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Enriched Fuzzy σ-Algebra and Fuzzy Rough Approximation Operators

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Abstract. In this paper, we introduce the concept of enriched fuzzy σ -algebra and establish its relation to fuzzy rough approximation operators. Firstly, the concept of enriched fuzzy σ -algebra is proposed as an extension of existing fuzzy σ -algebras, and several concrete examples are provided. Particularly, it has been proven that each σ -algebra can naturally generate an enriched fuzzy σ -algebra. Secondly, the methods for constructing enriched fuzzy σ -algebra using reflexive fuzzy rough approximation operators and serial rough fuzzy approximation operators are given, respectively. Finally, it is shown that enriched fuzzy σ -algebras generated by σ -algebras must be induced by some rough fuzzy approximation operator.

Keywords: Fuzzy σ -algebra; Fuzzy rough approximation operator; Rough set

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1. Introduction

Rough set theory was first proposed by Pawlak [8] as a formal tool for modeling and processing incomplete information. By utilizing the concept of upper and lower approximation in rough set theory,it is possible to unveil the knowledge concealed within an information system can be revealed in the form of decision rules [9,10]. The concept of σ -algebra [3], which underlies probability theory (especially measurable spaces [16]) and analytical mathematics such as Lebesgue integrals, can be interpreted as a collection of events, and probabilities can be assigned to each event. Pawlak was the earliest scholar to study the relationship between σ -algebra and rough set.He proved that the whole of the definable sets in Pawlak approximation space from an σ -algebra [9].

The combination of fuzzy sets and rough sets produces variously generalized rough sets. The common examples include rough fuzzy sets [1], $(\c{S},\check{T})$ -fuzzy rough sets [14] (where $\check{T}$ and $\c{S}$ represent t -norms and t -conorms , respectively) and more general L -fuzzy rough sets [4,11,18,15], etc. Wu [12] introduced the concept of fuzzy σ -algebra, and proved that the definable sets of rough fuzzy sets and $(\c{S},\check{T})$ -fuzzy rough sets both constitute fuzzy σ -algebra, respectively. It should be noted that Wu's definition of fuzzy σ -algebra only involves the min-max operation on [0,1], and does not touch more general t -norm and t -conorm operations. The original definition is too weak to match the much enriched structure of t -norm and t -conorm. Therefore, the main aim of this paper is to enrich the concept of fuzzy σ -algebra and discuss its relationship to rough fuzzy sets and $(\c{S},\check{T})$ -fuzzy rough sets.

The contents of this paper are arranged as follows. In section 2, we will review some concepts and symbols, including t -(co)norm, rough fuzzy sets and $(\c{S},\check{T})$ -fuzzy rough sets. In section 3, we introduce the concept of enriched fuzzy σ -algebra, give some examples, and establish the relationship between fuzzy σ -algebra and rough fuzzy sets and $(\c{S},\check{T})$ -fuzzy rough sets. Finally, we will summarize the full text and list some future work.

2. Preliminaries

This section introduces some concepts and symbols used in this paper, including t -(co)norm, rough fuzzy sets, and $(\c{S},\check{T})$ -fuzzy rough sets.

2.1 t -(co)norm and fuzzy sets

In this paper, we always use $\check{T},\c{S}:[0,1]\times[0,1]\to[0,1]$ to represent a continuous t -norm and t -conorms, respectively [7]. The most famous t -norms include: $\forall\alpha,\beta\in[0,1]$

(1) The Standard Min Operator $\alpha\check{T}_M\beta=\min\{\alpha,\beta\}$;

(2) The Algebraic Product $\alpha\check{T}_P\beta=\alpha*\beta$;

(3) The Lukasiewicz t -norm $\alpha\check{T}_L\beta=\max\{0,\alpha+\beta-1\}$;

The most important t -conorms include:

(1) The Standard Max Operator $\c{S}_M(\alpha,\beta)=\max\{\alpha,\beta\}$;

(2) The Probabilistic Sum $\c{S}_P(\alpha,\beta)=\alpha+\beta-\alpha*\beta$;

(3) The Bounded Sum $\c{S}_L(\alpha,\beta)=\min\{1,\alpha+\beta\}$;

A decreasing mapping $\sim:[0,1]\to[0,1]$ is referred to be an involutive negation when $\sim 1=0$, $\sim 0=1$ and $\forall\alpha\in[0,1]$, $\sim(\sim(\alpha))=\alpha$.

