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On the Diophantine Equation $n^x + 5^y = z^2$

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Abstract. In this paper, we study the Diophantine equation $n^x + 5^y = z^2$, where n is a positive integer and x, y, z are non-negative integers. We found that if $n \equiv 1 \pmod 4$, then the Diophantine equation has no non-negative integer solution. If $n \equiv 3 \pmod{20}$ or $n \equiv 7 \pmod{20}$, then the Diophantine equation has all non-negative integer solutions, which are $(n, x, y, z) = (n, 1, 0, (n+1)^{0.5})$, where $(n+1)^{0.5}$ is a positive integer.

Keywords: Diophantine equation; Mihailescu's theorem; non-negative integer solution

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In 2007, Acu [1] proved that the Diophantine equation $2^x + 5^y = z^2$ has exactly two solutions in non-negative integers $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$. In 2012, Sroysang [10] found that the Diophantine equation $3^x + 5^y = z^2$ has the unique non-negative integer solution $(x, y, z) = (1, 0, 2)$. In 2013, Rabago [8] found that the only solution (x, y, z) to the Diophantine equation $5^x + 31^y = z^2$ in non-negative integers is $(1, 1, 6)$. In the same year, Sroysang [11,12] proved that the Diophantine equations $5^x + 7^y = z^2$ and $5^x + 23^y = z^2$ have no non-negative integer solution. He proved also that the Diophantine equation $5^x + 43^y = z^2$ has no non-negative integer solution in [13] and the Diophantine equation $5^x + 63^y = z^2$ has the unique non-negative integer solution $(x, y, z) = (0, 1, 8)$ in [14]. In 2016, Khan, Rashid and Uddin [6] proved that the Diophantine equation $5^x + 9^y = z^2$ has no non-negative integer solution. In 2016, Cheenchan el at. [5] showed that the Diophantine equation $p^x + 5^y = z^2$, where p is prime and p satisfies; case 1: $p \equiv 1 \pmod 4$ or case 2: $p \equiv 3 \pmod 4$ and $p \equiv 2 \pmod 5$ or case 3: $p \equiv 3 \pmod 4$ and $p \equiv 3 \pmod 5$, has no non-negative integer solution.

In 2019, Burshtein [3] found that the Diophantine equation $5^x + 103^y = z^2$ has no positive integer solution. If y is even, then the Diophantine equation $5^x + 11^y = z^2$ also has no positive integer solution. Later in 2020, Burshtein [4] found also that the

Diophantine equation $5^x + 5^y = z^2$ has no positive integer solution. In the same year, Sangam [9] showed that the Diophantine equation $5^x + 8^y = z^2$ has no positive integer solution. In 2022, Tadee [15] found some conditions for non-existence of the non-negative integer solutions of the Diophantine equation $p^x + (p+14)^y = z^2$, where p and $p+14$ are prime. In the same year, Borah and Dutta [2] showed that the Diophantine equation $5^x + 24^y = z^2$ has the unique positive integer solution $(x, y, z) = (2, 1, 7)$.

In this article, we will solve the Diophantine equation $n^x + 5^y = z^2$, where n is a positive integer with $n \equiv 1 \pmod 4$ or $n \equiv 3, 7 \pmod{20}$ and x, y, z are non-negative integers, by using an elementary method and Mihailescu's theorem.

Theorem 1.1. (Mihailescu's theorem) [7] The Diophantine equation $a^x - b^y = 1$ has the unique integer solution $(a, b, x, y) = (3, 2, 2, 3)$, where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.

2. Main results

In this section, we present our results.

Theorem 2.1. Let n be a positive integer with $n \equiv 1 \pmod 4$. Then the Diophantine equation $n^x + 5^y = z^2$ has no non-negative integer solution.

Proof: Assume that x, y and z are non-negative integers such that $n^x + 5^y = z^2$. Since $n \equiv 1 \pmod 4$, we have $n^x + 5^y \equiv 2 \pmod 4$, and so $z^2 \equiv 2 \pmod 4$. This is impossible since $z^2 \equiv 0, 1 \pmod 4$.

By Theorem 2.1, we have the following corollaries.

Corollary 2.2. [4] The Diophantine equation $5^x + 5^y = z^2$ has no solution in positive integers x, y, z.

Corollary 2.3. [5] The Diophantine equation $p^x + 5^y = z^2$, where p is a prime number with $p \equiv 1 \pmod 4$, has no non-negative integer solution.

Corollary 2.4. [6] The Diophantine equation $5^x + 9^y = z^2$ has no non-negative integer solution.

