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An Approximate Technique for Solving Fully Fuzzy Complex Linear Systems

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Abstract. The study of coefficient matrices for fuzzy linear systems has expanded from simple fuzzy numbers to fuzzy complex numbers, and even fully fuzzy linear systems. In order to model and solve data more accurately, the solution of fully fuzzy linear systems is particularly important. This paper adopts the matrix embedding method proposed by Friedman et al. to study the solution of a fully fuzzy complex linear system with a fuzzy complex coefficient matrix and obtains the algebraic solution form and approximate solution calculation method of the linear system. At the same time, this paper adopts a wider range of pseudo geometric fuzzy numbers and corresponding fuzzy operations based on transmission average (TA) for calculation. Practice has shown that the method proposed in this paper has the same approximation effect as other existing solving methods.

Keywords: Fuzzy numbers, Pseudo-trapezoidal fuzzy numbers, Fully fuzzy complex linear systems, Approximate solutions.

AMS Mathematics Subject Classification (2010): 03E72, 08A72

1. Introduction

The theory of systems of equations has a wide range of applications in mathematics, production, and daily life. For instance, it is applied in physics, transportation management optimization problems, financial management, and current control, among others. In the process of analysis and modeling, there are many parameters and variables. The uncertainty and imprecision of these variables make the constructed model unable to fit the date well, while the introduction of fuzzy numbers can overcome these problem effectively.

Based on the above background, the concept of fuzzy numbers and arithmetic operations were firstly introduced and investigated by Zadeh [42], Dubois and Prade [19], and Nahmias [36]. After that, numerous researchers have carried out various study by means of the tool of fuzzy numbers. The most prominent contribution is that Friedman et al. [22] using the embedding method to solving an $n \times n$ fuzzy linear system. It also provides new ideas for solving other systems such as fuzzy linear system (FLS), dual fuzzy linear systems (DFLS), general fuzzy linear systems (GFLS), full fuzzy linear systems (FFLS), dual full fuzzy linear systems (DFFLS) and general dual fuzzy linear systems (GDFLS). Afterwards, researchers [15] expanded the number field under consideration from fuzzy number to the fuzzy complex number field, and also launched a series of studies

[38, 32, 11, 25, 14, 28, 27, 43, 23, 24, 4, 40] on fuzzy complex linear systems (FCLS) and fuzzy complex matrix equations (FCME). Among them, fully fuzzy linear systems are an important branch in the study of fuzzy equations. Many researchers [8, 5, 6, 9, 16, 17, 18] have done a lot of work on solving fully fuzzy linear systems. On the basis of Fridaman et al., Dehghan and Hashemi [16] extended some iterative methods, they discussed a new linear system named fully fuzzy linear system (FFLS), which all parameters are fuzzy number. Dehghan et al. in [16] and [17] proposed two numerical methods for solving fully fuzzy linear systems. In [35], authors used QR decomposition to solve it. Ezzati et al. [41] converted a fuzzy system into a crisp system to solving, and obtaining a positive solution. Behera and Chakraverty [13] proposed a double parametric approach for solving the FFSLE, where the parameters are only non-negative fuzzy numbers. Otadi and Mosleh [37] investigated a non-negative solution of a fully fuzzy matrix equation using an optimization technique. Jafarian and Jafari [31] presented a new computational method for fully fuzzy non-linear matrix equations. Malkawi et al. [34] discussed the necessary and sufficient conditions to achieve a positive solution for the FFSLE. In [3, 1, 10], Abbasi et al. propose pseudo geometric fuzzy numbers and demonstrate their relationship with fuzzy numbers, starting research on various linear systems under the concept of pseudo geometric fuzzy numbers. Recently, Abbasi and Allahviranloo [2] investigated and reported a new concept based on transmission average operations for solving the FFSLE. The fuzzy operations based on the transmission average (TA) have a special advantage over those based on extension principles or interval operations. This paper also conducts research on fully fuzzy complex linear system with coefficient matrices that are fuzzy complex numbers based on the algorithm and method proposed in this paper.

The rest of this paper is organized as follows. Section 2 presents some basic concepts of fuzzy numbers, fuzzy complex numbers and pseudo-trapezoidal fuzzy number. And introduce the fuzzy operations based on the transmission average (TA) and singular value decomposition theorem. Immediately, we provide the definition of fully fuzzy complex linear systems and study their solutions and approximate solutions. This paper will present the final research results and technique in the form of a theorem. Finally, numerical examples are given to illustrate the proposed method. Conclusions are brought.

2. Preliminaries

At first, we will recall some basic concepts associated with fuzzy numbers and fuzzy complex numbers.

Following [26] a fuzzy set $\tilde{x}$ with the membership function $\mu_{\tilde{x}}: \mathbb{R} \to [0,1]$ is a fuzzy number if

(1) There exists $t_0 \in \mathbb{R}$ such that $\mu_{\tilde{x}}(t_0) = 1$, $i.e.$, $\tilde{x}$ is normal.

(2) For an $\lambda \in [0,1]$ and $s, t \in \mathbb{R}$, we have $\mu_{\tilde{x}}(\lambda s + (1-\lambda)t) \geq \min\{\mu_{\tilde{x}}(s), \mu_{\tilde{x}}(t)\}$, $i.e.$, $\tilde{x}$ is a convey fuzzy set.

(3) For any $s \in \mathbb{R}$, the set $\{t \in \mathbb{R}: \mu_{\tilde{x}}(t) > s\}$ is an open set in $\mathbb{R}$, $i.e.$, $\mu_{\tilde{x}}$ is upper semi-continuous on $\mathbb{R}$.

(4) The set $\overline{\{t \in \mathbb{R}: \mu_{\tilde{x}}(t) > 0\}}$ is compact set in $\mathbb{R}$, where $\overline{A}$ denotes the closure of A.

In this paper, we represent the set of all fuzzy numbers by $\mathbb{R}_{\mathrm{F}}$. Clearly, we consider the $\mathbb{R} = \{x_t: t$ is real number$\}$, then it can be concluded that $\mathbb{R} \subset \mathbb{R}_{\mathrm{F}}$. For $0 < \alpha \leq 1$, $\alpha-$levels of the fuzzy numbers $\tilde{x}$ are defined as $[\tilde{x}]_\alpha = \{t \in \mathbb{R}: \mu_{\tilde{x}}(t) \geq \alpha\}$ and for

$\alpha = 0$ as $[\tilde{x}]_0 = \overline{\{t \in \mathbb{R}: \mu_{\tilde{x}}(t) > 0\}}$. Moreover, the support of the fuzzy numbers $\tilde{x}$ is defined as $supp(\tilde{x}) = [\tilde{x}]_0 = \overline{\{t \in \mathbb{R}: \mu_{\tilde{x}}(t) > 0\}}$.

Lemma 1. *[33] If $\tilde{x} \in \mathbb{R}_{\mathrm{F}}$ is a fuzzy number and $[\tilde{x}]_\alpha$ are its $\alpha-$ levels, then*

(1) $[\tilde{x}]_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)]$ is a bounded closed interval, for each $\alpha \in [0,1]$.

(2) $[\underline{x}(\alpha_1), \overline{x}(\alpha_1)] \supseteq [\underline{x}(\alpha_2), \overline{x}(\alpha_2)]$ for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$.

(3) $[\lim_{k\to\infty} \underline{x}(\alpha_k), \lim_{k\to\infty} \overline{x}(\alpha_k)] = [\underline{x}(\alpha), \overline{x}(\alpha)]$ whenever α_k is a non-decreasing sequence in $[0,1]$ converging to α.

Definition 1. *[2] (pseudo-geometric fuzzy numbers) A fuzzy number $\tilde{x}$ is called a pseudo-trapezoidal fuzzy number if its membership function*

$$\mu_{\tilde{x}}(t_0) = \begin{cases} \text{ł}_{\tilde{x}}(t_0), & \underline{x} \leq t_0 \leq x_1, \\ 1, & x_1 \leq t_0 \leq x_2, \\ r_{\tilde{x}}(t_0), & x_2 \leq t_0 \leq \overline{x}, \\ 0, & otherwise, \end{cases}$$

where $\text{ł}_{\tilde{x}}(t_0)$ and $r_{\tilde{x}}(t_0)$ are nondecreasing and non increasing functions. Respectively, the pseudo-trapezoidal fuzzy number $\tilde{x}$ is denoted by $\tilde{x} = (\underline{x}, x_1, x_2, \overline{x}, \text{ł}_{\tilde{x}}(t_0), r_{\tilde{x}}(t_0))$, and the trapezoidal fuzzy number by $\tilde{x} = (\underline{x}, x_1, x_2, \overline{x}, -, -)$ or $\tilde{x} = (\underline{x}, x_1, x_2, \overline{x})$, that $-,-$ means $\text{ł}_{\tilde{x}}(t_0)$ and $r_{\tilde{x}}(t_0)$ are linear.

A pseudo-triangular fuzzy number is a particular pseudo-trapezoidal fuzzy number when the $x_1 = x_2$. The pseudo-triangular fuzzy number $\tilde{x}$ is denoted by $\tilde{x} = (\underline{x}, x, \overline{x}, \text{ł}_{\tilde{x}}(t_0), r_{\tilde{x}}(t_0))$ and the triangular fuzzy number by $\tilde{x} = (\underline{x}, x, \overline{x}, -, -)$ or $\tilde{x} = (\underline{x}, x, \overline{x})$.

Definition 2. *[40] (fuzzy complex numbers) A fuzzy complex set $\tilde{z}$ is called an fuzzy complex number if the following conditions are satisfied*

(1) $\tilde{z}$ is a upper semi-continuous function.

(2) $\tilde{z}_\alpha$ is a compact set for $0 \leq \alpha \leq 1$.

(3) $\tilde{z}$ is normal, i.e., there exists a $t_0 \in \mathbb{R}$ such that $\tilde{z}(t_0) = 1$.

(4) $\tilde{z}$ is a fuzzy convex set, i.e., $\tilde{z}(\lambda z_1 + (1-\lambda) z_2) > \min\{\tilde{z}(z_1), \tilde{z}(z_2)\}$ for all $z_1, z_2 \in \mathbb{C}$, $\lambda \in [0,1]$.

We write complex numbers as $z = x + iy$, where x and y are real numbers. Correspondingly, according to the definition of fuzzy complex numbers, we write them in the form $\tilde{z} = \tilde{x} + i\tilde{y}$, where $\tilde{x}$ and $\tilde{y}$ are fuzzy numbers. We use $\mathbb{C}$ and $\tilde{\mathbb{C}}$ to represent respectively the set of all complex numbers and fuzzy complex numbers.