In this paper, we also assume that $\check{T}$ and $\c{S}$ are dual, i.e., $\forall a,b\in[0,1]$,

$$\sim(a\check{T}b)=\sim a\c{S}\sim b,\ \sim(a\c{S}b)=\sim a\check{T}\sim b.$$

Let U be a nonempty finite set, all subsets in U are denoted as $P(U)$. By a fuzzy set in U we mean a mapping $A:U\to[0,1]$. The family of all fuzzy sets in U is denoted as $F(U)$. For a crisp set $X\subseteq U$, we also use X to denote its characteristic function. For $\alpha\in[0,1]$, we use $\hat{\alpha}$ to denote the constant vale fuzzy set valued α.

For $A,B\in F(U)$, we define some operations of fuzzy sets as follows: for any $x\in U$,

(1) $A^{\sim}(x) = \sim(A(x))$; (2) $(A\check{T}B)(x) = A(x)\check{T}B(x)$; (3) $(AŞB)(x) = A(x)ŞB(x)$.

2.2. Rough fuzzy set and（Ş,Ť）-fuzzy rough set

Definition 2.1. [13] Let $(U,\acute{R})$ be a generalized approximation space (GAPS), i.e. $\acute{R}$ is a binary relation on U. For $A \in F(U)$, the lower approximation $\underline{\acute{R}F}(A)$ and upper approximation $\overline{\acute{R}F}(A)$ are defined as follows: $\forall x \in U$

$$\underline{\acute{R}F}(A)(x) = \wedge_{y\in \acute{R}_s(x)} A(y),\ \overline{\acute{R}F}(A)(x) = \vee_{y\in \acute{R}_s(x)} A(y),$$

where $\acute{R}_s(x) = \{y \in U \mid (x,y) \in \acute{R}\}$. The pair $(\overline{\acute{R}F}(A), \underline{\acute{R}F}(A))$ is called the generalized rough fuzzy set of A, and the operators $\underline{\acute{R}F}$, $\overline{\acute{R}F} : F(U) \to F(U)$ are called generalized rough fuzzy lower and upper approximation operators, respectively.

Theorem 2.1. [13] Let $(U,\acute{R})$ be a GAPS. Then: $\forall A, B, A_j (j \in J) \in F(U)$, $\forall \alpha \in [0,1]$,

(RFLD) $\underline{\acute{R}F}(A) = \sim \overline{\acute{R}F}(\sim A)$, (RFUD) $\overline{\acute{R}F}(A) = \sim \underline{\acute{R}F}(\sim A)$;

(RFLI) $\underline{\acute{R}F}(A \cup \hat{\alpha}) = \underline{\acute{R}F}(A) \cup \hat{\alpha}\,\theta$, (RFU1) $\overline{\acute{R}F}(A \cap \hat{\alpha}) = \overline{\acute{R}F}(A) \cap \hat{\alpha}$;

(RFL2) $\underline{\acute{R}F}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \underline{\acute{R}F}(A_j)$, (FRU2) $\overline{\acute{R}F}(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} \overline{\acute{R}F}(A_j)$;

(RFL3) $A \subseteq B \Rightarrow \underline{\acute{R}F}(A) \subseteq \underline{\acute{R}F}(B)$, (FRU3) $A \subseteq B \Rightarrow \overline{\acute{R}F}(A) \subseteq \overline{\acute{R}F}(B)$.

