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On Constructing Uninorms via Closure (Interior) Operators on a Bounded Lattice

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Abstract. Quite recently, Ouyang-Zhang proposed an interesting approach to construct uninorms via closure (interior) operators on a bounded lattice. In which, they defined an important closure (interior) operator by the notion of universally comparable element when the bounded lattice is complete. However, their operators are not well defined in general when the bounded lattice is not complete. In this paper, we define a closure (interior) operator on bounded lattice, and prove that these operators reduce into Ouyang-Zhang's operators when the bounded lattice is complete. Hence our closure (interior) operator can be regarded as the extension of Ouyang-Zhang's closure (interior) operator to non-complete bounded lattice. At last, the uninorms corresponding to the closure (resp., interior) operator are constructed and discussed.

Keywords: Triangular norm; Uninorm; Lattice; Closure operator; interior operator.

AMS Mathematics Subject Classification (2010): 03E72, 94D05

1. Introduction

Triangular norms (t-norms) and triangular conorms (t-conorms) on the unit interval [0, 1] have been applied in many realms such as fuzzy logic [4], fuzzy rough set-topology [8,12], L-fuzzy covering rough sets [16] and fuzzy measure integrals [10]. As a unification of t-norms and t-conorms, uninorms also play important role in many fields.

Nowadays, t-norm and t-conorms have been extended from [0, 1] to more general lattice-ordered structure (e.g., bounded lattice) [1,3,6,11]. Since 2015, the research on uninorms has also been extended to the bounded lattice [2,7,13-15]. Among them, the construction of uninorms on a bounded lattice L has always been a challenging problem due to the poor structures of L compared with [0, 1].

Quite recently, Ouyang-Zhang [9] proposed an effective approach to construct uninorms via closure (interior) operators on a bounded lattice. In which, for a complete (bounded) lattice L, they defined a closure operator $\Uparrow: L \to L$ and an interior operator $\Downarrow: L \to L$ as follows:

$$\forall x \in L, \Uparrow x = \wedge\left\{a \in UC(L) \mid a \geq x\right\}, \Downarrow x = \vee\left\{a \in UC(L) \mid a \leq x\right\}.$$

The operators $\Downarrow$ and $\Uparrow$ play important role in [9], two-thirds of the literature is devoted to these two operators and the associated uninorms (indeed four uninorms are constructed by them).

Notice that when L is not complete, the two operators $\Uparrow, \Downarrow$ (and so the resulted uninorms) may be not well defined since $\Uparrow x$ and $\Downarrow x$ may not exist. Hence the main aim of the paper is to extend these two operators $\Uparrow, \Downarrow$ from complete lattice to arbitrary bounded lattice— the unified lattice environment expected in [9].

The contents are listed as below. In Section 2, we recall some concepts and symbols as preliminary. In Section 3, we define a closure operator $\tilde{\Uparrow}$ (resp., interior operator $\tilde{\Downarrow}$) on a bounded lattice L and prove that $\tilde{\Uparrow}=\Uparrow$ (resp., $\tilde{\Downarrow}=\Downarrow$) when L is a complete lattice. Hence $\tilde{\Uparrow}$ (resp., $\tilde{\Downarrow}$) can be regarded as the extension of Ouyang-Zhang's closure operator $\Uparrow$ (resp., interior operator $\Downarrow$) from complete lattice to bounded lattice. In Section 4, we discuss the uninorms constructed by our closure (interior) operator. Of course, these uninorms can be regarded as the extensions of Ouyang-Zhang's uninorms on completed lattice. In Section 5, we make a conclusion.

2. Preliminaries

By a lattice L we mean a quadruple $(L, \leq, \wedge, \vee)$, where $(L, \leq)$ is a partial order set and for any $x, y \in L$, the meet $x \wedge y$ and the join $x \vee y$ always exist. A lattice L is called bounded if it has a top element 1 and a bottom element 0. In addition, a lattice L is called complete if for each $A \subseteq L$, the meet $\wedge A$ and the join $\vee A$ always exist. A complete lattice is naturally a bounded lattice.

