Journal of Mathematics and Informatics
Vol. 28, 2025, 33-37
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 29 May 2025
www.researchmathsci.org
DOI:http://dx.doi.org/10.22457/jmi.v28a03256

Journal of **Mathematics and Informatics**

Harmonic Downhill Index of Graphs

V.R.Kulli

Department of Mathematics
Gulbarga University, Gulbarga 585 106, India
E-mail: vrkulli@gmail.com

Received 10 March 2025; accepted 28 May 2025

Abstract. In this study, we introduce the harmonic downhill index and its corresponding polynomial of a graph. Furthermore, we compute this index for some standard graphs, wheel graphs, gear graphs and helm graphs.

Keywords: harmonic downhill index, harmonic downhill polynomial, graphs.

AMS Mathematics Subject Classification (2010): 05*C*07, 05*C*09

1. Introduction

In this paper, G denotes a finite, simple, connected graph, $V(G)$ and $E(G)$ denote the vertex set and edge set of G. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u.

A u-v path P in G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, ... v_{k+1}$ in G is a downhill path if for every i, $1 \le i \le k$, $d_G(v_i) \ge d_G(v_{i+1})$.

A vertex v is downhill dominating a vertex u if there exists a downhill path originating from u to v. The downhill neighbourhood of a vertex v is denoted by $N_{dn}(v)$ and defined as: $N_{dn}(v) = \{u$: v downhill dominates $u\}$. The downhill degree $d_{dn}(v)$ of a vertex v is the number of downhill neighbours of v [1].

The first and second downhill indices were introduced in [1], and they are defined as

$$DWM_1(G) = \sum_{u \in V(G)} d_G(u)^2, \qquad DWM_2(G) = \sum_{u \in V(G)} d_G(u) d_G(v).$$

Some downhill indices were recently studied in [2, 3, 4].

The harmonic index [5] of a graph G is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)}.$$

We introduce the harmonic downhill index of a graph, it is defined as

$$HDW(G) = \sum_{uv \in E(G)} \frac{2}{d_{dn}(u) + d_{dn}(v)}.$$

Considering the harmonic downhill index, we introduce the harmonic downhill polynomial of a graph G and it is defined as

$$HDW(G,x)=\sum_{uv\in E(G)} x^{\frac{2}{d_{dn}(u)+d_{dn}(v)}}.$$

Some topological indices were recently studied in [6, 7, 8].

This paper computes the harmonic downhill index and its corresponding polynomial of certain graphs.

2. Results for some standard graphs

Proposition 1. Let G be r-regular with n vertices and $r \geq 2$. Then

$$HDW(G)=\frac{nr\sqrt{(n-1)}}{\sqrt{2}}.$$

Proof: Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v)=n-1$ for every v in G.

$$HDW(G)=\sum_{uv\in E(G)}\frac{2}{d_{dn}(u)+d_{dn}(v)}=\frac{nr}{2}\frac{2}{(n-1)+(n-1)}=\frac{nr}{2(n-1)}.$$

Corollary 1.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$HDW(C_n)=\frac{n}{(n-1)}.$$

Corollary 1.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$HDW(K_n)=\frac{n}{2}.$$

Proposition 2. Let P be a path with $n \geq 3$ vertices. Then

$$HDW(P)==\frac{n+1}{n-1}.$$

Proof: Let P be a path with $n \geq 3$ vertices. We obtain two partitions of the edge set of P as follows:

$E_1 = \{uv \in E(P) \mid d_{dn}(u)=0,\ d_{dn}(v)=n-1\}$, $|E_1| = 2$.

$E_2 = \{uv \in E(P) \mid d_{dn}(u)= d_{dn}(v)=n-1\}$, $|E_2| = n-3$.

$$HDW(G)=\sum_{uv\in E(G)}\frac{2}{d_{dn}(u)+d_{dn}(v)}=\frac{2\times 2}{0+(n-1)}+\frac{(n-3)2}{(n-1)+(n-1)}=\frac{n+1}{n-1}.$$

3. Results for wheel graphs

The wheel W_n is the join of C_n and K_1. Clearly, W_n has $n+1$vertices and $2n$ edges. Then W_n has two types of edges based on the downhill degree of the vertices of each edge as follows:

$E_1 = \{uv \in E(W_n) \mid d_{dn}(u) = n,\ d_{dn}(v) = n-1\}$, $|E_1| = n$.