Definition 3. *(pseudo-geometric fuzzy complex numbers) A fuzzy complex number $\tilde{z} = \tilde{x} + i\tilde{y}$ is called a pseudo-trapezoidal fuzzy number, its membership function is $\mu_{\tilde{z}}(t_0) = \mu_{\tilde{x}}(t_0) + i\mu_{\tilde{y}}(t_0)$, where*

$$\mu_{\tilde{x}}(t_0) = \begin{cases} \text{ł}_{\tilde{x}}(t_0), & \underline{x} \leq t_0 \leq x_1, \\ 1, & x_1 \leq t_0 \leq x_2, \\ r_{\tilde{x}}(t_0), & x_2 \leq t_0 \leq \overline{x}, \\ 0, & otherwise, \end{cases}$$

$$\mu_{\tilde{y}}(t_0)=\begin{cases}\text{ł}_{\tilde{y}}(t_0), & \underline{y}\le t_0\le y_1,\\ 1, & y_1\le t_0\le y_2,\\ r_{\tilde{y}}(t_0), & y_2\le t_0\le \overline{y},\\ 0, & otherwise,\end{cases}$$

where $\text{ł}_{\tilde{x}}(t_0),\text{ł}_{\tilde{y}}(t_0)$ and $r_{\tilde{x}}(t_0),r_{\tilde{y}}(t_0)$ are nondecreasing and non increasing functions. Respectively, the pseudo-trapezoidal fuzzy complex number $\tilde{z}$ is denoted by $\tilde{z}=(\underline{x},x_1,x_2,\overline{x},\text{ł}_{\tilde{x}}(t_0),r_{\tilde{x}}(t_0))+i(\underline{y},y_1,y_2,\overline{y},\text{ł}_{\tilde{y}}(t_0),r_{\tilde{y}}(t_0))$,

As for the operations of fuzzy numbers, here we introduce the fuzzy number operation based on interval operation and the transmission average (TA) respectively.

Definition 4. *[23] For any two arbitrary fuzzy complex numbers $\tilde{z}_1=\tilde{x}_1+i\tilde{y}_1$ and $\tilde{z}_2=\tilde{x}_2+i\tilde{y}_2$ and complex number $c=a+ib$, the $\alpha-$levels of the sum $\tilde{z}_1+\tilde{z}_2$ and the product $c\cdot\tilde{z}_1$ are defined based on interval arithmetic as follows*

$$\begin{aligned}[\tilde{z}_1+\tilde{z}_2]_\alpha &= ([\tilde{x}_1]_\alpha+[\tilde{x}_2]_\alpha)+i([\tilde{y}_1]_\alpha+[\tilde{y}_2]_\alpha)\\ &= [\underline{x}_1(\alpha)+\underline{x}_2(\alpha),\overline{x}_1(\alpha)+\overline{x}_2(\alpha)]+i[\underline{y}_1(\alpha)+\underline{y}_2(\alpha),\overline{y}_1(\alpha)+\overline{y}_2(\alpha)]\end{aligned}$$

and

$$\begin{aligned}[c\cdot\tilde{z}_1]_\alpha &= [(a+ib)\cdot\tilde{z}_1]_\alpha=(a+ib)\cdot([\tilde{x}_1]_\alpha+i[\tilde{y}_1]_\alpha)\\ &= (a[\tilde{x}_1]_\alpha-b[\tilde{y}_1]_\alpha)+i(a[\tilde{x}_1]_\alpha-b[\tilde{y}_1]_\alpha).\end{aligned}$$

Remark 1. *Two fuzzy complex numbers $\tilde{z}_1=\tilde{x}_1+i\tilde{y}_1$ and $\tilde{z}_2=\tilde{x}_2+i\tilde{y}_2$ are equal, if and only if $\tilde{x}_1=\tilde{x}_2$ and $\tilde{y}_1=\tilde{y}_2$, i.e., $[\tilde{x}_1]_\alpha=[\tilde{x}_2]_\alpha$ and $[\tilde{y}_1]_\alpha=[\tilde{y}_2]_\alpha$ for each $\alpha\in$* [0,1].

Definition 5. *[2] (Fuzzy arithmetic operations based on TA) Consider two pseudo-trapezoidal fuzzy numbers, $\tilde{x}=(\underline{x},x_1,x_2,\overline{x},\text{ł}_{\tilde{x}}(t_0),r_{\tilde{x}}(t_0))$, $\tilde{y}=(\underline{y},y_1,y_2,\overline{y},\text{ł}_{\tilde{y}}(t_0),r_{\tilde{y}}(t_0))$, with the following $\alpha-$cut forms*

$$x=\cup_{\alpha\in(0,1]}\alpha.x_\alpha,\quad x_\alpha=[\underline{x}_\alpha,\overline{x}_\alpha],\quad y=\cup_{\alpha\in(0,1]}\alpha.y_\alpha,\quad y_\alpha=[\underline{y}_\alpha,\overline{y}_\alpha].$$

Let $\phi=\frac{x_1+x_2}{2}$, $\psi=\frac{y_1+y_2}{2}$, then,

(1) addition,

$$x+y=\cup_{\alpha\in(0,1]}\alpha.(x+y)_\alpha,\quad (x+y)_\alpha=[\underline{(x+y)}_\alpha,\overline{(x+y)}_\alpha],$$

where $\underline{(x+y)}_\alpha=\frac{\phi+\psi}{2}+\frac{\underline{x}_\alpha+\underline{y}_\alpha}{2}$, $\overline{(x+y)}_\alpha=\frac{\phi+\psi}{2}+\frac{\overline{x}_\alpha+\overline{y}_\alpha}{2}$.

(2) subtraction,

$$-y=\cup_{\alpha\in(0,1]}\alpha.(-y)_\alpha,\quad (-y)_\alpha=[\underline{(-y)}_\alpha,\overline{(-y)}_\alpha],$$

where $\underline{(-y)}_\alpha=-2\psi+\underline{y}_\alpha$, $\overline{(-y)}_\alpha=-2\psi+\overline{y}_\alpha$.

So

$$x-y=x+(-y)=\cup_{\alpha\in(0,1]}\alpha.(x-y)_\alpha,\quad (x-y)_\alpha=[\underline{(x-y)}_\alpha,\overline{(x-y)}_\alpha],$$

where $\underline{(x-y)}_\alpha=\frac{\phi-3\psi}{2}+\frac{\underline{x}_\alpha+\underline{y}_\alpha}{2}$, $\overline{(x+y)}_\alpha=\frac{\phi-3\psi}{2}+\frac{\overline{x}_\alpha+\overline{y}_\alpha}{2}$.

(3) multiplication,

$$xy = \cup_{\alpha\in(0,1]} \alpha.(xy)_\alpha, \qquad (xy)_\alpha = [\underline{(xy)}_\alpha, \overline{(xy)}_\alpha],$$

where

$$[\underline{(xy)}_\alpha, \overline{(xy)}_\alpha] = \begin{cases} [\frac{\psi}{2}\underline{x}_\alpha + \frac{\phi}{2}\underline{y}_\alpha, \frac{\psi}{2}\overline{x}_\alpha + \frac{\phi}{2}\overline{y}_\alpha], & \phi \geq 0, \psi \geq 0, \\ [\frac{\psi}{2}\overline{x}_\alpha + \frac{\phi}{2}\underline{y}_\alpha, \frac{\psi}{2}\underline{x}_\alpha + \frac{\phi}{2}\overline{y}_\alpha], & \phi \geq 0, \psi \leq 0, \\ [\frac{\psi}{2}\overline{x}_\alpha + \frac{\phi}{2}\overline{y}_\alpha, \frac{\psi}{2}\underline{x}_\alpha + \frac{\phi}{2}\underline{y}_\alpha], & \phi \leq 0, \psi \leq 0, \\ [\frac{\psi}{2}\underline{x}_\alpha + \frac{\phi}{2}\overline{y}_\alpha, \frac{\psi}{2}\overline{x}_\alpha + \frac{\phi}{2}\underline{y}_\alpha], & \phi \leq 0, \psi \geq 0. \end{cases}$$

(4) division,

$$y^{-1} = \cup_{\alpha\in(0,1]} \alpha.(y^{-1})_\alpha, \qquad (y^{-1})_\alpha = [\underline{(y^{-1})}_\alpha, \overline{(y^{-1})}_\alpha],$$

where $\underline{(y^{-1})}_\alpha = \frac{1}{\psi^2}\underline{y}_\alpha$, $\overline{(y^{-1})}_\alpha = \frac{1}{\psi^2}\overline{y}_\alpha$.

So

$$xy^{-1} = \cup_{\alpha\in(0,1]} \alpha.(xy^{-1})_\alpha, \qquad (xy^{-1})_\alpha = [\underline{(xy^{-1})}_\alpha, \overline{(xy^{-1})}_\alpha],$$

where

$$[\underline{(xy^{-1})}_\alpha, \overline{(xy^{-1})}_\alpha] = \begin{cases} [\frac{1}{2\psi}\underline{x}_\alpha + \frac{\phi}{2\psi^2}\underline{y}_\alpha, \frac{1}{2\psi}\overline{x}_\alpha + \frac{\phi}{2\psi^2}\overline{y}_\alpha], & \phi \geq 0, \psi \geq 0, \\ [\frac{1}{2\psi}\overline{x}_\alpha + \frac{\phi}{2\psi^2}\underline{y}_\alpha, \frac{1}{2\psi}\underline{x}_\alpha + \frac{\phi}{2\psi^2}\overline{y}_\alpha], & \phi \geq 0, \psi \leq 0, \\ [\frac{1}{2\psi}\overline{x}_\alpha + \frac{\phi}{2\psi^2}\overline{y}_\alpha, \frac{1}{2\psi}\underline{x}_\alpha + \frac{\phi}{2\psi^2}\underline{y}_\alpha], & \phi \leq 0, \psi \leq 0, \\ [\frac{1}{2\psi}\underline{x}_\alpha + \frac{\phi}{2\psi^2}\overline{y}_\alpha, \frac{1}{2\psi}\overline{x}_\alpha + \frac{\phi}{2\psi^2}\underline{y}_\alpha], & \phi \leq 0, \psi \geq 0. \end{cases}$$

Pseudo-triangular fuzzy numbers, trapezoidal fuzzy numbers, and triangular fuzzy numbers are all special forms of pseudo-trapezoidal fuzzy numbers, so they can also be operated using defined operations.

Definition 6. *[29] For an $m \times n$ matrix A of real elements $(m \geq n)$ the SVD is defined by*

$$A = USV^\intercal,$$

where U is an $m \times n$ matrix having the property that $U^\intercal U = I_n$, where I_n is the $n \times n$ identity matrix, V is an $n \times n$ matrix such that $V^\intercal V = I_n$, and S is a diagonal $n \times n$ matrix of nonnegative elements. The diagonal elements of S are called the singular values of A and will be denoted by $s_k, k \in \{1, \cdots, n\}$. The columns of U and V are called the left and right singular vectors of A, respectively. The singular values may be ordered (along with the corresponding columns of U and V) so that $s_1 \geq s_2 \geq \cdots \geq s_n \geq 0$. With this ordering, the largest index r such that $s_r > 0$ is the rank of A, and the first r columns of U comprise an orthonormal basis of the space spanned by the columns of A. An important property of the SVD is that for all $k \leq r$, the first k columns of U, along with the corresponding columns of V and rows and columns of S, provide the best least squares approximation to the matrix A having a rank of k.