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In this paper, we always assume L is a bounded lattice unless otherwise stated.

For $a,b \in L$ with $a \le b$, we define the subintervals $[a,b],[a,b),(a,b)$ as

$[a,b]=\{x \in L \mid a \le x \le b\},[a,b)=\{x \in L \mid a \le x< b\},$

$(a,b]=\{x \in L \mid a< x \le b\},(a,b)=\{x \in L \mid a< x< b\}.$

Definition 2.1. [2] Let $[a,b]$ be a subinterval of L.

(1)A function $T:[a,b]\times[a,b]\to[a,b]$ is called a t-norm on $[a,b]$ if T is commutative, monotonic, associative and has neutral element b. When $[a,b]=[0,1]$, i.e., $[a,b]=L$, a t-norm on $[a,b]$ is called a t-norm on L.

(2) A function $S:[a,b]\times[a,b]\to[a,b]$ is called a t-conorm on $[a,b]$ if S is commutative, monotonic, associative and has neutral element a. When $[a,b]=[0,1]$, i.e., $[a,b]=L$, a t-conorm on $[a,b]$ is called a t-conorm on L.

Definition 2.2. [7] For $e \in L$, a function $U:L\times L \to L$ is called a uninorm on L if U is commutative, monotonic, associative and has neutral element e.

Definition 2.3. [5] (1) A function $C:L\to L$ is called a closure operator on L if for any $x,y \in L$: $(C1)\ x \le C(x)$; $(C2)\ C(x \vee y)= C(x)\vee C(y)$; $(C3)\ CC(x)=C(x)$.

(2) A function $I:L\to L$ is called a interior operator on L if for any $x,y \in L$:

$(I1)\ x \ge I(x)$; $(I2)\ I(x \wedge y)= I(x)\wedge I(y)$; $(I3)\ II(x)=I(x)$.

The following proposition shows that each closure operator (resp., interior operator) can generate a t-conorm (resp., t- norm).

Proposition 2.4. (1) ([9] for $e=0$) For a closure operator $C:L\to L$ and $e \in L$, the function $S_C:[e,1]\times[e,1]\to[e,1]$ determined by

$$S_C(x,y)=\begin{cases} x\vee y, & e\in\{x,y\};\\ C(x)\vee C(y), & \text{otherwise.}\end{cases}$$

is a t-conorm on $[e,1]$.

(3) ([9] for $e=1$) For a interior operator $I:L\to L$ and $e\in L$, the function $T_I:[0,e]\times[0,e]\to[0,e]$ determined by $T_I(x,y)=\begin{cases} x\wedge y, & e\in\{x,y\};\\ I(x)\vee I(y), & \text{otherwise.}\end{cases}$ is a t-norm on $[0,e]$.

Ouyang-Zhang shown that one can construct uninorms via closure and interior operators.

Theorem 2.5. [9] Let $C:L\to L$ be a closure operator, $e\in L$ and $T:[0,e]\times[0,e]\to[0,e]$ be a t-norm. Then the function $U_{C,T}:L\times L\to L$ determined by

$$U_{C,T}(x,y)=\begin{cases} T(x,y), & x,y\in[0,e];\\ y, & x\in[0,e], y\notin[0,e];\\ x, & x\notin[0,e], y\in[0,e];\\ C(x)\vee C(y), & \text{otherwise.}\end{cases}$$

is a uninorm on L has neutral element e.

Theorem 2.6. [9] Let $I:L\to L$ be a interior operator, $e\in L$ and $S:[e,1]\times[e,1]\to[e,1]$ be a t-conorm. Then the function $U_{I,S}:L\times L\to L$ determined by

$$U_{I,S}(x,y)=\begin{cases} S(x,y), & x,y\in[e,1];\\ y, & x\in[e,1], y\notin[e,1];\\ x, & x\notin[e,1], y\in[e,1];\\ I(x)\wedge I(y), & \text{otherwise.}\end{cases}$$

is a uninorm on L has neutral element e.

