G) in a graph G is called an independent set if any two vertices in the subset are not adjacent. An independent set in a graph G is called a maximum independent set if its size is maximum. The independence number of a graph G is defined as the size of a maximum independent set in G and is denoted by $\beta(G)$. For two graphs H_1 and H_2, we define $H_1 \leq H_2$ as $V(H_1) = V(H_2)$ and $E(H_1) \subseteq E(H_2)$. The join of two disjoint graphs H_1 and H_2 is denoted by $H_1 \vee H_2$. For two disjoint vertex subsets S and T of V(G), we define E(S, T) as $\{e : e = ab \in E(G), a \in S, b \in T\}$. Namely, E(S, T) is the set of all the edges in E(G) such that one end vertex of each edge is in S and another end vertex of the edge is in T. We use K_p to denote a complete graph of order p. We also use $K_{a,\,b}$ to denote a complete bipartite graph with two partition sets X and Y such that |X| = a and |Y| = b. The Laplacian matrix of a graph G, denoted L(G), is defined as D(G)- A(G), where D(G) is the diagonal matrix diag(d_1, d_2, ..., d_n) and A(G) is the adjacency matrix of G. We use $0 = \lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_n(G)$ to denote the eigenvalues of L(G). The second smallest eigenvalue $\lambda_2(G)$ of L(G) is called the algebraic connectivity of G (see [3]) and the largest eigenvalue $\lambda_n(G)$ of L(G) is called the Lapalcian spectral radius of G. A cycle C in a graph G is called a Hamilton cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamilton cycle. A path P in a graph G is called a Hamilton path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamilton path. In this note, we present sufficient conditions based on the algebraic connectivity or the Laplacian spectral radius for Hamiltonian graphs and traceable graphs. Below are the results of this paper.

Theorem 1.1. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges [1].
If

$$\lambda_2 \geq \frac{ne}{(k+1)(n-\beta)},$$

then *G* is Hamiltonian or G is $K_{k,\, k+1}$ [2].
If

$$\lambda_n \leq \frac{2n\delta}{\delta+n-k-1}\sqrt{\frac{\delta}{n-k-1}},$$

then *G* is Hamiltonian or $G \in \{H: K_{k,\, k+1} \leq H \leq K_{k+1}^c \vee K_k\}$.

Theorem 1.2. Let G be a k-connected ($k \geq 1$) graph with $n \geq 9$ vertices and e edges [1]. If

$$\lambda_2 \geq \frac{ne}{(k+2)(n-\beta)},$$

then *G* is traceable or G is $K_{k,\, k+2}$ [2].
If

$$\lambda_n \leq \frac{2n\delta}{\delta+n-k-2}\sqrt{\frac{\delta}{n-k-2}},$$

then *G* is traceable or $G \in \{H: K_{k,\, k+2} \leq H \leq K_{k+2}^c \vee K_k\}$.

2. Lemmas

We will use the following results as our lemmas. Each of them (except for Lemma 2.3) has been used in author's previous papers such as [5]. The first two are from [2].

Lemma 2.1. [2] Let G be a k-connected graph of order $n \geq 3$. If $\beta \leq k$, then G is Hamiltonian.

Lemma 2.2. [2] Let G be a k-connected graph of order n. If $\beta \leq k + 1$, then G is traceable.
Lemma 2.3 below is Proposition 2.1 on Page 173 in [6].

Lemma 2.3. [6] Let G be a graph of order n. For any nontrivial subset X of vertices of G, $X \neq \emptyset$, $X \neq V(G)$, we have

$$\lambda_2 \leq \frac{n\,|E(X, V-X)|}{|X|\,|V-X|} \leq \lambda_n.$$

Lemma 2.4 below is from [7].

Lemma 2.4. [7] Let G be a balanced bipartite graph of order 2n with bipartition (A, B). If $d(x) + d(y) \geq n + 1$ for any $x \in A$ and any $y \in B$ with $xy \notin E$, then G is Hamiltonian.

Lemma 2.5 below is from [4].