Lemma 2.5. Let n be a positive integer. Then the Diophantine equation $n^x + 1 = z^2$ has all non-negative integer solutions $(n, x, z) = (2, 3, 3)$ and $(n, x, z) = (n, 1, \sqrt{n+1})$, where $\sqrt{n+1}$ is a positive integer.

Proof: Let x, y and z be non-negative integers such that $n^x + 1 = z^2$. It is easy to check that $z > 1, n > 1$ and $x > 0$. If $x = 1$, then $z^2 = n + 1$, and so $(n, x, z) = (n, 1, \sqrt{n+1})$, where $\sqrt{n+1}$ is a positive integer. If $x > 1$, then $\min\{z, n, 2, x\} > 1$. By Theorem 1.1, we obtain $(n, x, z) = (2, 3, 3)$.

Lemma 2.6. Let n be a positive integer with $n \equiv 2,3 \pmod{5}$. If the Diophantine equation $n^x + 5^y = z^2$ has a non-negative integer solution and $y > 0$, then x is even.

Proof: Let x, y and z be non-negative integers such that $n^x + 5^y = z^2$. Since $y > 0$, we get $5^y \equiv 0 \pmod{5}$. Assume that x is odd. Then there exists a non-negative integer k such that $x = 2k + 1$. Since $n \equiv 2,3 \pmod{5}$, it implies that $n^x = n^{2k+1} \equiv 2,3 \pmod{5}$, and so $z^2 = n^x + 5^y \equiv 2, 3 \pmod{5}$. This is impossible since $z^2 \equiv 0, 1, 4 \pmod{5}$. Hence, x is even.

Theorem 2.7. Let n be a positive integer with $n \equiv 3,7 \pmod{20}$. Then the Diophantine equation $n^x + 5^y = z^2$ has all non-negative integer solutions $(n, x, y, z) = (n, 1, 0, \sqrt{n+1})$, where $\sqrt{n+1}$ is a positive integer.

Proof: Let x, y and z be non-negative integers such that $n^x + 5^y = z^2$. Since $n \equiv 3,7 \pmod{20}$, it implies that $n \equiv 3 \pmod{4}$ and $n \equiv 2,3 \pmod{5}$. Assume that $y > 0$. By Lemma 2.6, it follows that x is even. There exists a non-negative integer k such that $x = 2k$. Therefore $(z - n^k)(z + n^k) = 5^y$. Since 5 is prime, we obtain $z - n^k = 5^u$ and $z + n^k = 5^{y-u}$ for some non-negative integer u. Then $y > 2u$ and $2n^k = 5^u(5^{y-2u} - 1)$. Since $n \equiv 2,3 \pmod{5}$, we obtain that $u = 0$, and so $2n^k = 5^y - 1 \equiv 0 \pmod{4}$. Then n is even. This is impossible since $n \equiv 3 \pmod{4}$. Thus $y = 0$. Since $n \equiv 3 \pmod{4}$, we have $n \neq 2$. By Lemma 2.5, it implies that $(n, x, y, z) = (n, 1, 0, \sqrt{n+1})$, where $\sqrt{n+1}$ is a positive integer.

By Theorem 2.7, we can easily show that some previous researches are true.

Corollary 2.8. [10] $(1, 0, 2)$ is the unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$, where x, y and z are non-negative integers.

Corollary 2.9. [11] The Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution.

Corollary 2.10. [12] The Diophantine equation $5^x + 23^y = z^2$ has no non-negative integer solution.

Corollary 2.11. [13] The Diophantine equation $5^x + 43^y = z^2$ has no non-negative integer solution.

Corollary 2.12. [14] $(0, 1, 8)$ is the unique solution (x, y, z) for the Diophantine equation $5^x + 63^y = z^2$, where x, y and z are non-negative integers.

Corollary 2.13. [3] The Diophantine equation $5^x + 103^y = z^2$ has no positive integer solution.

3. Conclusion

In this paper, by using an elementary method and Mihailescu's theorem, we showed that the Diophantine equation $n^x + 5^y = z^2$ has no non-negative integer solution, where $n \equiv 1 \pmod 4$. Moreover, if $n \equiv 3, 7 \pmod{20}$, then all non-negative integer solutions of the equation are $(n, x, y, z) = (n, 1, 0, \sqrt{n+1})$, where $\sqrt{n+1}$ is a positive integer.

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Conflict of interest. The paper is written by a single author so there is no conflict of interest.

Authors' Contributions. It is a single-author paper. So, full credit goes to the author.

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