In [9], Ouyang-Zhang introduced the notion of universally comparable element in L. For

$x, y \in L$, we say x is comparable with y if $x \le y$ or $y \le x$, otherwise we say x is not comparable with y and denoted it as $x||y$. If x is comparable with any $y \in L$, then we say x is an universally comparable element in L. The set of all universally comparable elements in L is denoted by $UC(L) := \{x \in L \mid x \text{ is comparable with } y \in L\}$. When L is a complete lattice, Ouyang-Zhang defined new (interior) closure operator via $UC(L)$.

Definition 2.7. ([9]) Let L be a completed lattice. The functions $\Uparrow, \Downarrow : L \to L$ defined by

$$\forall x \in L, \Uparrow x = \wedge\{a \in UC(L) \mid a \ge x\}, \Downarrow x = \vee\{a \in UC(L) \mid a \le x\}$$

is a closure operator and interior operation on L, respectively.

Obviously, the operators $\Uparrow, \Downarrow$ may be not well defined if L is not complete. So it is a natural requirement to extend them from complete lattice to non-complete bounded lattice.

3. The closure (interior) operators on a bounded lattice via $UC(L)$

In this section, we shall extend Ouyang-Zhang's closure (interior) operator by two approaches s.t. they can be defined for any bounded lattice.

Proposition 3.1. For any $A \subseteq UC(L)$, if $\wedge A$, $\vee A$ exist, then $\wedge A, \vee A \in UC(L)$.

Proof: Let $A \subseteq UC(L)$.

(1) Let $\wedge A$ exist. Take $a := \wedge A$ and $b \in L$, we check that a is comparable with b. Obviously, if $A = \varnothing$, then $1 = \wedge A \in UC(L)$. So we assume A is nonempty.

Case 1: There exists $x \in A$, s.t. $x \le b$. Then $a \le x \le b$, and so $a \le b$.

Case 2: There exists no $x \in A$, s.t. $x \le b$. Then $x > b$ for any $x \in A$ since x is comparable with b, and so $a = \wedge A \ge b$.

(2) Let $\vee A$ exist. Take $a := \vee A$ and $b \in L$, we check that a is comparable with b. Obviously, if $A = \varnothing$, then $0 = \wedge A \in UC(L)$. We assume that A is nonempty.

Case 1: There exists $x \in A$, s.t. $x \ge b$. Then we have $a \ge x \ge b$, and so $a \ge b$.

Case 2: There exists no $x \in A$, s.t. $x \ge b$. Then $x < b$ for any $x \in A$ since x is comparable with b, and so $a = \vee A \le b$.

Hence $a = \wedge A \in UC(L)$ and $a = \vee A \in UC(L)$.

Corollary 3.2. [9] If L is complete, then so is $UC(L)$ with the inherited order from L.

Proposition.3.3. Let $x, y \in L$.

(1) If $x \le y$, then $\{a \in UC(L) \mid a \ge x\} \supseteq \{a \in UC(L) \mid a \ge y\}$.

(2) If $x \,||\, y$, then

$\{a \in UC(L) \mid a \ge x\} = \{a \in UC(L) \mid a > x\} = \{a \in UC(L) \mid a > y\} = \{a \in UC(L) \mid a \ge y\}$.

(3) If $x \le y$, then $\{a \in UC(L) \mid a \le x\} \subseteq \{a \in UC(L) \mid a \le y\}$.

(4) If $x \parallel y$, then

$\{a \in UC(L) \mid a \le x\} = \{a \in UC(L) \mid a < x\} = \{a \in UC(L) \mid a < y\} = \{a \in UC(L) \mid a \le y\}$.

Proof: (1) and (3) are obvious.