Moreover, we have that $\acute{R}$ is serial ($\forall x \in U$, $\acute{R}_s(x) \neq \varnothing$.) iff one of the following holds:

(RFL0) $\underline{\acute{R}F}(\hat{\alpha}) = \hat{\alpha}$, $\forall \alpha \in [0,1]$; (RFU0) $\overline{\acute{R}F}(\hat{\alpha}) = \hat{\alpha}$, $\forall \alpha \in [0,1]$;

(RFLU0) $\underline{\acute{R}F}(A) \subseteq \overline{\acute{R}F}(A), \forall A \in F(U)$.

Definition 2.2. [14] Let $(U, Я)$ is a fuzzy approximation space (FAPS), i.e., $Я$ is a fuzzy relation on U. For $A \in F(U)$, the Ş -lower approximation $\underline{ŞЯ}(A)$ and Ť -upper approximation $\overline{\check{T}Я}(A)$ are defined as follows: $\forall x \in U$

$$\underline{ŞЯ}(A)(x) = \wedge_{y\in U}(\sim Я(x,y)ŞA(y));\ \overline{\check{T}Я}(A)(x) = \vee_{y\in U}(Я(x,y)\check{T}A(y)),$$

The pair $(\underline{Я}(A), \overline{Я}(A))$ is called (Ş,Ť) -fuzzy rough set of A, $\underline{ŞЯ}$ (resp., $\overline{\check{T}Я}$) is called Ş - fuzzy rough lower (resp., Ť -fuzzy rough upper) approximation operator.

Theorem 2.2. [14] Let $(U, Я)$ be a FAPS. Then $\forall A, B, A_j (j \in J) \in F(U)$,

(FL1) $\underline{ŞЯ}(A) = \sim \overline{\check{T}Я}(\sim A)$, (FU1) $\overline{\check{T}Я}(A) = \sim \underline{ŞЯ}(\sim A)$;

(FL2) $\underline{ŞЯ}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \underline{ŞЯ}(A_j)$, (FU2) $\overline{\check{T}Я}(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} \overline{\check{T}Я}(A_j)$;

(FL3) $A \subseteq B \Rightarrow \underline{Ş Я}(A) \subseteq \underline{Ş Я}(B)$, (FU3) $A \subseteq B \Rightarrow \overline{Ť Я}(A) \subseteq \overline{Ť Я}(B)$.

Moreover, we have $Я$ is a reflexive ($\forall x \in U, Я(x,x)=1$.) iff one of the following holds:

(FLR) $\underline{Ş Я}(A) \subseteq A, \forall A \in F(U)$; (FUR) $A \subseteq \overline{Ť Я}(A), \forall A \in F(U)$.

3. Enriched fuzzy σ-algebra

In this section, we will investigate a novel fuzzy σ-algebra, called enriched fuzzy σ-algebra, and discuss the relationships between it and rough fuzzy sets and $(Ş,Ť)$-fuzzy rough sets.

Definition 3.1. [3] A family $Є \subseteq P(U)$ is called an σ-algebra on U whenever:

(1) $U \in Є$; (2) $\{X_n \mid n \in N\} \subseteq Є \Rightarrow \bigcup_{n \in N} X_n \in Є$; (3) $X \in Є \Rightarrow \sim X \in Є$.

Where the set in $Є$ is called a measurable set, and the pair $(U, Є)$ is a measurable space.

Definition 3.2. [6] A family $\Im \subseteq F(U)$ is called a fuzzy σ-algebra on U provided:

(1) $\hat{\alpha} \in \Im$, $\forall \alpha \in [0,1]$; (2) $\{A_n | n \in N\} \subseteq \Im \Rightarrow \bigcup_{n \in N} A_n \in \Im$; (3) $A \in \Im \Rightarrow \sim A \in \Im$.

Each member of $\Im$ is called a fuzzy measurable set, and the pair $(U, \Im)$ is a fuzzy measurable space.

As we mentioned earlier, the definition of fuzzy σ-algebra only involves the min-max operation, and does not involve the more general t-norm and t-conomn operation. Therefore, it cannot be matched with $(Ş,Ť)$-fuzzy rough set. So, we introduce the following enriched concept.