$E_2 = \{uv \in E(W_n) \mid d_{dn}(u) = d_{dn}(v) = n-1\}$, $|E_2| = n$.

Theorem 1. Let W_n be a wheel with n+1vertices and 2n edges, n≥4. Then the harmonic downhill index of W_n is

$$HDW(W_n) = \frac{2n}{2n-1} + \frac{n}{n-1}.$$

Proof: We deduce

$$HDW(W_n) = \sum_{uv \in E(W_n)} \frac{2}{d_{dn}(u) + d_{dn}(v)}$$

$$= \frac{2n}{n+(n-1)} + \frac{2n}{(n-1)+(n-1)} = \frac{2n}{2n-1} + \frac{n}{n-1}.$$

Theorem 2. Let W_n be a wheel with n+1vertices, n≥ 4. Then the harmonic downhill polynomial of W_n is

$$HDW(W_n, x) = nx^{\frac{2}{2n-1}} + nx^{\frac{1}{n-1}}.$$

Proof: We obtain

$$HDW(W_n, x) = \sum_{uv \in E(W_n)} x^{\frac{2}{d_{dn}(u) + d_{dn}(v)}}$$

$$= nx^{\frac{2}{n+(n-1)}} + nx^{\frac{2}{(n-1)+(n-1)}} = nx^{\frac{2}{2n-1}} + nx^{\frac{1}{n-1}}.$$

4. Results for gear graphs

A bipartite wheel graph is a graph obtained from W_n with n+1 vertices adding a vertex between each pair of adjacent rim vertices and this graph is denoted by G_n and also called as a gear graph. Clearly, $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n$. A gear graph G_n is depicted in Figure 1.

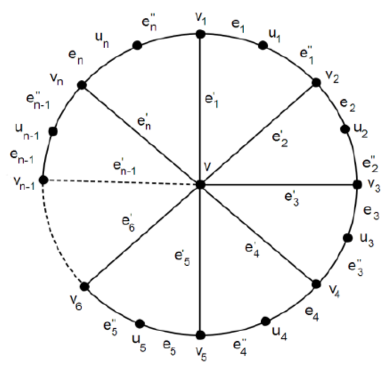

Figure 1: Gear graph G_n

Let G_n be a gear graph with 2n+1vertices, 3n edges, n≥4. Then G_n has two types of edges based on the downhill degree of the vertices of each edge as follows:

$E_1 = \{u \in E(G_n) \mid d_{dn}(u) = 2n, d_{dn}(v) = 2\}$, $|E_1| = n$.

$E_2 = \{u \in E(G_n) \mid d_{dn}(u) = 2, , d_{dn}(v) = 0\}$, $|E_2| = 2n$.

Theorem 3. Let G_n be a gear graph with $2n+1$vertices, $3n$ edges, $n\geq 4$. Then the harmonic downhill index of G_n is

$$HDW(G_n) = \frac{n}{n+1} + 2n.$$

Proof: We deduce

$$HDW(G_n) = \sum_{uv\in E(G_n)} \frac{2}{d_{dn}(u)+d_{dn}(v)} = \frac{2n}{2n+2} + \frac{4n}{2+0} = \frac{n}{n+1} + 2n.$$

Theorem 4. Let G_n be a gear graph with $2n+1$vertices, $3n$ edges, $n\geq 4$. Then the harmonic downhill polynomial of G_n is

$$HDW(G_n, x) = nx^{\frac{1}{n+1}} + 2nx^{1}.$$

Proof: We deduce

$$HDW(G_n, x) = \sum_{uv\in E(G_n)} x^{\frac{2}{d_{dn}(u)+d_{dn}(v)}} = nx^{\frac{2}{2n+2}} + 2nx^{\frac{2}{2+0}} = nx^{\frac{1}{n+1}} + 2nx^{1}.$$

5. Results for helm graphs

The helm graph H_n is a graph obtained from W_n (with $n+1$ vertices) by attaching an end edge to each rim vertex of W_n. Clearly, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. A graph H_n is shown in Figure 2.