(2) Suppose that $x||y$. Then for any $a \in UC(L)$ with $a \ge x$, we have $a > x$ (otherwise $a = x$ means that x is comparable with y). By a is comparable with y we have $a \le y$ or $a > y$. But if $a \le y$, then together with $x < a$ we have $x < y$, which contradicts with $x \parallel y$. So, $a > y$. Now we have proved that

$$\{a \in UC(L) \mid a \ge x\} = \{a \in UC(L) \mid a > x\} \subseteq \{a \in UC(L) \mid a > y\}.$$

Similarly, we can prove that

$$\{a \in UC(L) \mid a \ge y\} = \{a \in UC(L) \mid a > y\} \subseteq \{a \in UC(L) \mid a > x\}.$$

A combination of the above two inequality, we get the desired equality.

(4) Suppose that $x||y$. Then for any $a \in UC(L)$ with $a \le x$, we have $a < x$ (otherwise $a = x$ means that x is comparable with y). By a is comparable with y we have $a \ge y$ or $a < y$. But if $a \ge y$, then together with $x > a$ we have $x > y$, which contradicts with $x \,||\, y$. So, $a < y$. Now we have proved that

$$\{a \in UC(L) \mid a \le x\} = \{a \in UC(L) \mid a < x\} \subseteq \{a \in UC(L) \mid a < y\}.$$

Similarly, we can prove that

$$\{a \in UC(L) \mid a \le y\} = \{a \in UC(L) \mid a < y\} \subseteq \{a \in UC(L) \mid a < x\}.$$

A combination of the above two inequality, we get the desired equality.

In the following, we discuss the closure operator on a bounded lattice via $UC(L)$.

Definition 3.4. A function $\tilde{\Uparrow}: L \to L$ is defined as: $\forall x \in L$,

$$\tilde{\Uparrow} x = \begin{cases} \wedge\{a \in UC(L) \mid a \geq x\}, & \wedge\{a \in UC(L) \mid a \geq x\} \text{ exists;} \\ x, & \text{otherwise.} \end{cases}$$

Remark 3.5. When L is complete (particularly, L is finite), then $\wedge A$, $\vee A$ always exist for any $A \subseteq UC(L)$. In this case, the function $\tilde{\Uparrow} = \Uparrow$.

Proposition 3.6. The function $\tilde{\Uparrow}: L \to L$ has the following properties.

(1) For $x \in L, x \leq \tilde{\Uparrow} x$. (2) For $x \in L$, if $x \in UC(L)$, then $\tilde{\Uparrow} x = x$.

(3) For $x \in L$, if $\wedge\{a \in UC(L) \mid a \geq x\}$ exists, then $\tilde{\Uparrow} x \in UC(L)$.

(4) For $x \in L$, $\tilde{\Uparrow}\tilde{\Uparrow} x = \tilde{\Uparrow} x$. (5) If $x \leq y$ implies $\tilde{\Uparrow} x \leq \tilde{\Uparrow} y$.

Proof. (1) and (2) are obvious, and (3) follows by Proposition 3.1.
(4)Let $x \in L$, we divided into two cases .

Case 1: $\wedge\{a \in UC(L) \mid a \geq x\}$ exists. Then it follows by (2), (3) that $\tilde{\Uparrow}\tilde{\Uparrow} x = \tilde{\Uparrow} x$.

Case 2: $\wedge\{a \in UC(L) \mid a \geq x\}$ not exists. Then $\tilde{\Uparrow} x = x$ and so $\tilde{\Uparrow}\tilde{\Uparrow} x = \tilde{\Uparrow} x$.

(1) Let $x \leq y$, we divided into four cases to prove that $\tilde{\Uparrow} x \leq \tilde{\Uparrow} y$.

Case 1: $\wedge\{a \in UC(L) \mid a \geq x\}$ and $\wedge\{a \in UC(L) \mid a \geq y\}$ exist. It follows by Proposition 3.3 (1) that $\tilde{\Uparrow} x = \wedge\{a \in UC(L) \mid a \geq x\} \leq \wedge\{a \in UC(L) \mid a \geq y\} = \tilde{\Uparrow} y$.

Case 2: $\wedge\{a \in UC(L) \mid a \geq x\}$ exists and $\wedge\{a \in UC(L) \mid a \geq y\}$ not exist. It follows by (3) $\tilde{\Uparrow} x \in UC(L)$, so $\tilde{\Uparrow} x \leq y$ or $\tilde{\Uparrow} x \geq y$.