Definition 3.3. A family $\Im \subseteq F(U)$ is called an enriched fuzzy σ-algebra on U whenever:

(1) $U \in \Im$; (2) $A \in \Im$, $\alpha \in [0,1] \Rightarrow \hat{\alpha} Ť A \in \Im$,

(3) $\{A_n | n \in N\} \subseteq \Im \Rightarrow \bigcup_{n \in N} A_n \in \Im$; (4) $A \in \Im \Rightarrow \sim A \in \Im$.

Each member of $\Im$ is called a fuzzy measurable set, and the pair $(U, \Im)$ is an enriched fuzzy measurable space.

Remark 3.1. In Definition 3.1, for any $\alpha \in [0,1]$, from $U \in \Im$ and (2) we know that $\hat{\alpha} = \hat{\alpha} Ť U \in \Im$, that is, each enriched fuzzy σ-algebra is a fuzzy σ-algebra in [11].

Example 3.1. (1) All fuzzy sets $F(U)$ forms the largest enriched fuzzy σ-algebra on U.
(2) All constant fuzzy sets form the minimum enriched fuzzy σ-algebra on U.
Next, we show that each σ-algebra can generate an enriched fuzzy σ-algebra.

Lemma 3.1. Let $\Im_i\left(i\in I\right)$ be a family of enriched fuzzy σ-algebra on U, then $\bigcap_{i\in I}\Im_i$ is also an enriched fuzzy σ-algebra.

Proof: ① $U\in\Im_i\Rightarrow U\in\bigcap_{i\in I}\Im_i$,

② $\forall A\in\bigcap_{i\in I}\Im_i$, $\alpha\in[0,1]$, due to $\hat{\alpha}\check{T}A\in\Im_i$, therefore $\hat{\alpha}\check{T}A\in\bigcap_{i\in I}\Im_i$,

③ $\left\{A_n\middle|n\in N\right\}\subseteq\bigcap_{i\in I}\Im_i$, due to $\bigcup_{n\in N}A_n\in\Im_i$, therefore $\bigcup_{n\in N}A_n\in\bigcap_{i\in I}\Im_i$,

④ $\forall A\in\bigcap_{i\in I}\Im_i$,so $A\in\Im_i$ for $\sim A\in\Im_i$, therefore $\sim A\in\bigcap_{i\in I}\Im_i$.

That is to say, $\bigcap_{i\in I}\Im_i$ is an enriched fuzzy σ-algebra.

Lemma 3.2. Let $\Subset$ be a σ-algebra over U, $\forall x\in U$, denoted $[x]=\cap\left\{C\middle|x\in C\in\Subset\right\}$, then $\left\{[x]\middle|x\in U\right\}$ forms a partition of U.

Proof: Noting that $\forall x\in U$, $x\in[x]$ and $\cup\left\{[x]\middle|x\in U\right\}=U$, so we need only to check that when $[x]\cap[y]\neq\varnothing$, then $[x]=[y]$. Indeed, if we suppose $[x]\neq[y]$ then $x\notin[y]$ or $y\notin[x]$. Without loss of generality, we assume that $x\notin[y]$, then $x\in[y]'$, thus $[x]\subseteq[y]'$, hence $[x]\cap[y]=\varnothing$, a contradiction! Therefore $[x]=[y]$.

As we all know, there is a one-to-one correspondence between partition and equivalent relation on U. Therefore, σ-algebra $\Subset$ (through the above partition) also determines an equivalent relation, which is denoted as $\acute{R}_{\Subset}$. For any $x\in U$, $[x]$ is precise its equivalent class.