3. The fully fuzzy complex linear systems and its approximate solution

In this section, we give the definitions of the fully fuzzy complex linear systems. Some related properties are discussed.

Definition 7. *The* $n \times n$ *linear system*

$$\begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{c}_{n1} & \tilde{c}_{n2} & \cdots & \tilde{c}_{nn} \end{pmatrix} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \vdots \\ \tilde{z}_n \end{pmatrix} = \begin{pmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \\ \vdots \\ \widetilde{w}_n \end{pmatrix} \tag{3.1}$$

is called a fully fuzzy complex linear system, in which $\tilde{c}_{kj} = \tilde{a}_{kj} + i\tilde{b}_{kj}, \widetilde{w}_k = \tilde{p}_k + i\tilde{q}_k, 1 \le k, j \le n$ are known fuzzy complex numbers and $\tilde{z}_k = \tilde{x}_k + i\tilde{y}_k, 1 \le k \le n$ are unknown fuzzy complex numbers.

We present the matrix form of the system (3.1) as follows

$$\tilde{C} \cdot \tilde{Z} = \widetilde{W}, \tag{3.2}$$

where $\tilde{C} = (\tilde{c}_{kj})_{n\times n} = (\tilde{a}_{kj} + i\tilde{b}_{kj})_{n\times n}$ are fuzzy complex-valued $n \times n$ matrices, $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n)^T = (\tilde{x}_1 + i\tilde{y}_1, \tilde{x}_2 + i\tilde{y}_2, \dots, \tilde{x}_n + i\tilde{y}_n)^T$ and $\widetilde{W} = (\widetilde{w}_1, \widetilde{w}_2, \dots, \widetilde{w}_n)^T = (\tilde{p}_1 + i\tilde{q}_1, \tilde{p}_2 + i\tilde{q}_2, \dots, \tilde{p}_n + i\tilde{q}_n)^T$ are two column vectors of rectangular fuzzy complex numbers. Also, if $\tilde{C} = \tilde{A} + i\tilde{B}$, $\tilde{Z} = \tilde{X} + i\tilde{Y}$ and $\widetilde{W} = \tilde{P} + i\tilde{Q}$, then the system can be rewritten as

$$(\tilde{A} + i\tilde{B}) \cdot (\tilde{X} + i\tilde{Y}) = (\tilde{P} + i\tilde{Q}), \tag{3.3}$$

where $\tilde{A}, \tilde{B}$ are $n \times n$ fuzzy numbers matrices, and $\tilde{X}$, $\tilde{Y}$, $\tilde{P}$ and $\tilde{Q}$ are $n \times 1$ fuzzy number vectors.

Definition 8. *(Algebraic solution of fully fuzzy complex linear system) A fuzzy complex vector* $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n)^T = (\tilde{x}_1 + i\tilde{y}_1, \tilde{x}_2 + i\tilde{y}_2, \dots, \tilde{x}_n + i\tilde{y}_n)^T$ *is called the algebraic solution of (3.2) if*

$$\underline{\sum_{j=1}^{n} \tilde{a}_{kj} \cdot \tilde{x}_j - \sum_{j=1}^{n} \tilde{b}_{kj} \cdot \tilde{y}_j}_{\alpha} = \underline{p_k}_{\alpha}, \quad \overline{\sum_{j=1}^{n} \tilde{a}_{kj} \cdot \tilde{x}_j - \sum_{j=1}^{n} \tilde{b}_{kj} \cdot \tilde{y}_j}_{\alpha} = \overline{p_k}_{\alpha},$$

$$\underline{\sum_{j=1}^{n} \tilde{a}_{kj} \cdot \tilde{y}_j + \sum_{j=1}^{n} \tilde{b}_{kj} \cdot \tilde{x}_j}_{\alpha} = \underline{q_k}_{\alpha}, \quad \overline{\sum_{j=1}^{n} \tilde{a}_{kj} \cdot \tilde{y}_j - \sum_{j=1}^{n} \tilde{b}_{kj} \cdot \tilde{x}_j}_{\alpha} = \overline{q_k}_{\alpha}.$$

Lemma 2. *[2] Let* $\tilde{x}_k, k = 1,2,\cdots,n$ *are fuzzy numbers and* $\tilde{x}_k = \bigcup_{\alpha\in(0,1]} \alpha.x_{k\alpha}$, $x_\alpha = [\underline{x}_{k\alpha}, \overline{x}_{k\alpha}], x_k = \frac{x_{k1}+\overline{x}_{k1}}{2}$, *then* $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \cdots + \tilde{x}_n = \bigcup_{\alpha\in(0,]} \alpha.x_\alpha, x_\alpha = [\underline{x}_\alpha, \overline{x}_\alpha]$, *where*

$$\underline{x}_\alpha = \frac{1}{2}\{\frac{1}{2}\{\cdots\{\frac{1}{2}\{\sum_{k=1}^{2} x_k + \sum_{k=1}^{2} \underline{x}_{k\alpha}\} + \sum_{k=1}^{3} x_k + \underline{x}_{3\alpha}\} + \sum_{k=1}^{4} x_k + \underline{x}_{4\alpha} + \cdots\} + \sum_{k=1}^{n} x_k + \underline{x}_{n\alpha}\},$$

$$\overline{x}_\alpha = \frac{1}{2}\{\frac{1}{2}\{\cdots\{\frac{1}{2}\{\sum_{k=1}^{2} x_k + \sum_{k=1}^{2} \overline{x}_{k\alpha}\} + \sum_{k=1}^{3} x_k + \overline{x}_{3\alpha}\} + \sum_{k=1}^{4} x_k + \overline{x}_{4\alpha} + \cdots\} + \sum_{k=1}^{n} x_k + \overline{x}_{n\alpha}\}.$$

Theorem 1. *Suppose that the fuzzy number vector* $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n)^T = (\tilde{x}_1 + i\tilde{y}_1, \tilde{x}_2 + i\tilde{y}_2, \dots, \tilde{x}_n + i\tilde{y}_n)^T$ *be an algebraic solution of the fully fuzzy complex linear system, where the solution with the following* $\alpha-$*forms*

$$\tilde{z}_k = \cup_{\alpha\in(0,1]} \alpha . x_{k\alpha} + i \cup_{\alpha\in(0,1]} \alpha . y_{k\alpha},$$

$$x_{k\alpha} = [\underline{x}_{k\alpha}, \overline{x}_{k\alpha}], \quad x_k = \frac{\underline{x}_{k1} + \overline{x}_{k1}}{2}, \quad y_{k\alpha} = [\underline{y}_{k\alpha}, \overline{y}_{k\alpha}], \quad y_k = \frac{\underline{y}_{k1} + \overline{y}_{k1}}{2}.$$

The operation algorithm is based on TA, so the following conclusion holds

(1) For any $0 < \alpha < 1$, the $n \times n$ fully fuzzy complex linear system is transformed into an equivalent $2n \times 2n$ fully fuzzy linear system according to the definition of fuzzy complex number equality. Then, based on the fuzzy number operation of TA, it is transformed into an $4n \times 4n$ real linear system, where the coefficient matrix is a real number and the unknown vector is a column vector composed of the upper and lower bounds of the real and imaginary parts of the solution.

(2) The complex column vector $z = (z_1, z_2, \dots, z_n)^T = (x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n)^T$ is an exact solution of the $n \times n$ crisp complex linear system as follows

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}.$$

Proof Considering that the Theorem contains a large number of types and cannot list them all in their entirety, we prove here that for analyzing and reasoning the solutions of linear systems, specific number solution need to be distinguished. The article proves that the fuzzy number operation involved is based on TA.

For 2×2: Consider the 2×2 fully fuzzy complex linear system

$$\begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} \\ \tilde{c}_{21} & \tilde{c}_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix},$$

$$\begin{cases} (\tilde{a}_{11} + i\tilde{b}_{11})(\tilde{x}_1 + i\tilde{y}_1) + (\tilde{a}_{12} + i\tilde{b}_{12})(\tilde{x}_2 + i\tilde{y}_2) = \tilde{p}_1 + i\tilde{q}_1, \\ (\tilde{a}_{21} + i\tilde{b}_{21})(\tilde{x}_1 + i\tilde{y}_1) + (\tilde{a}_{22} + i\tilde{b}_{22})(\tilde{x}_2 + i\tilde{y}_2) = \tilde{p}_2 + i\tilde{q}_2, \end{cases}$$

we have

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 - \tilde{b}_{11}\tilde{y}_1 + \tilde{a}_{12}\tilde{x}_2 - \tilde{b}_{12}\tilde{y}_2 = \tilde{p}_1, \\ \tilde{a}_{21}\tilde{x}_1 - \tilde{b}_{21}\tilde{y}_1 + \tilde{a}_{22}\tilde{x}_2 - \tilde{b}_{22}\tilde{y}_2 = \tilde{p}_2, \\ \tilde{a}_{11}\tilde{y}_1 + \tilde{b}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{y}_2 - \tilde{b}_{12}\tilde{x}_2 = \tilde{q}_1, \\ \tilde{a}_{21}\tilde{y}_1 + \tilde{b}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{y}_2 - \tilde{b}_{22}\tilde{x}_2 = \tilde{q}_2. \end{cases}$$

First, we define $\tilde{A}_{kj}, \tilde{A}^*_{kj}, \tilde{B}_{kj}$ and $\tilde{B}^*_{kj}$, k, j = 1, 2, as follows

$$\tilde{A}_{kj} = \tilde{a}_{kj}\tilde{x}_j, \qquad \tilde{A}^*_{kj} = \tilde{a}_{kj}\tilde{y}_j, \qquad \tilde{B}_{kj} = \tilde{b}_{kj}\tilde{y}_j, \qquad \tilde{B}^*_{kj} = \tilde{b}_{kj}\tilde{x}_j.$$

Suppose $a_{kj}, b_{kj}, x_j, y_j \geq 0, k, j = 1, 2$, i.e. the core of the real and imaginary parts of the coefficients and the solution are all positive.