Let $\tilde{\Uparrow} x = \wedge\{a \in UC(L) \mid a \geq x\}$, then $\{a \in UC(L) \mid a \geq x\} \subseteq \{a \in UC(L) \mid a \geq y\}$. Then by Proposition 3.3(1) $\{a \in UC(L) \mid a \geq x\} = \{a \in UC(L) \mid a \geq y\}$, which implies that $\wedge\{a \in UC(L) \mid a \geq y\}$ exists, a contradiction. Hence $\tilde{\Uparrow} x \leq y = \tilde{\Uparrow} y$

because $\wedge\{a \in UC(L) \mid a \geq y\}$ not exists.

Case 3: $\wedge\{a \in UC(L) \mid a \geq x\}$ not exists and $\wedge\{a \in UC(L) \mid a \geq y\}$ exists. Then $\tilde{\Uparrow} x = x \leq y \leq \tilde{\Uparrow} y$.

Case 4: $\wedge\{a \in UC(L) \mid a \geq x\}$ and $\wedge\{a \in UC(L) \mid a \geq y\}$ not exist. Then $\tilde{\Uparrow} x = x \leq y = \tilde{\Uparrow} y$.

Theorem 3.7. The function $\tilde{\Uparrow} L \to L$ is a closure operator.

Proof: By Proposition 3.6 (1) (4), we only prove $\forall x, y \in L$, $\tilde{\Uparrow} x \vee \tilde{\Uparrow} y = \tilde{\Uparrow}(x \vee y)$.

(1) x is comparable with y. It follows by the order-preserving of $\tilde{\Uparrow}$ that

$$\tilde{\Uparrow} x \vee \tilde{\Uparrow} y = \tilde{\Uparrow} y = \tilde{\Uparrow}(x \vee y) \text{ or } \tilde{\Uparrow} x \vee \tilde{\Uparrow} y = \tilde{\Uparrow} x = \tilde{\Uparrow}(x \vee y)$$

(2) $x \,||\, y$. We divide it into three cases.

Case 1: $\wedge\{a \in UC(L) \mid a \geq x\}$ exists. Then $\tilde{\Uparrow} x \in UC(L)$ is comparable with y, so $\tilde{\Uparrow} x \geq y$ (otherwise, we have $x \leq \tilde{\Uparrow} x < y$, which contradicts with $x \parallel y$). It follows that $\tilde{\Uparrow} x \geq x \vee y$, and so $\tilde{\Uparrow} x = \tilde{\Uparrow}\tilde{\Uparrow} x \geq \tilde{\Uparrow}(x \vee y)$, that means $\tilde{\Uparrow} x \vee \tilde{\Uparrow} y \geq \tilde{\Uparrow}(x \vee y)$. Since $\tilde{\Uparrow}(x \vee y) \geq \tilde{\Uparrow} x \vee \tilde{\Uparrow} y$ holds, then $\tilde{\Uparrow}(x \vee y) = \tilde{\Uparrow} x \vee \tilde{\Uparrow} y$.

Case 2: $\wedge\{a \in UC(L) \mid a \geq y\}$ exists. Similarly, $\tilde{\Uparrow}(x \vee y) = \tilde{\Uparrow} x \vee \tilde{\Uparrow} y$.

Case 3: $\wedge\{a \in UC(L) \mid a \geq x\}$ and $\wedge\{a \in UC(L) \mid a \geq y\}$ not exist. From $x \parallel y$ and Proposition 3.3(2) we obtain

$$\{a \in UC(L) \mid a \geq x\} = \{a \in UC(L) \mid a \geq y\}.$$

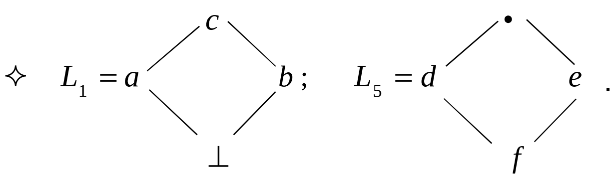
It follows by

$$\{a \in UC(L) \mid a \geq x \vee y\} = \{a \in UC(L) \mid a \geq x\} \cap \{a \in UC(L) \mid a \geq y\}$$