Definition 3.4. Let $\Subset$ be a σ-algebra over U and define

$$\omega(\Subset)=\left\{\Im|\Im\text{ is a fuzzy }\sigma\text{-algebra and }\Im\supseteq\{\hat{\alpha},C_i\}\right\},$$

where $\alpha\in[0,1]$, C_i are the equivalent classes in $\acute{R}_{\Subset}$. By Lemma 3.1, we know that $\omega(\Subset)$ is the smallest enriched fuzzy σ-algebra containing $\{\hat{\alpha},C_i\}$, which is called the fuzzy σ-algebra generated by $\Subset$.

The following theorem shows that each serial rough fuzzy approximation spaces can induce an enriched fuzzy σ-algebra.

Theorem 3.1. Let $\left(U,\acute{R}\right)$ be a serial GAPS, denote

$$\Im_{\acute{R}}=\left\{A\in F\left(U\right)|\underline{\acute{R}F}\left(A\right)=A=\overline{\acute{R}F}\left(A\right)\right\},$$

Then $\Im_{\acute{R}}$ forms an enriched fuzzy σ-algebra over U.

Proof: (1) According to the definition of rough fuzzy approximation, we get $\forall x\in U$,

$$\underline{\acute{R}F}(U)(x)=\bigwedge_{y\in\acute{R}_s(x)}U(y)=U(x)=\bigvee_{y\in\acute{R}_s(x)}U(y)=\overline{\acute{R}F}(U)(x),$$

which means $\underline{\acute{R}F}(U)=U=\overline{\acute{R}F}(U)$, hence $U\in\Im_{\acute{R}}$.

(2) For $A\in\Im_{\acute{R}}$, $\alpha\in[0,1]$,then $\underline{\acute{R}F}(A)=A=\overline{\acute{R}F}(A)$, it follows that $\forall x\in U$,

$$\underline{\acute{R}F}(\hat{\alpha}\check{T}A)(x)=\bigwedge_{y\in\acute{R}_s(x)}\hat{\alpha}\check{T}A(y)=\hat{\alpha}\check{T}\bigwedge_{y\in\acute{R}_s(x)}A(y)=\hat{\alpha}\check{T}\underline{\acute{R}F}A(x)=(\hat{\alpha}\check{T}A)(x).$$

and

$$\overline{\acute{R}F}(\hat{\alpha}\check{T}A)(x)=\bigvee_{y\in\acute{R}_s(x)}\left(\hat{\alpha}\check{T}A\right)(y)=\hat{\alpha}\check{T}\bigvee_{y\in\acute{R}_s(x)}A(y)=\hat{\alpha}\check{T}\overline{\acute{R}F}A(x)=(\hat{\alpha}\check{T}A)(x).$$

we have that $\underline{\acute{R}F}(\hat{\alpha}\check{T}A)=\hat{\alpha}\check{T}A=\overline{\acute{R}F}(\hat{\alpha}\check{T}A)$, hence $\hat{\alpha}\check{T}A\in\Im_{\acute{R}}$.

(3) For $A_n\in\Im_{\acute{R}}$, $n\in N$, we have that $\underline{\acute{R}F}(A_n)=A_n=\overline{\acute{R}F}(A_n)$, since $\acute{R}$ is serial, known by Theorem 2.1 and Theorem 2.2,

$$\underline{\acute{R}F}(\bigcup_{n\in N}A_n)\subseteq\overline{\acute{R}F}(\bigcup_{n\in N}A_n)=\bigcup_{n\in N}\overline{\acute{R}F}(A_n)=\bigcup_{n\in N}A_n.$$

On the other hand, it is known by Theorem 2.1

$$\bigcup_{n\in N}A_n=\bigcup_{n\in N}\underline{\acute{R}F}(A_n)\subseteq\underline{\acute{R}F}(\bigcup_{n\in N}A_n),$$

from the above two formulas,

$$\underline{\acute{R}F}(\bigcup_{n\in N}A_n)=\bigcup_{n\in N}A_n=\overline{\acute{R}F}(\bigcup_{n\in N}A_n),$$

which means, $\bigcup_{n\in N}A_n\in\Im_{\acute{R}}$.