Using the fuzzy arithmetic operations based on TA, we have

$$\begin{cases} \tilde{A}_{kj} = [\underline{\tilde{A}}_{kj\alpha}, \overline{\tilde{A}}_{kj\alpha}] = [\frac{1}{2}\{\underline{a}_{kj\alpha}x_j + \underline{x}_{j\alpha}a_{kj}\}, \frac{1}{2}\{\overline{a}_{kj\alpha}x_j + \overline{x}_{j\alpha}a_{kj}\}], & k, j = 1,2, \\ \tilde{B}_{kj} = [\underline{\tilde{B}}_{kj\alpha}, \overline{\tilde{B}}_{kj\alpha}] = [\frac{1}{2}\{\underline{b}_{kj\alpha}y_j + \underline{y}_{j\alpha}b_{kj}\}, \frac{1}{2}\{\overline{b}_{kj\alpha}y_j + \overline{y}_{j\alpha}b_{kj}\}], & k, j = 1,2, \\ \tilde{A}^*_{kj} = [\underline{\tilde{A}}^*_{kj\alpha}, \overline{\tilde{A}}^*_{kj\alpha}] = [\frac{1}{2}\{\underline{a}_{kj\alpha}y_j + \underline{y}_{j\alpha}a_{kj}\}, \frac{1}{2}\{\overline{a}_{kj\alpha}y_j + \overline{y}_{j\alpha}a_{kj}\}], & k, j = 1,2, \\ \tilde{B}^*_{kj} = [\underline{\tilde{B}}^*_{kj\alpha}, \overline{\tilde{B}}^*_{kj\alpha}] = [\frac{1}{2}\{\underline{b}_{kj\alpha}x_j + \underline{x}_{j\alpha}b_{kj}\}, \frac{1}{2}\{\overline{b}_{kj\alpha}x_j + \overline{x}_{j\alpha}b_{kj}\}], & k, j = 1,2. \end{cases}$$

Taking $\alpha = 1$, the centers of the fuzzy number are obtained as follows

$$A_{kj} = a_{kj}x_j, \qquad B_{kj} = b_{kj}y_j, \qquad A^*_{kj} = a_{kj}y_j, \qquad B^*_{kj} = b_{kj}x_j, \qquad k, j = 1,2.$$

For $k = 1,2$, calculate

$$\begin{aligned} \underline{\textstyle\sum_{j=1}^2 \tilde{A}_{kj} - \sum_{j=1}^2 \tilde{B}_{kj}}_{\alpha} &= \tfrac{1}{2}\{\{a_{k1}x_1 + a_{k2}x_2\} - \{b_{k1}y_1 + b_{k2}y_2\} \\ &+ \tfrac{3}{8}\{\underline{a}_{k1\alpha}x_1 + \underline{x}_{1\alpha}a_{k1}\} + \tfrac{3}{8}\{\underline{a}_{k2\alpha}x_2 + \underline{x}_{2\alpha}a_{k2}\} + \tfrac{1}{8}\{\overline{a}_{k1\alpha}x_1 + \overline{x}_{1\alpha}a_{k1}\} + \tfrac{1}{8}\{\overline{a}_{k2\alpha}x_2 + \overline{x}_{2\alpha}a_{k2}\} \\ &- \tfrac{1}{8}\{\underline{b}_{k1\alpha}y_1 + \underline{y}_{1\alpha}b_{k1}\} - \tfrac{1}{8}\{\underline{b}_{k2\alpha}y_2 + \underline{y}_{2\alpha}b_{k2}\} - \tfrac{3}{8}\{\overline{b}_{k1\alpha}y_1 + \overline{y}_{1\alpha}b_{k1}\} - \tfrac{3}{8}\{\overline{b}_{k2\alpha}y_2 + \overline{y}_{2\alpha}b_{k2}\}\}, \\ \overline{\textstyle\sum_{j=1}^2 \tilde{A}_{kj} - \sum_{j=1}^2 \tilde{B}_{kj}}_{\alpha} &= \tfrac{1}{2}\{\{a_{k1}x_1 + a_{k2}x_2\} - \{b_{k1}y_1 + b_{k2}y_2\} \\ &+ \tfrac{1}{8}\{\underline{a}_{k1\alpha}x_1 + \underline{x}_{1\alpha}a_{k1}\} + \tfrac{1}{8}\{\underline{a}_{k2\alpha}x_2 + \underline{x}_{2\alpha}a_{k2}\} + \tfrac{3}{8}\{\overline{a}_{k1\alpha}x_1 + \overline{x}_{1\alpha}a_{k1}\} + \tfrac{3}{8}\{\overline{a}_{k2\alpha}x_2 + \overline{x}_{2\alpha}a_{k2}\} \\ &- \tfrac{3}{8}\{\underline{b}_{k1\alpha}y_1 + \underline{y}_{1\alpha}b_{k1}\} - \tfrac{3}{8}\{\underline{b}_{k2\alpha}y_2 + \underline{y}_{2\alpha}b_{k2}\} - \tfrac{1}{8}\{\overline{b}_{k1\alpha}y_1 + \overline{y}_{1\alpha}b_{k1}\} - \tfrac{1}{8}\{\overline{b}_{k2\alpha}y_2 + \overline{y}_{2\alpha}b_{k2}\}\}, \end{aligned}$$

$$\begin{aligned} \underline{\textstyle\sum_{j=1}^2 \tilde{A}^*_{kj} + \sum_{j=1}^2 \tilde{B}^*_{kj}}_{\alpha} &= \tfrac{1}{2}\{\{a_{k1}y_1 + a_{k2}y_2\} + \{b_{k1}x_1 + b_{k2}x_2\} \\ &+ \tfrac{3}{8}\{\underline{a}_{k1\alpha}y_1 + \underline{y}_{1\alpha}a_{k1}\} + \tfrac{3}{8}\{\underline{a}_{k2\alpha}y_2 + \underline{y}_{2\alpha}a_{k2}\} + \tfrac{1}{8}\{\overline{a}_{k1\alpha}y_1 + \overline{y}_{1\alpha}a_{k1}\} + \tfrac{1}{8}\{\overline{a}_{k2\alpha}y_2 + \overline{y}_{2\alpha}a_{k2}\} \\ &+ \tfrac{3}{8}\{\underline{b}_{k1\alpha}x_1 + \underline{x}_{1\alpha}b_{k1}\} + \tfrac{3}{8}\{\underline{b}_{k2\alpha}x_2 + \underline{x}_{2\alpha}b_{k2}\} + \tfrac{1}{8}\{\overline{b}_{k1\alpha}x_1 + \overline{x}_{1\alpha}b_{k1}\} + \tfrac{1}{8}\{\overline{b}_{k2\alpha}x_2 + \overline{x}_{2\alpha}b_{k2}\}\}, \\ \overline{\textstyle\sum_{j=1}^2 \tilde{A}^*_{kj} + \sum_{j=1}^2 \tilde{B}^*_{kj}}_{\alpha} &= \tfrac{1}{2}\{\{a_{k1}y_1 + a_{k2}y_2\} + \{b_{k1}x_1 + b_{k2}x_2\} \\ &+ \tfrac{1}{8}\{\underline{a}_{k1\alpha}y_1 + \underline{y}_{1\alpha}a_{k1}\} + \tfrac{1}{8}\{\underline{a}_{k2\alpha}y_2 + \underline{y}_{2\alpha}a_{k2}\} + \tfrac{3}{8}\{\overline{a}_{k1\alpha}y_1 + \overline{y}_{1\alpha}a_{k1}\} + \tfrac{3}{8}\{\overline{a}_{k2\alpha}y_2 + \overline{y}_{2\alpha}a_{k2}\} \\ &+ \tfrac{1}{8}\{\underline{b}_{k1\alpha}x_1 + \underline{x}_{1\alpha}b_{k1}\} + \tfrac{1}{8}\{\underline{b}_{k2\alpha}x_2 + \underline{x}_{2\alpha}b_{k2}\} + \tfrac{3}{8}\{\overline{b}_{k1\alpha}x_1 + \overline{x}_{1\alpha}b_{k1}\} + \tfrac{3}{8}\{\overline{b}_{k2\alpha}x_2 + \overline{x}_{2\alpha}b_{k2}\}\}, \end{aligned}$$

$$\begin{cases} \underline{\textstyle\sum_{j=1}^2 \tilde{A}_{kj} - \sum_{j=1}^2 \tilde{B}_{kj}}_{\alpha} = \underline{p_k}_{\alpha}, k = 1,2, \\ \overline{\textstyle\sum_{j=1}^2 \tilde{A}_{kj} - \sum_{j=1}^2 \tilde{B}_{kj}}_{\alpha} = \overline{p_k}_{\alpha}, k = 1,2, \\ \underline{\textstyle\sum_{j=1}^2 \tilde{A}^*_{kj} + \sum_{j=1}^2 \tilde{B}^*_{kj}}_{\alpha} = \underline{q_k}_{\alpha}, k = 1,2, \\ \overline{\textstyle\sum_{j=1}^2 \tilde{A}^*_{kj} + \sum_{j=1}^2 \tilde{B}^*_{kj}}_{\alpha} = \overline{q_k}_{\alpha}, k = 1,2, \end{cases}$$