That $\{a \in UC(L) \mid a \geq x \vee y\} = \{a \in UC(L) \mid a \geq x\} = \{a \in UC(L) \mid a \geq y\}$. So $\wedge\{a \in UC(L) \mid a \geq x \vee y\}$ not exists. Hence $\tilde{\Uparrow} x \vee \tilde{\Uparrow} y = x \vee y = \tilde{\Uparrow}(x \vee y)$.

A combination of the above (1)-(3), we show that $x, y \in L, \tilde{\Uparrow} x \vee \tilde{\Uparrow} y = \tilde{\Uparrow}(x \vee y)$.

Example 3.8. Let $L = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5$ and



✧ $L_2 = [0,1)$ with the usual order; $L_4 = (2,3]$ with the usual order;

✧ $L_3 = \{m + n \mid m, n \in (1,2)\}$ with $m_1 + n_1 i \leq m_2 + n_2 i$ iff $m_1 \leq m_2, n_1 \leq n_2$;

✧ $x < y$ for any $x \in L_j, y \in L_{j+1}, j = 1,2,3,4$.

It is easily seen that L is bounded but not complete (for example $\vee(1,2)$ and $\wedge(1,2)$ not exist), and $UC(L) = \{\perp, C\} \cup [0,1) \cup (2,3] \cup \{f, \bullet\}$. Furthermore:

(1)for $x \in L_1, \wedge\{y \in UC(L) \mid y \geq x\}$ exists, and $\tilde{\Uparrow} x = c$ if $x \in \{a, b\}$ and $\tilde{\Uparrow} x = x$ otherwise,

(2) for $x \in L_2, \wedge\{y \in UC(L) \mid y \geq x\}$ exists, and $\tilde{\Uparrow} x = x$,

(3) for $x \in L_3, \wedge\{y \in UC(L) \mid y \geq x\}$ not exists, and so $\tilde{\Uparrow} x = x$,

(4) for $x \in L_4, \wedge\{y \in UC(L) \mid y \geq x\}$ exists, and $\tilde{\Uparrow} x = x$,

(5) for $x \in L_5, \wedge\{y \in UC(L) \mid y \geq x\}$ exists, and $\tilde{\Uparrow} x = \bullet$ if $x \in \{d, e\}$ and $\tilde{\Uparrow} x = x$ otherwise.

Hence $\tilde{\Uparrow} x = \begin{cases} c, & x \in \{a,b\}; \\ \bullet, & x \in \{d,e\}; \\ x, & \text{otherwise.} \end{cases}$

Similar to closure operator, we can extend Ouyang-Zhang's interior operator $\Downarrow$ s.t. it can be defined for any bounded lattice.

Definition 3.9. A function $\tilde{\Downarrow}: L \to L$ is defined as: $\forall x \in L$,

$$\tilde{\Downarrow} x = \begin{cases} \vee\{a \in UC(L) \mid a \le x\}, & \vee\{a \in UC(L) \mid a \le x\} \text{ exists;} \\ x, & \text{otherwise.} \end{cases}$$

Theorem 3.10. The function $\tilde{\Downarrow} L \to L$ is a interior operator.

Proof: It is similar to Proposition 3.7.

4. Uninorms constructed by the closure (interior) operator via $UC(L)$

In this section, we shall extend Ouyang-Zhang's constructions for uninorms via $UC(L)$ from complete lattice to arbitrary bounded lattice.

From Theorem 2.6 and Theorem 2.7 we know that we can construct uninorms from closure (interior) operators compatible with $UC(L)$. Indeed, we may construct at most twelve uninorms as the following three theorems show.