Lemma 2.5. [4] Let G be a 2-connected bipartite graph with bipartition (A, B), where $|A| \geq |B|$. If each vertex in A has degree at least s and each vertex in B has degree at least t, then G contains a cycle of length at least 2min ($|B|$, $s + t - 1$, $2s - 2$).

3. Proofs

Proof of Theorem 1.1. In this proof, we will use some ideas in author's previous papers such as the ones in the proofs of Theorem 1 in [5]. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges satisfying exactly one of two conditions in Theorem 1.1. Suppose G is not Hamiltonian. Then Lemma 2.1 implies that $\beta \geq k + 1$. Also, we have that $n \geq 2\delta + 1 \geq 2k + 1$ otherwise $\delta \geq k \geq n/2$ and G is Hamiltonian. Let $I := \{u_1, u_2, ..., u_\beta\}$ be a maximum independent set in G. Set $V - I := \{v_1, v_2, ..., v_{n - \beta}\}$. Thus

$$\sum_{u \in I} d(u) = |E(I, V - I)| \leq \sum_{v \in V - I} d(v).$$

Since

$$\sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) = 2e.$$

we have that

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V - I} d(v).$$

[1]. From Lemma 2.3 and the given condition in this case, we have that

$$\frac{ne}{(k+1)(n-\beta)} \leq \lambda_2 \leq \frac{n\,|E(I, V - I)|}{|I|\,|V - I|}$$

$$= \frac{n \sum_{u \in I} d(u)}{|I|\,|V - I|} \leq \frac{ne}{(k+1)(n-\beta)}.$$

Thus $|I| = \beta = k + 1$ and $\sum_{u \in I} d(u) = e$. Note that $\sum_{u \in I} d(u) = e$ implies that $\sum_{v \in V - I} d(v) = e$ and G is a bipartite graph with partition sets of I and V - I. Since V - I is an independent set in G and I with $|I| = \beta$ is a maximum independent set in G, we have that $n - (k + 1) = |V - I| \leq |I| = (k + 1)$. Thus $n \leq 2k + 2$. Therefore $n = 2k + 2$ or $n = 2k + 1$. If $n = 2k + 2$, then Lemma 2.4 implies that G is Hamiltonian, a contradiction. If $n = 2k + 1$, then G is $K_{k, k+1}$.

This completes the proof of [1] in Theorem 1.1.

[2]. From Lemma 2.3 and the given condition in this case, we have that

$$\frac{2n\delta}{\delta + n - k - 1}\sqrt{\frac{\delta}{n - k - 1}} \geq \lambda_n \geq \frac{n\,|E(I, V - I)|}{|I|\,|V - I|}$$

$$= \frac{n \sum_{u \in I} d(u)}{\beta(n-\beta)} \geq \frac{n\beta\delta}{\beta(n-\beta)} = \frac{n\delta}{(n-\beta)}$$

$$\geq \frac{2n\delta}{\delta+n-\beta}\sqrt{\frac{\delta}{n-\beta}} \geq \frac{2n\delta}{\delta+n-k-1}\sqrt{\frac{\delta}{n-k-1}}.$$

Thus all the inequalities above become an equality. Hence we have $|I| = \beta = k + 1$, $d(u) = \delta$ for each $u \in I$, and

$$\frac{n\delta}{(n-\beta)} = \frac{2n\delta}{\delta+n-\beta}\sqrt{\frac{\delta}{n-\beta}}$$

which implies that $n - \beta = \delta$. Note that if $n \geq 2k + 2$, then $\delta = n - \beta = n - k - 1 \geq n/2$ and G is Hamiltonian, a contradiction. Thus $n = 2k + 1$ and $G \in \{H : K_{k,\, k+1} \leq H \leq K_{k+1}^{c} \vee K_k\}$.

This completes the proof of [2] in Theorem 1.1.