we obtain

$$
\begin{aligned}
& 3\{\underline{x_1}_\alpha a_{11} + \underline{x_2}_\alpha a_{12} - \overline{y_1}_\alpha b_{11} - \overline{y_2}_\alpha b_{12}\} + \{\overline{x_1}_\alpha a_{11} + \overline{x_2}_\alpha a_{12} - \underline{y_1}_\alpha b_{11} - \underline{y_2}_\alpha b_{12}\} \\
= {} & 8[2\underline{p_1}_\alpha - \{a_{11}x_1 + a_{12}x_2 - b_{11}y_1 - b_{12}y_2\}] - 3\{\underline{a}_{11\alpha}x_1 + \underline{a}_{12\alpha}x_2 - \overline{b}_{11\alpha}y_1 - \overline{b}_{12\alpha}y_2\} \\
& -\{\overline{a}_{11\alpha}x_1 + \overline{a}_{12\alpha}x_2 - \underline{b}_{11\alpha}y_1 - \underline{b}_{12\alpha}y_2\}, \\
& \{\underline{x_1}_\alpha a_{11} + \underline{x_2}_\alpha a_{12} - \overline{y_1}_\alpha b_{11} - \overline{y_2}_\alpha b_{12}\} + 3\{\overline{x_1}_\alpha a_{11} + \overline{x_2}_\alpha a_{12} - \underline{y_1}_\alpha b_{11} - \underline{y_2}_\alpha b_{12}\} \\
= {} & 8[2\overline{p_1}_\alpha - \{a_{11}x_1 + a_{12}x_2 - b_{11}y_1 - b_{12}y_2\}] - \{\underline{a}_{11\alpha}x_1 + \underline{a}_{12\alpha}x_2 - \overline{b}_{11\alpha}y_1 - \overline{b}_{12\alpha}y_2\} \\
& -3\{\overline{a}_{11\alpha}x_1 + \overline{a}_{12\alpha}x_2 - \underline{b}_{11\alpha}y_1 - \underline{b}_{12\alpha}y_2\}, \\
& 3\{\underline{x_1}_\alpha a_{21} + \underline{x_2}_\alpha a_{22} - \overline{y_1}_\alpha b_{21} - \overline{y_2}_\alpha b_{22}\} + \{\overline{x_1}_\alpha a_{21} + \overline{x_2}_\alpha a_{22} - \underline{y_1}_\alpha b_{21} - \underline{y_2}_\alpha b_{22}\} \\
= {} & 8[2\underline{p_2}_\alpha - \{a_{21}x_1 + a_{22}x_2 - b_{21}y_1 - b_{22}y_2\}] - 3\{\underline{a}_{21\alpha}x_1 + \underline{a}_{22\alpha}x_2 - \overline{b}_{21\alpha}y_1 - \overline{b}_{22\alpha}y_2\} \\
& -\{\overline{a}_{21\alpha}x_1 + \overline{a}_{22\alpha}x_2 - \underline{b}_{21\alpha}y_1 - \underline{b}_{22\alpha}y_2\}, \\
& \{\underline{x_1}_\alpha a_{21} + \underline{x_2}_\alpha a_{22} - \overline{y_1}_\alpha b_{21} - \overline{y_2}_\alpha b_{22}\} + 3\{\overline{x_1}_\alpha a_{21} + \overline{x_2}_\alpha a_{22} - \underline{y_1}_\alpha b_{21} - \underline{y_2}_\alpha b_{22}\} \\
= {} & 8[2\overline{p_2}_\alpha - \{a_{21}x_1 + a_{22}x_2 - b_{21}y_1 - b_{22}y_2\}] - \{\underline{a}_{21\alpha}x_1 + \underline{a}_{22\alpha}x_2 - \overline{b}_{21\alpha}y_1 - \overline{b}_{22\alpha}y_2\} \\
& -3\{\overline{a}_{21\alpha}x_1 + \overline{a}_{22\alpha}x_2 - \underline{b}_{21\alpha}y_1 - \underline{b}_{22\alpha}y_2\}, \\
& 3\{\underline{y_1}_\alpha a_{11} + \underline{y_2}_\alpha a_{12} + \underline{x_1}_\alpha b_{11} + \underline{x_2}_\alpha b_{12}\} + \{\overline{y_1}_\alpha a_{11} + \overline{y_2}_\alpha a_{12} + \overline{x_1}_\alpha b_{11} + \overline{x_2}_\alpha b_{12}\} \\
= {} & 8[2\underline{q_1}_\alpha - \{a_{11}y_1 + a_{12}y_2 + b_{11}x_1 + b_{12}x_2\}] - 3\{\underline{a}_{11\alpha}y_1 + \underline{a}_{12\alpha}y_2 + \underline{b}_{11\alpha}x_1 + \underline{b}_{12\alpha}x_2\} \\
& -\{\overline{a}_{11\alpha}y_1 + \overline{a}_{12\alpha}y_2 + \overline{b}_{11\alpha}x_1 + \overline{b}_{12\alpha}x_2\}, \\
& \{\underline{y_1}_\alpha a_{11} + \underline{y_2}_\alpha a_{12} + \underline{x_1}_\alpha b_{11} + \underline{x_2}_\alpha b_{12}\} + 3\{\overline{y_1}_\alpha a_{11} + \overline{y_2}_\alpha a_{12} + \overline{x_1}_\alpha b_{11} + \overline{x_2}_\alpha b_{12}\} \\
= {} & 8[2\overline{q_1}_\alpha - \{a_{11}y_1 + a_{12}y_2 + b_{11}x_1 + b_{12}x_2\}] - \{\underline{a}_{11\alpha}y_1 + \underline{a}_{12\alpha}y_2 + \underline{b}_{11\alpha}x_1 + \underline{b}_{12\alpha}x_2\} \\
& -3\{\overline{a}_{11\alpha}y_1 + \overline{a}_{12\alpha}y_2 + \overline{b}_{11\alpha}x_1 + \overline{b}_{12\alpha}x_2\}, \\
& 3\{\underline{y_1}_\alpha a_{21} + \underline{y_2}_\alpha a_{22} + \underline{x_1}_\alpha b_{21} + \underline{x_2}_\alpha b_{22}\} + \{\overline{y_1}_\alpha a_{21} + \overline{y_2}_\alpha a_{22} + \overline{x_1}_\alpha b_{21} + \overline{x_2}_\alpha b_{22}\} \\
= {} & 8[2\underline{q_2}_\alpha - \{a_{21}y_1 + a_{22}y_2 + b_{21}x_1 + b_{22}x_2\}] - 3\{\underline{a}_{21\alpha}y_1 + \underline{a}_{22\alpha}y_2 + \underline{b}_{21\alpha}x_1 + \underline{b}_{22\alpha}x_2\} \\
& -\{\overline{a}_{21\alpha}y_1 + \overline{a}_{22\alpha}y_2 + \overline{b}_{21\alpha}x_1 + \overline{b}_{22\alpha}x_2\}, \\
& \{\underline{y_1}_\alpha a_{21} + \underline{y_2}_\alpha a_{22} + \underline{x_1}_\alpha b_{21} + \underline{x_2}_\alpha b_{22}\} + 3\{\overline{y_1}_\alpha a_{21} + \overline{y_2}_\alpha a_{22} + \overline{x_1}_\alpha b_{21} + \overline{x_2}_\alpha b_{22}\} \\
= {} & 8[2\overline{q_2}_\alpha - \{a_{21}y_1 + a_{22}y_2 + b_{21}x_1 + b_{22}x_2\}] - \{\underline{a}_{21\alpha}y_1 + \underline{a}_{22\alpha}y_2 + \underline{b}_{21\alpha}x_1 + \underline{b}_{22\alpha}x_2\} \\
& -3\{\overline{a}_{21\alpha}y_1 + \overline{a}_{22\alpha}y_2 + \overline{b}_{21\alpha}x_1 + \overline{b}_{22\alpha}x_2\}.
\end{aligned}
$$

We briefly denoted $d_k(x_1, x_2, y_1, y_2)$ as $d_k, 1 \leq k \leq 8$, then the above equation can be written in matrix form as follows

$$
\begin{pmatrix}
3a_{11} & 3a_{12} & -b_{11} & -b_{12} & a_{11} & a_{12} & -3b_{11} & -3b_{12} \\
3a_{21} & 3a_{22} & -b_{21} & -b_{22} & a_{21} & a_{22} & -3b_{21} & -3b_{22} \\
3b_{11} & 3b_{12} & 3a_{11} & 3a_{12} & b_{11} & b_{12} & a_{11} & a_{12} \\
3b_{21} & 3b_{22} & 3a_{21} & 3a_{22} & b_{21} & b_{22} & a_{21} & a_{22} \\
a_{11} & a_{12} & -3b_{11} & -3b_{12} & 3a_{11} & 3a_{12} & -b_{11} & -b_{12} \\
a_{21} & a_{22} & -3b_{21} & -3b_{22} & 3a_{21} & 3a_{22} & -b_{21} & -b_{22} \\
b_{11} & b_{12} & a_{11} & a_{12} & 3b_{11} & 3b_{12} & 3a_{11} & 3a_{12} \\
b_{21} & b_{22} & a_{21} & a_{22} & 3b_{21} & 3b_{22} & 3a_{21} & 3a_{22}
\end{pmatrix}
\begin{pmatrix}
\underline{x_1}_\alpha \\ \underline{x_2}_\alpha \\ \underline{y_1}_\alpha \\ \underline{y_2}_\alpha \\ \overline{x_1}_\alpha \\ \overline{x_2}_\alpha \\ \overline{y_1}_\alpha \\ \overline{y_2}_\alpha
\end{pmatrix}
=
\begin{pmatrix}
d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8
\end{pmatrix},
$$

where

$$
\begin{aligned}
d_1(x_1,x_2,y_1,y_2) = \ & 8[2\underline{p_1}_\alpha - \{a_{11}x_1 + a_{12}x_2 - b_{11}y_1 - b_{12}y_2\}] - 3\{\underline{a}_{11\alpha}x_1 + \underline{a}_{12\alpha}x_2 - \\
& \overline{b}_{11\alpha}y_1 - \overline{b}_{12\alpha}y_2\} - \{\overline{a}_{11\alpha}x_1 + \overline{a}_{12\alpha}x_2 - \underline{b}_{11\alpha}y_1 - \underline{b}_{12\alpha}y_2\}, \\
d_2(x_1,x_2,y_1,y_2) = \ & 8[2\underline{p_2}_\alpha - \{a_{21}x_1 + a_{22}x_2 - b_{21}y_1 - b_{22}y_2\}] - 3\{\underline{a}_{21\alpha}x_1 + \underline{a}_{22\alpha}x_2 - \\
& \overline{b}_{21\alpha}y_1 - \overline{b}_{22\alpha}y_2\} - \{\overline{a}_{21\alpha}x_1 + \overline{a}_{22\alpha}x_2 - \underline{b}_{21\alpha}y_1 - \underline{b}_{22\alpha}y_2\}, \\
d_3(x_1,x_2,y_1,y_2) = \ & 8[2\underline{q_1}_\alpha - \{a_{11}y_1 + a_{12}y_2 + b_{11}x_1 + b_{12}x_2\}] - 3\{\underline{a}_{11\alpha}y_1 + \underline{a}_{12\alpha}y_2 + \\
& \underline{b}_{11\alpha}x_1 + \underline{b}_{12\alpha}x_2\} - \{\overline{a}_{11\alpha}y_1 + \overline{a}_{12\alpha}y_2 + \overline{b}_{11\alpha}x_1 + \overline{b}_{12\alpha}x_2\}, \\
d_4(x_1,x_2,y_1,y_2) = \ & 8[2\underline{q_2}_\alpha - \{a_{21}y_1 + a_{22}y_2 + b_{21}x_1 + b_{22}x_2\}] - 3\{\underline{a}_{21\alpha}y_1 + \underline{a}_{22\alpha}y_2 + \\
& \underline{b}_{21\alpha}x_1 + \underline{b}_{22\alpha}x_2\} - \{\overline{a}_{21\alpha}y_1 + \overline{a}_{22\alpha}y_2 + \overline{b}_{21\alpha}x_1 + \overline{b}_{22\alpha}x_2\}, \\
d_5(x_1,x_2,y_1,y_2) = \ & 8[2\overline{p_1}_\alpha - \{a_{11}x_1 + a_{12}x_2 - b_{11}y_1 - b_{12}y_2\}] - \{\underline{a}_{11\alpha}x_1 + \underline{a}_{12\alpha}x_2 - \\
& \overline{b}_{11\alpha}y_1 - \overline{b}_{12\alpha}y_2\} - 3\{\overline{a}_{11\alpha}x_1 + \overline{a}_{12\alpha}x_2 - \underline{b}_{11\alpha}y_1 - \underline{b}_{12\alpha}y_2\}, \\
d_6(x_1,x_2,y_1,y_2) = \ & 8[2\overline{p_2}_\alpha - \{a_{21}x_1 + a_{22}x_2 - b_{21}y_1 - b_{22}y_2\}] - \{\underline{a}_{21\alpha}x_1 + \underline{a}_{22\alpha}x_2 - \\
& \overline{b}_{21\alpha}y_1 - \overline{b}_{22\alpha}y_2\} - 3\{\overline{a}_{21\alpha}x_1 + \overline{a}_{22\alpha}x_2 - \underline{b}_{21\alpha}y_1 - \underline{b}_{22\alpha}y_2\}, \\
d_7(x_1,x_2,y_1,y_2) = \ & 8[2\overline{q_1}_\alpha - \{a_{11}y_1 + a_{12}y_2 + b_{11}x_1 + b_{12}x_2\}] - \{\underline{a}_{11\alpha}y_1 + \underline{a}_{12\alpha}y_2 + \\
& \underline{b}_{11\alpha}x_1 + \underline{b}_{12\alpha}x_2\} - 3\{\overline{a}_{11\alpha}y_1 + \overline{a}_{12\alpha}y_2 + \overline{b}_{11\alpha}x_1 + \overline{b}_{12\alpha}x_2\}, \\
d_8(x_1,x_2,y_1,y_2) = \ & 8[2\overline{q_2}_\alpha - \{a_{21}y_1 + a_{22}y_2 + b_{21}x_1 + b_{22}x_2\}] - \{\underline{a}_{21\alpha}y_1 + \underline{a}_{22\alpha}y_2 + \\
& \underline{b}_{21\alpha}x_1 + \underline{b}_{22\alpha}x_2\} - 3\{\overline{a}_{21\alpha}y_1 + \overline{a}_{22\alpha}y_2 + \overline{b}_{21\alpha}x_1 + \overline{b}_{22\alpha}x_2\},
\end{aligned}
$$