Theorem 4.1. Let $T : [0,e] \times [0,e] \to [0,e]$ be a t-norm and $e \in L$. Then the function $U_{\tilde{\Uparrow},T} : L \times L \to L$ determined by

$$U_{\tilde{\Uparrow},T}(x,y) = \begin{cases} T(x,y), & x,y \in [0,e]; \\ y, & x \in [0,e], y \notin [0,e]; \\ x, & x \notin [0,e], y \in [0,e]; \\ \tilde{\Uparrow} x \vee \tilde{\Uparrow} y, & \text{otherwise.} \end{cases}$$

is a uninorm on L has neutral element e.

Theorem 4.2. Let $S : [e,1] \times [e,1] \to [e,1]$ be a t-conorm and $e \in L$. Then the function

$U_{\tilde{\Downarrow},S}: L \times L \to L$ defined by $U_{\tilde{\Downarrow},S}(x,y) = \begin{cases} S(x,y), & x,y \in [e,1]; \\ y, & x \in [e,1], y \notin [e,1]; \\ x, & x \notin [e,1], y \in [e,1]; \\ \tilde{\Downarrow} x \wedge \tilde{\Downarrow} y, & \text{otherwise}. \end{cases}$

is a uninorm on L has neutral element e.

From Proposition 2.4 (1) and Theorem 4.1 (resp., Proposition 2.4 (2) and Theorem 4.2), we obtain the following corollary.

Corollary 4.3. Let $e \in L$

(1) The function $U_{\tilde{\Uparrow},T_{\tilde{\Downarrow}}}: L \times L \to L$ determined by

$$U_{\tilde{\Uparrow},T_{\tilde{\Downarrow}}}(x,y) = \begin{cases} \tilde{\Downarrow} x \wedge \tilde{\Downarrow} y, & x,y \in [0,e); \\ y, & x = e \text{ and } y \in L, \text{or } x \in [0,e) \text{ and } y \notin [0,e]; \\ x, & y = e \text{ and } x \in L, \text{or } y \in [0,e) \text{ and } x \notin [0,e]; \\ \tilde{\Uparrow} x \vee \tilde{\Uparrow} y, & \text{otherwise}. \end{cases}$$

is a uninorm on L has neutral element e.

(2) The function $U_{\tilde{\Downarrow},S_{\tilde{\Uparrow}}}: L \times L \to L$ determined by

$$U_{\tilde{\Downarrow},S_{\tilde{\Uparrow}}}(x,y) = \begin{cases} \tilde{\Uparrow} x \vee \tilde{\Uparrow} y, & x,y \in (e,1]; \\ y, & x = e \text{ and } y \in L, \text{or } x \in (e,1] \text{ and } y \notin [e,1]; \\ x, & y = e \text{ and } x \in L, \text{or } y \in (e,1] \text{ and } x \notin [e,1]; \\ \tilde{\Downarrow} x \wedge \tilde{\Downarrow} y, & \text{otherwise}. \end{cases}$$

is a uninorm on L has neutral element e.

Remark 4.4. If L is complete, then $U_{\tilde{\Uparrow},T} = U_{\Uparrow,T}$, $U_{\tilde{\Downarrow},S} = U_{\Downarrow,S}$, $U_{\tilde{\Uparrow},T_{\tilde{\Downarrow}}} = U_{\Uparrow,T_{\Downarrow}}$ and $U_{\tilde{\Downarrow},S_{\tilde{\Uparrow}}} = U_{\Downarrow,S_{\Uparrow}}$. In this case, Theorem 4.1, 4.2, and Corollary 4.3 (1), (2) degenerate as Theorem 4.7, 5.6, 5.8 and 5.10 in [9], respectively.

5. Concluding remarks

In this paper, on a bounded lattice L, we defined a closure operator $\tilde{\Uparrow}$ and an interior operator $\tilde{\Downarrow}$, and discussed the uninorms constructed by them. We verified that when L is complete, it holds that $\tilde{\Uparrow}=\Uparrow$ and $\tilde{\Downarrow}=\Downarrow$. Hence our study generalizes and enriches Ouyang-Zhang's work on complete (bounded) lattice in [9].

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