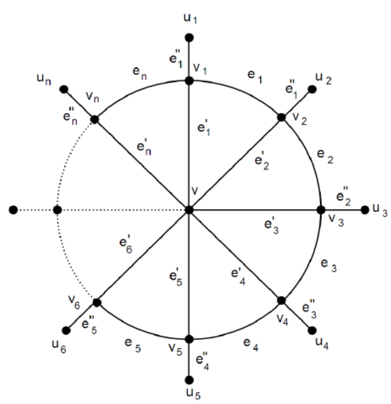

Figure 2: Helm graph H_n

Let H_n be a helm graph with $3n$ edges, $n\geq 5$. Then H_n has three types of edges based on the downhill degree of the vertices of each edge as follows:

$E_1 = \{uv\in E(H_n) \mid d_{dn}(u) = 2n,\ d_{dn}(v) = 2n-1\}$, $\qquad |E_1| = n$.
$E_2 = \{uv \in E(H_n) \mid d_{dn}(u) = d_{dn}(v) = 2n-1\}$, $\qquad |E_2| = n$.
$E_3 = \{uv \in E(H_n) \mid d_{dn}(u) = 2n-1,\ d_{dn}(v) = 0\}$, $\qquad |E_3| = n$.

Theorem 5. Let H_n be a helm graph with $2n+1$ vertices, $n\geq 5$. Then the harmonic downhill index of H_n is

$$HDW(H_n) = \frac{2n}{4n-1} + \frac{3n}{2n-1}.$$

Proof: We obtain

$$HDW(H_n)=\sum_{uv\in E(H_n)}\frac{2}{d_{dn}(u)+d_{dn}(v)}$$

$$=\frac{2n}{2n+(2n-1)}+\frac{2n}{(2n-1)+(2n-1)}+\frac{2n}{(2n-1)+0}=\frac{2n}{4n-1}+\frac{3n}{2n-1}.$$

Theorem 6. Let H_n be a helm graph with $2n$+1vertices, $3n$ edges, $n\geq 5$. Then the harmonic downhill polynomial of H_n is

$$HDW(H_n,x)=nx^{\frac{2}{4n-1}}+nx^{\frac{1}{2n-1}}+nx^{\frac{2}{2n-1}}.$$

Proof: We deduce

$$HDW(H_n,x)=\sum_{uv\in E(H_n)}x^{\frac{2}{d_{dn}(u)+d_{dn}(v)}}=nx^{\frac{2}{2n+(2n-1)}}+nx^{\frac{2}{(2n-1)+(2n-1)}}+nx^{\frac{2}{(2n-1)+0}}$$

$$=nx^{\frac{2}{4n-1}}+nx^{\frac{1}{2n-1}}+nx^{\frac{2}{2n-1}}.$$

5. Conclusion

In this study, the harmonic downhill index and its corresponding polynomial are defined and studied.

Acknowledgements. The author is thankful to the reviewers for their constructive comments for the paper.

Conflicts of interest. The author declares no conflicts of interest.

Authors' contributions. It is a single-author paper, and the author makes the full contribution.

REFERENCES

1. B.Al-Ahmadi, A.Saleh and W.Al-Shammakh, Downhill Zagreb topological indices of graphs, *International Journal of Analysis and Applications*, 19(2) (2021) 205-227.
2. B.Al-Ahmadi, A.Saleh and W.Al-Shammakh, Downhill Zagreb polynomials of graphs, *Research & Reviews: Discrete Mathematical Structures*, 7 (2020) 15-26.
3. V.R.Kulli, Downhill Nirmala indices of graphs, *International Journal of Mathematics and Computer Research*, 13(4) (2025) 5126-5131.
4. V.R.Kulli, Downhill Sombor modified downhill Sombor indices of graphs, submitted.
5. S.Fajtlowicz, On conjectures of Graffiti-II, *Congr. Numer.* 60(1987) 187.
6. V.R.Kulli, Computation of multiplicative minus F-indices of titania nanotubes, *Journal of Mathematics and Informatics* 19(2020) 135-140.
7. V.R.Kulli, Sum augmented and multiplicative sum augmented indices of some nanostructures, *J. of Mathematics and Informatics*, 24 (2023) 27-31.
8. V.R.Kulli, Domination Dharwad indices of graphs, *Journal of Mathematics and Informatics*, 25 (2023) 71-76.