we obtain the 8×8 crisp linear system.

By considering $\alpha = 1$ and then summing levels 1 and levels 5 and dividing them by 8, the same operation is performed on the corresponding other rows, and we obtain the simplified equation as follows

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 - b_{11}y_1 - b_{12}y_2 = p_1 \\
a_{21}x_1 + a_{22}x_2 - b_{21}y_1 - b_{22}y_2 = p_2 \\
a_{11}y_1 + a_{12}y_2 + b_{11}x_1 + b_{12}x_2 = q_1 \\
a_{21}y_1 + a_{22}y_2 + b_{21}x_1 + b_{22}x_2 = q_2
\end{cases},
$$

i.e.

$$
\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{21} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.
$$

The proof is completed.

Corollary 1. *Computational procedure for solving fully fuzzy complex linear system. Consider the* $n \times n$ *fully fuzzy complex linear system*

$$
\begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{c}_{n1} & \tilde{c}_{n2} & \cdots & \tilde{c}_{nn} \end{pmatrix} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \vdots \\ \tilde{z}_n \end{pmatrix} = \begin{pmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \\ \vdots \\ \widetilde{w}_n \end{pmatrix},
$$

where $\tilde{c}_{kj} = \tilde{a}_{kj} + i\tilde{b}_{kj}, \widetilde{w}_k = \tilde{p}_k + i\tilde{q}_k, 1 \le k, j \le n$ are known fuzzy complex numbers and $\tilde{z}_k = \tilde{x}_k + i\tilde{y}_k, 1 \le k \le n$ are unknown fuzzy complex numbers.

The steps of the proposed method are as follow

STEP 1: Solving the $n \times n$ crisp complex linear system as follows

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}.$$

We obtain the center of the algebraic solution by solving the crisp system. If $z = (z_1, z_2, \dots, z_n)^T = (x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n)^T$ does not exist uniquely then means end, else we get in STEP 2.

STEP 2: By means of the interval arithmetic, fuzzy arithmetic operations based on TA and embedding approach, the $n \times n$ fully fuzzy complex linear systems could be converted into $4n \times 4n$ linear systems.

STEP 3: Solving the $4n \times 4n$ crisp linear systems, then the left and right boundary is obtained.

It should be noted that if the solution $(\underline{x_1}_\alpha, \underline{x_2}_\alpha, \underline{y_1}_\alpha, \underline{y_2}_\alpha, \overline{x_1}_\alpha, \overline{x_2}_\alpha, \overline{y_1}_\alpha, \overline{y_2}_\alpha)^T$ obtained in the third step does not exist or is not unique, then the solution of the original system also does not exist or is not unique. If it exists $k, j, 1 \le k, j \le n, \alpha \in (0,1]$, such that $\underline{x}_{kj\alpha} > \overline{x}_{kj\alpha}$ or $\underline{y}_{kj\alpha} > \overline{y}_{kj\alpha}$, this result differs from the definition of fuzzy numbers, cannot form fuzzy numbers, and therefore is not a solution for linear systems. If the center of the obtained fuzzy solution is the same as the center of the algebraic solution, then the solution is called an approximate solution of a linear system.

4. Numerical examples

Example 1. *Let us consider the following* 2×2 *fully fuzzy complex linear system*

$$\begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} \\ \tilde{c}_{21} & \tilde{c}_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = \begin{pmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \end{pmatrix},$$

Where

$\tilde{c}_{11} = \bigcup_{\alpha \in (0,1]} \alpha . c_{11\alpha}, c_{11\alpha} = a_{11\alpha} + ib_{11\alpha} = [2\alpha, 4 - 2\alpha] + i[1 + \alpha, 3 - \alpha], c_{11} = 2 + 2i,$

$\tilde{c}_{12} = \bigcup_{\alpha \in (0,1]} \alpha . c_{12\alpha}, c_{12\alpha} = a_{12\alpha} + ib_{12\alpha} = [3 - \alpha, 5 - 3\alpha] + i[2 - 2\alpha, 4 - 4\alpha], c_{12} = 2 + 0i,$

$\tilde{c}_{21} = \bigcup_{\alpha \in (0,1]} \alpha . c_{21\alpha}, c_{21\alpha} = a_{21\alpha} + ib_{21\alpha} = [5\alpha - 1, 1 + 3\alpha] + i[4 - 4\alpha, 6 - 6\alpha], c_{21} = 4 + 0i,$

$\tilde{c}_{22} = \bigcup_{\alpha \in (0,1]} \alpha . c_{22\alpha}, c_{22\alpha} = a_{22\alpha} + ib_{22\alpha} = [4 - 3\alpha, 6 - 5\alpha] + i[2 - \alpha, 4 - 3\alpha], c_{22} = 1 + i,$

$\widetilde{w}_1 = \bigcup_{\alpha \in (0,1]} \alpha . w_{1\alpha}, w_{1\alpha} = p_{1\alpha} + iq_{1\alpha} = [\frac{165}{8} + \frac{11}{8}\alpha, \frac{217}{8} - \frac{41}{8}\alpha] + i[\frac{121}{8} - \frac{13}{8}\alpha, \frac{165}{8} - \frac{49}{8}\alpha], w_1 = 21 + 14i,$

$\widetilde{w}_2 = \bigcup_{\alpha \in (0,1]} \alpha . w_{2\alpha}, w_{2\alpha} = p_{2\alpha} + iq_{2\alpha} = [\frac{211}{8} + \frac{21}{8}\alpha, \frac{251}{8} - \frac{19}{8}\alpha] + i[\frac{126}{8} - \frac{38}{8}\alpha, \frac{166}{8} - \frac{78}{8}\alpha], w_2 = 29 + 11i.$

First, let $\alpha = 1$, we can obtain the following 2×2 complex linear system

$$\begin{cases}(2+2i)(x_1+iy_1)+(2+0i)(x_2+iy_2)=21+14i\\(4+0i)(x_1+iy_1)+(1+i)(x_2+iy_2)=29+11i\end{cases},$$

i.e.

$$\begin{cases}2x_1+2x_2-2y_1-0y_2=21\\4x_1+x_2-0y_1-y_2=29\\2x_1+0x_2+2y_1+2y_2=14\\0x_1+x_2+4y_1+y_2=11\end{cases},$$

by solving, we can get $(x_1, x_2, y_1, y_2)^\intercal = (5.75; 6; 1.25; 0)^\intercal$.

According to the symbol of $a_{ij}, b_{ij}, x_j, y_j, i, j = 1, 2$, we obtain the 8×8 real linear system

$$\begin{pmatrix}6&6&-2&0&2&2&-6&0\\12&3&0&-1&4&1&0&-3\\6&0&6&6&2&0&2&2\\0&3&12&3&0&1&4&1\\2&2&-6&0&6&6&-2&0\\4&1&0&-3&12&3&0&-1\\2&0&2&2&6&0&6&6\\0&1&4&1&0&3&12&3\end{pmatrix}\begin{pmatrix}\underline{x_1}_\alpha\\\underline{x_2}_\alpha\\\underline{y_1}_\alpha\\\underline{y_2}_\alpha\\\overline{x_1}_\alpha\\\overline{x_2}_\alpha\\\overline{y_1}_\alpha\\\overline{y_2}_\alpha\end{pmatrix}=\begin{pmatrix}d_1(x_1,x_2,y_1,y_2)\\d_2(x_1,x_2,y_1,y_2)\\d_3(x_1,x_2,y_1,y_2)\\d_4(x_1,x_2,y_1,y_2)\\d_5(x_1,x_2,y_1,y_2)\\d_6(x_1,x_2,y_1,y_2)\\d_7(x_1,x_2,y_1,y_2)\\d_8(x_1,x_2,y_1,y_2)\end{pmatrix}.$$

According to the calculation methods of $d_1(x_1, x_2, y_1, y_2) - d_8(x_1, x_2, y_1, y_2)$ mentioned above and the values of x_1, x_2, y_1, y_2, the following results are obtained.

$$d_1(x_1,x_2,y_1,y_2)=\frac{135}{2}+\frac{65}{2}\alpha,\qquad d_2(x_1,x_2,y_1,y_2)=121-5\alpha,$$
$$d_3(x_1,x_2,y_1,y_2)=\frac{61}{2}+\frac{35}{2}\alpha,\qquad d_4(x_1,x_2,y_1,y_2)=3+41\alpha,$$
$$d_5(x_1,x_2,y_1,y_2)=\frac{193}{2}+\frac{7}{2}\alpha,\qquad d_6(x_1,x_2,y_1,y_2)=149-33\alpha,$$
$$d_7(x_1,x_2,y_1,y_2)=\frac{123}{2}+\frac{5}{2}\alpha,\qquad d_8(x_1,x_2,y_1,y_2)=31+13\alpha.$$