Proof of Theorem 1.2. In this proof, we will use some ideas in author's previous papers such as the ones in the proofs of Theorem 2 in [5]. Let G be a k-connected ($k \geq 1$) graph with $n \geq 9$ vertices and e edges satisfying exactly one of two conditions in Theorem 1.2. If $n = 1$ or $n = 2$, then G is traceable. From now one, we assume that $n \geq 3$. Suppose G is not traceable. Then Lemma 2.2 implies that $\beta \geq k + 2$. Also, we have that $n \geq 2\delta + 2 \geq 2k + 2$ otherwise $\delta \geq k \geq (n - 1)/2$ and G is traceable. Let $I := \{u_1, u_2, ..., u_\beta\}$ be a maximum independent set in G. Set $V - I := \{v_1, v_2, ..., v_{n-\beta}\}$. Thus

$$\sum_{u \in I} d(u) = |E(I, V-I)| \leq \sum_{v \in V-I} d(v).$$

Since

$$\sum_{u \in I} d(u) + \sum_{v \in V-I} d(v) = 2e.$$

we have that

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V-I} d(v).$$

[1]. From Lemma 2.3 and the given condition in this case, we have that

$$\frac{ne}{(k+2)(n-\beta)} \leq \lambda_2 \leq \frac{n\,|E(I, V-I)|}{|I|\,|V-I|}$$

$$= \frac{n \sum_{u \in I} d(u)}{|I|\,|V-I|} \leq \frac{ne}{(k+2)(n-\beta)}.$$

Thus $|I| = \beta = k + 2$ and $\sum_{u \in I} d(u) = e$. Note that $\sum_{u \in I} d(u) = e$ implies that $\sum_{v \in V-I} d(v) = e$ and G is a bipartite graph with partition sets of I and V - I. Since V - I is an independent set in G and I with $|I| = \beta$ is a maximum independent set in G, we have that $n - (k + 2) = |V - I| \leq |I| = (k + 2)$. Thus $n \leq 2k + 4$. Therefore $n = 2k + 4$, $n = 2k + 3$, or $n = 2k + 2$. If $n = 2k + 4$, then Lemma 2.4 implies that G is Hamiltonian and thereby G is traceable, a contradiction. If n = 2k + 3, them Lemma 2.5 implies that G has a cycle of length at least (n – 1) and thereby G is traceable, a contradiction. If n = 2k + 2, then G is $K_{k,\, k+2}$.

This completes the proof of [1] in Theorem 1.2.

[2]. From Lemma 2.3 and the given condition in this case, we have that

$$\frac{2n\delta}{\delta+n-k-2}\sqrt{\frac{\delta}{n-k-2}} \geq \lambda_n \geq \frac{n\,|E(I, V-I)|}{|I|\,|V-I|}$$

$$= \frac{n\sum_{u\in I} d(u)}{\beta(n-\beta)} \geq \frac{n\beta\delta}{\beta(n-\beta)} = \frac{n\delta}{(n-\beta)}$$

$$\geq \frac{2n\delta}{\delta+n-\beta}\sqrt{\frac{\delta}{n-\beta}} \geq \frac{2n\delta}{\delta+n-k-2}\sqrt{\frac{\delta}{n-k-2}}.$$

Thus all the inequalities above become an equality. Hence we have |I| = β = k + 1, d(u) = δ for each u ∈ I, and

$$\frac{n\delta}{(n-\beta)} = \frac{2n\delta}{\delta+n-\beta}\sqrt{\frac{\delta}{n-\beta}}$$

which implies that n - β = δ. Note that if n ≥ 2k + 3, then δ = n - β = n - k - 2 ≥ (n - 1/2 and G is traceable, a contradiction. Thus n = 2k + 2 and G ∈{H : $K_{k,\, k+2} \leq H \leq K^{c}_{k+2} \vee K_k$}.

This completes the proof of [2] in Theorem 1.2.

4. Conclusion

In this note, we obtain sufficient conditions based on the algebraic connectivity or the Laplacian spectral radius for Hamiltonian graphs and traceable graphs. It is noted that there are k-connected (k ≥ 2) Hamiltonian graphs (resp., k-connected (k ≥ 1) traceable graphs) which don't satisfy the conditions in Theorem 1.1 (resp., Theorem 1.2).

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Authors' Contributions. It is a single-author paper. So, full credit goes to the author.

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