Due to

$$det\begin{pmatrix}6&6&-2&0&2&2&-6&0\\12&3&0&-1&4&1&0&-3\\6&0&6&6&2&0&2&2\\0&3&12&3&0&1&4&1\\2&2&-6&0&6&6&-2&0\\4&1&0&-3&12&3&0&-1\\2&0&2&2&6&0&6&6\\0&1&4&1&0&3&12&3\end{pmatrix}=2.2729\neq 0,$$

so

$$\begin{pmatrix} \underline{x_1}_\alpha \\ \underline{x_2}_\alpha \\ \underline{y_1}_\alpha \\ \underline{y_2}_\alpha \\ \overline{x_1}_\alpha \\ \overline{x_2}_\alpha \\ \overline{y_1}_\alpha \\ \overline{y_2}_\alpha \end{pmatrix} = \begin{pmatrix} 6 & 6 & -2 & 0 & 2 & 2 & -6 & 0 \\ 12 & 3 & 0 & -1 & 4 & 1 & 0 & -3 \\ 6 & 0 & 6 & 6 & 2 & 0 & 2 & 2 \\ 0 & 3 & 12 & 3 & 0 & 1 & 4 & 1 \\ 2 & 2 & -6 & 0 & 6 & 6 & -2 & 0 \\ 4 & 1 & 0 & -3 & 12 & 3 & 0 & -1 \\ 2 & 0 & 2 & 2 & 6 & 0 & 6 & 6 \\ 0 & 1 & 4 & 1 & 0 & 3 & 12 & 3 \end{pmatrix}^{-1} \begin{pmatrix} \frac{135}{2} + \frac{65}{2}\alpha \\ 121 - 5\alpha \\ \frac{61}{2} + \frac{35}{2}\alpha \\ 3 + 41\alpha \\ \frac{193}{2} + \frac{7}{2}\alpha \\ 149 - 33\alpha \\ \frac{123}{2} + \frac{5}{2}\alpha \\ 31 + 13\alpha \end{pmatrix}.$$

Through calculation, it is found that the coefficient matrix of the 8×8 real linear system is close to singularity. Here, singular value decomposition is used to solve the pseudo-inverse of the coefficient matrix. Let

$$H = \begin{pmatrix} 6 & 6 & -2 & 0 & 2 & 2 & -6 & 0 \\ 12 & 3 & 0 & -1 & 4 & 1 & 0 & -3 \\ 6 & 0 & 6 & 6 & 2 & 0 & 2 & 2 \\ 0 & 3 & 12 & 3 & 0 & 1 & 4 & 1 \\ 2 & 2 & -6 & 0 & 6 & 6 & -2 & 0 \\ 4 & 1 & 0 & -3 & 12 & 3 & 0 & -1 \\ 2 & 0 & 2 & 2 & 6 & 0 & 6 & 6 \\ 0 & 1 & 4 & 1 & 0 & 3 & 12 & 3 \end{pmatrix} = USV^{\mathsf{T}},$$

where U is an 8×8 orthogonal matrix, S is an 8×8diagonal matrix, and the elements on the diagonal are singular values. V is an 8×8 orthogonal matrix, These can be calculated using a MATLAB program as follows

$$U =$$
$$\begin{pmatrix} -0.2542 & -0.3509 & -0.3536 & 0.0000 & 0.2132 & -0.5166 & 0.5000 & 0.3536 \\ -0.0483 & -0.5567 & -0.3536 & -0.5000 & -0.3478 & 0.2584 & -0.0000 & -0.3536 \\ 0.3509 & -0.2542 & -0.3536 & -0.0000 & 0.5166 & 0.2132 & -0.5000 & 0.3536 \\ 0.5567 & -0.0483 & -0.3536 & 0.5000 & -0.2584 & -0.3478 & -0.0000 & -0.3536 \\ -0.2542 & -0.3509 & 0.3536 & 0.0000 & 0.2132 & -0.5166 & -0.5000 & -0.3536 \\ -0.0483 & -0.5567 & 0.3536 & 0.5000 & -0.3478 & 0.2584 & -0.0000 & 0.3536 \\ 0.3509 & -0.2542 & 0.3536 & 0.0000 & 0.5166 & 0.2132 & 0.5000 & -0.3536 \\ 0.5567 & -0.0483 & 0.3536 & -0.5000 & -0.2584 & -0.3478 & 0.0000 & 0.3536 \end{pmatrix},$$

$$S =$$
$$\begin{pmatrix} 20.8006 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 20.8006 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12.6491 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.8800 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.8800 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0000 \end{pmatrix},$$

$$V=\begin{pmatrix}0.0000 & -0.6609 & -0.4472 & -0.5000 & 0.0397 & 0.2481 & -0.0000 & -0.2236\\ 0.0000 & -0.2513 & -0.2236 & 0.0000 & -0.1045 & -0.6526 & 0.5000 & 0.4472\\ 0.6609 & 0.0000 & -0.4472 & 0.5000 & -0.2481 & 0.0397 & -0.0000 & -0.2236\\ 0.2513 & 0.0000 & -0.2236 & 0.0000 & 0.6526 & -0.1045 & -0.5000 & 0.4472\\ 0.0000 & -0.6609 & 0.4472 & 0.5000 & 0.0397 & 0.2481 & 0.0000 & 0.2236\\ -0.0000 & -0.2513 & 0.2236 & 0.0000 & -0.1045 & -0.6526 & -0.5000 & -0.4472\\ 0.6609 & 0.0000 & 0.4472 & -0.5000 & -0.2481 & 0.0397 & 0.0000 & 0.2236\\ 0.2513 & 0.0000 & 0.2236 & 0.0000 & 0.6526 & -0.1045 & 0.5000 & -0.4472\end{pmatrix}.$$

Furthermore, the pseudo-inverse can be calculated. According to $H=USV^{\intercal}$, we can obtain $H^{\dagger}=VS^{\dagger}U^{\intercal}$, where $S^{\dagger}$ is the pseudo-inverse of S, and there is

$$H^{\dagger}=\begin{pmatrix}-2.3002 & 2.3002 & -2.3002 & 2.3002 & 2.3002 & -2.3002 & 2.3002 & -2.3002\\ 4.6003 & -4.6003 & 4.6003 & -4.6003 & -4.6003 & 4.6003 & -4.6003 & 4.6003\\ -2.3002 & 2.3002 & -2.3002 & 2.3002 & 2.3002 & -2.3002 & 2.3002 & -2.3002\\ 4.6003 & -4.6003 & 4.6003 & -4.6003 & -4.6003 & 4.6003 & -4.6003 & 4.6003\\ 2.3002 & -2.3002 & 2.3002 & -2.3002 & -2.3002 & 2.3002 & -2.3002 & 2.3002\\ -4.6003 & 4.6003 & -4.6003 & 4.6003 & 4.6003 & -4.6003 & 4.6003 & -4.6003\\ 2.3002 & -2.3002 & 2.3002 & -2.3002 & -2.3002 & 2.3002 & -2.3002 & 2.3002\\ -4.6003 & 4.6003 & -4.6003 & 4.6003 & 4.6003 & -4.6003 & 4.6003 & -4.6003\end{pmatrix}.$$

The solution of the 8×8 linear system can be obtained as follows

$$\begin{pmatrix}\underline{x_1}_{\alpha}\\ \underline{x_2}_{\alpha}\\ \underline{y_1}_{\alpha}\\ \underline{y_2}_{\alpha}\\ \overline{x_1}_{\alpha}\\ \overline{x_2}_{\alpha}\\ \overline{y_1}_{\alpha}\\ \overline{y_2}_{\alpha}\end{pmatrix}=\begin{pmatrix}1766529464463599 17/64\alpha+58884315482121627/64\\ -88326473223179749/16\alpha-29442157741060619/16\\ 88326473223179791/32\alpha+29442157741060605/32\\ -44163236611589977/8\alpha-14721078870530369/8\\ -176652946446359913/64\alpha-58884315482120719/64\\ 176652946446359689/32\alpha+58884315482121335/32\\ -176652946446359681/64\alpha-58884315482120943/64\\ 44163236611590015/8\alpha+14721078870530361/8\end{pmatrix}.$$

Finally, the solution of the 2×2 fully fuzzy complex linear system are respectively $\tilde{z}_1=\bigcup_{\alpha\in(0,1]}\alpha.z_{1\alpha},\tilde{z}_2=\bigcup_{\alpha\in(0,1]}\alpha.z_{2\alpha}$. where

$$z_{1\alpha}=x_{1\alpha}+iy_{1\alpha}=[\frac{176652946446359917}{64}\alpha+\frac{58884315482121627}{64},-\frac{176652946446359913}{64}\alpha-\frac{58884315482120719}{64}]+i[\frac{88326473223179791}{32}\alpha+\frac{29442157741060605}{32},-\frac{176652946446359681}{64}\alpha-\frac{58884315482120943}{64}],$$

$$z_{2\alpha}=x_{2\alpha}+iy_{2\alpha}=[-\frac{88326473223179749}{16}\alpha-\frac{29442157741060619}{16},\frac{176652946446359689}{32}\alpha+\frac{58884315482121335}{32}]+i[-\frac{44163236611589977}{8}\alpha-\frac{14721078870530369}{8},\frac{44163236611590015}{8}\alpha+\frac{14721078870530361}{8}].$$

When $\alpha=1$, the centers of the fuzzy complex number solutions are obtained as follows $z_1=5.75+1.25i,z_2=6+0i$. Although the transformed linear system can yield solution through the pseudo-inverse, we can verify that for any $\alpha \in (0,1]$, equation $\underline{x_1}_\alpha \geq \overline{x_1}_\alpha$, $\underline{y_1}_\alpha \geq \overline{y_1}_\alpha$ holds. Therefore, this solution is not the exact solution of the corresponding system in the problem.

It is worth noting that we can also use the method proposed by $D.Behera$ and $S.Chakraverty$ [12] to solve the problem.

The original fully fuzzy complex linear system in the problem is

$$\begin{cases} & ([2\alpha,4-2\alpha]+i[1+\alpha,3-\alpha])\tilde{z}_1+([3-\alpha,5-3\alpha]+i[2-2\alpha,4-4\alpha])\tilde{z}_2 \\ = & [\frac{165}{8}+\frac{11}{8}\alpha,\frac{217}{8}-\frac{41}{8}\alpha]+i[\frac{121}{8}-\frac{13}{8}\alpha,\frac{165}{8}-\frac{49}{8}\alpha], \\ & ([5\alpha-1,1+3\alpha]+i[4-4\alpha,6-6\alpha])\tilde{z}_1+([4-3\alpha,6-5\alpha]+i[2-\alpha,4-3\alpha])\tilde{z}_2 \\ = & [\frac{211}{8}+\frac{21}{8}\alpha,\frac{251}{8}-\frac{19}{8}\alpha]+i[\frac{126}{8}-\frac{38}{8}\alpha,\frac{166}{8}-\frac{78}{8}\alpha]. \end{cases}$$

Base on the TA −based fuzzy number operations and the correspondence between the real and imaginary parts of fuzzy complex numbers, the following results are obtained

$$\begin{cases} & [2\alpha,4-2\alpha]\tilde{x}_1+[3-\alpha,5-3\alpha]\tilde{x}_2-[1+\alpha,3-\alpha])\tilde{y}_1-[2-2\alpha,4-4\alpha]\tilde{y}_2 \\ = & \frac{165}{8}+\frac{11}{8}\alpha,\frac{217}{8}-\frac{41}{8}\alpha], \\ & [5\alpha-1,1+3\alpha]\tilde{x}_1+[4-3\alpha,6-5\alpha]\tilde{x}_2-[4-4\alpha,6-6\alpha])\tilde{y}_1-[2-\alpha,4-3\alpha]\tilde{y}_2 \\ = & [\frac{211}{8}+\frac{21}{8}\alpha,\frac{251}{8}-\frac{19}{8}\alpha], \\ & [1+\alpha,3-\alpha])\tilde{x}_1+[2-2\alpha,4-4\alpha]\tilde{x}_2+[2\alpha,4-2\alpha]\tilde{y}_1+[3-\alpha,5-3\alpha]\tilde{y}_2 \\ = & [\frac{121}{8}-\frac{13}{8}\alpha,\frac{165}{8}-\frac{49}{8}\alpha], \\ & [4-4\alpha,6-6\alpha]\tilde{x}_1+[2-\alpha,4-3\alpha]\tilde{x}_2+[5\alpha-1,1+3\alpha]\tilde{y}_1+[4-3\alpha,6-5\alpha]\tilde{y}_2 \\ = & [\frac{126}{8}-\frac{38}{8}\alpha,\frac{166}{8}-\frac{78}{8}\alpha]. \end{cases}$$

Because $\tilde{x}_j=[\underline{x}_j,\overline{x}_j]$, $\tilde{y}_j=[\underline{y}_j,\overline{y}_j]$, $j=1,2$, according to the interval operation rules in the reference, we can obtain

$$\begin{cases} [2\alpha\underline{x}_1+(3-\alpha)\underline{x}_2+(\alpha-3)\underline{y}_1+(4\alpha-4)\underline{y}_2,(4-2\alpha)\overline{x}_1+(5-3\alpha)\overline{x}_2 \\ +(-1-\alpha)\overline{y}_1+(2\alpha-2)\overline{y}_2]=\frac{165}{8}+\frac{11}{8}\alpha,\frac{217}{8}-\frac{41}{8}\alpha], \\ [(5\alpha-1)\underline{x}_1+(4-3\alpha)\underline{x}_2+(6\alpha-6)\underline{y}_1+(3\alpha-4)\underline{y}_2,(1+3\alpha)\overline{x}_1+(6-5\alpha)\overline{x}_2 \\ +(4\alpha-4)\overline{y}_1(\alpha-2)\overline{y}_2]=[\frac{211}{8}+\frac{21}{8}\alpha,\frac{251}{8}-\frac{19}{8}\alpha], \\ [(1+\alpha)\underline{x}_1+(2-2\alpha)\underline{x}_2+2\alpha\underline{y}_1+(3-\alpha)\underline{y}_2,(3-\alpha)\overline{x}_1+(4-4\alpha)\overline{x}_2 \\ +(4-2\alpha)\overline{y}_1+(5-3\alpha)\overline{y}_2]=[\frac{121}{8}-\frac{13}{8}\alpha,\frac{165}{8}-\frac{49}{8}\alpha], \\ [(4-4\alpha)\underline{x}_1+(2-\alpha)\underline{x}_2+(5\alpha-1)\underline{y}_1+(4-3\alpha)\underline{y}_2,(6-6\alpha)\overline{x}_1+(4-3\alpha)\overline{x}_2 \\ +(1+3\alpha)\overline{y}_1+(6-5\alpha)\overline{y}_2]=[\frac{126}{8}-\frac{38}{8}\alpha,\frac{166}{8}-\frac{78}{8}\alpha], \end{cases}$$

at the same time, the following equations hold

$$\begin{cases} \psi[2\alpha\underline{x}_1+(3-\alpha)\underline{x}_2+(\alpha-3)\underline{y}_1+(4\alpha-4)\underline{y}_2]+(1-\psi)[(4-2\alpha)\overline{x}_1+(5-3\alpha)\overline{x}_2 \\ +(-1-\alpha)\overline{y}_1+(2\alpha-2)\overline{y}_2]=\psi[\frac{165}{8}+\frac{11}{8}\alpha]+(1-\psi)[\frac{217}{8}-\frac{41}{8}\alpha], \\ \psi[(5\alpha-1)\underline{x}_1+(4-3\alpha)\underline{x}_2+(6\alpha-6)\underline{y}_1+(3\alpha-4)\underline{y}_2]+(1-\psi)[(1+3\alpha)\overline{x}_1 \\ +(6-5\alpha)\overline{x}_2+(4\alpha-4)\overline{y}_1+(\alpha-2)\overline{y}_2]=\psi[\frac{211}{8}+\frac{21}{8}\alpha]+(1-\psi)[\frac{251}{8}-\frac{19}{8}\alpha], \\ \psi[(1+\alpha)\underline{x}_1+(2-2\alpha)\underline{x}_2+2\alpha\underline{y}_1+(3-\alpha)\underline{y}_2]+(1-\psi)[(3-\alpha)\overline{x}_1+(4-4\alpha)\overline{x}_2 \\ +(4-2\alpha)\overline{y}_1+(5-3\alpha)\overline{y}_2]=\psi[\frac{121}{8}-\frac{13}{8}\alpha]+(1-\psi)[\frac{165}{8}-\frac{49}{8}\alpha], \end{cases}$$

and

$$\begin{cases} \psi[(4-4\alpha)\underline{x}_1+(2-\alpha)\underline{x}_2+(5\alpha-1)\underline{y}_1+(4-3\alpha)\underline{y}_2]+(1-\psi)[(6-6\alpha)\overline{x}_1 \\ +(4-3\alpha)\overline{x}_2+(1+3\alpha)\overline{y}_1+(6-5\alpha)\overline{y}_2]=\psi[\frac{126}{8}-\frac{38}{8}\alpha]+(1-\psi)[\frac{166}{8}-\frac{78}{8}\alpha]. \end{cases}$$

Taking $\psi=0.2, 0.4$ respectively, we can obtain a system of linear equations regarding the unknown $\underline{x}_1, \underline{x}_2, \underline{y}_1, \underline{y}_2, \overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2$. Solving the system we can obtain

$$\begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{y}_1 \\ \underline{y}_2 \\ \overline{x}_1 \\ \overline{x}_2 \\ \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} = \begin{pmatrix} \frac{1416\alpha^4-9611\alpha^3+18785\alpha^2-7385\alpha+395}{1368\alpha^4-6000\alpha^3+10096\alpha^2-6000\alpha+1176} \\ \frac{2090\alpha^4-20339\alpha^3+52489\alpha^2-38713\alpha+8633}{1368\alpha^4-6000\alpha^3+10096\alpha^2-6000\alpha+1176} \\ \frac{367\alpha^4-1417\alpha^3+5265\alpha^2-3987\alpha+492}{1368\alpha^4-6000\alpha^3+10096\alpha^2-6000\alpha+1176} \\ \frac{193\alpha^4+3967\alpha^3-6895\alpha^2+2693\alpha+42}{1368\alpha^4-6000\alpha^3+10096\alpha^2-6000\alpha+1176} \\ \frac{8298\alpha^4-40493\alpha^3+57491\alpha^2-22927\alpha+1327}{8600\alpha^4-36304\alpha^3+53008\alpha^2-30864\alpha+6200} \\ \frac{9816\alpha^4-79959\alpha^3+173265\alpha^2-131221\alpha+32131}{8600\alpha^4-36304\alpha^3+53008\alpha^2-30864\alpha+6200} \\ \frac{617\alpha^4+1309\alpha^3-1325\alpha^2-513\alpha+600}{8600\alpha^4-36304\alpha^3+53008\alpha^2-30864\alpha+6200} \\ \frac{1293\alpha^4+10355\alpha^3-19587\alpha^2+9601\alpha-1406}{8600\alpha^4-36304\alpha^3+53008\alpha^2-30864\alpha+6200} \end{pmatrix},$$

the final solution is written as

$$\tilde{z}_1=[\frac{1416\alpha^4-9611\alpha^3+18785\alpha^2-7385\alpha+395}{1368\alpha^4-6000\alpha^3+10096\alpha^2-6000\alpha+1176}, \frac{8298\alpha^4-40493\alpha^3+57491\alpha^2-22927\alpha+1327}{8600\alpha^4-36304\alpha^3+53008\alpha^2-30864\alpha+6200}]+ i[\frac{367\alpha^4-1417\alpha^3+5265\alpha^2-3987\alpha+492}{1368\alpha^4-6000\alpha^3+10096\alpha^2-6000\alpha+1176}, \frac{617\alpha^4+1309\alpha^3-1325\alpha^2-513\alpha+600}{8600\alpha^4-36304\alpha^3+53008\alpha^2-30864\alpha+6200}],$$

$$\tilde{z}_2=[\frac{2090\alpha^4-20339\alpha^3+52489\alpha^2-38713\alpha+8633}{1368\alpha^4-6000\alpha^3+10096\alpha^2-6000\alpha+1176}, \frac{9816\alpha^4-79959\alpha^3+173265\alpha^2-131221\alpha+32131}{8600\alpha^4-36304\alpha^3+53008\alpha^2-30864\alpha+6200}]+ i[\frac{193\alpha^4+3967\alpha^3-6895\alpha^2+2693\alpha+42}{1368\alpha^4-6000\alpha^3+10096\alpha^2-6000\alpha+1176}, \frac{1293\alpha^4+10355\alpha^3-19587\alpha^2+9601\alpha-1406}{8600\alpha^4-36304\alpha^3+53008\alpha^2-30864\alpha+6200}].$$

When $\alpha=1$, the center of the fuzzy solutions are

$$z_1=5.7+1.1i, \qquad z_2=6.4+0.2i.$$

The exact solutions of this example are $\tilde{z}_1=[2+4\alpha, 4+2\alpha]+i[3-2\alpha, 5-4\alpha]$, $\tilde{z}_2=[5+\alpha, 7-\alpha]+i[4-3\alpha, 6-5\alpha]$. When $\alpha=1$, the center of the exact solutions are $z_1=6+i$, $z_2=6+i$. By observation, the center of the fuzzy solutions obtained by the method proposed in [12] is almost the same as the result calculated by the method proposed in this paper. We can consider that the method proposed in this paper, use TA −based to solve the approximate solution of the fully fuzzy complex linear systems, is equally effective.

5. Conclusions and remarks

In this paper, we conduct research on solving fully fuzzy complex linear systems, and provide specific solution methods based on fuzzy number operations using transmission average (TA) and matrix embedding methods. At the same time, specific numerical examples are provided to verify the proposed method, and the effectiveness of the proposed method is compared with other methods proposed by scholars.

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