(4) If $A\in\Im_{\acute{R}}$,that is, $\underline{\acute{R}F}(A)=A=\overline{\acute{R}F}(A)$, known by the duality of $\overline{\acute{R}F}$ and $\underline{\acute{R}F}$,

$$\underline{\acute{R}F}(\sim A)=\sim\overline{\acute{R}F}(A)=\sim A=\sim\underline{\acute{R}F}(A)=\overline{\acute{R}F}(\sim A),$$

which means, $\sim A\in\Im_{\acute{R}}$.

To sum up, $\Im_{\acute{R}}$ is an enriched fuzzy σ-algebra over U.

Next, we show that each enriched fuzzy σ-algebra generated by a σ-algebra can be induced by rough fuzzy approximation operator.

Theorem 3.2. Let $\Subset$ be a σ-algebra over U, then there is a serial binary relation $\acute{R}$ on U such that $\omega(\Subset)=\Im_{\acute{R}}$.

Proof: Let $\acute{R}$ be the equivalent relation determine by $\Subset$, and $\{C_1,C_2,\cdots\}$ denote its equivalent classes. Obviously, $\acute{R}$ is serial. The next we check that $\omega(\Subset)=\Im_{\acute{R}}$.

(1) $\omega(\Subset)\subseteq\Im_{\acute{R}}$. Noting that $\omega(\Subset)$ is the smallest enriched fuzzy σ-algebra containing $\{\alpha,C_i\}$, we need only check that $\{\alpha,C_i\}\subseteq\Im_{\acute{R}}$. Indeed, $\forall\alpha\in[0,1]$,

$$\underline{\acute{R}F}(\hat{\alpha})(x)=\bigwedge_{y\in\acute{R}_s(x)}\hat{\alpha}(y)=\hat{\alpha}(x)=\bigvee_{y\in\acute{R}_s(x)}\hat{\alpha}(y)=\overline{\acute{R}F}(\hat{\alpha})(x),$$

so $\underline{\acute{R}F}(\hat{\alpha}) = \hat{\alpha} = \overline{\acute{R}F}(\hat{\alpha})$, i.e. $\hat{\alpha} \in \Im_{\acute{R}}$.

For each equivalent class C_i and $x \in U$, if $x \in C_i$ then

$$\underline{\acute{R}F}(C_i)(x) = \wedge_{y \in C_i} C_i(y) = 1 = C_i(x) = 1 = \vee_{y \in C_i} C_i(y) = \overline{\acute{R}F}(C_i)(x),$$

otherwise

$$\underline{\acute{R}F}(C_i)(x) = \wedge_{y \in C_i} C_i(y) = 0 = C_i(x) = \vee_{y \in C_i} C_i(y) = \overline{\acute{R}F}(C_i)(x),$$

so $C_i \in \Im_{\acute{R}}$.

(2) $\Im_{\acute{R}} \subseteq \omega(\epsilon)$. Let $B \in \Im_{\acute{R}}$, i.e., $\underline{\acute{R}F}(B) = B = \overline{\acute{R}F}(B)$. Then for any $x \in U$, there is C_i such that $x \in C_i = \acute{R}_s(x)$ since $\{C_1, C_2, \cdots\}$ is the equivalent classes of $\acute{R}$. It follows by

$$\underline{\acute{R}F}(B)(x) = \wedge_{y \in \acute{R}_s(x)} B(y) = B(x) = \overline{\acute{R}F}(B)(x) = \vee_{y \in \acute{R}_s(x)} B(y),$$

we have

$$\wedge_{y \in C_i} B(y) = B(x) = \vee_{y \in C_i} B(y),$$

and denote $B(x) = \alpha_i$, then $B = \vee_{i \in I} \alpha_i \check{T} C_i$, which means $B \in \omega(\epsilon)$, therefore $\Im_{\acute{R}} \subseteq \omega(\epsilon)$.

To sum up, it can be proved $\omega(\epsilon) = \Im_{\acute{R}}$.

Next, we discuss the enriched fuzzy σ-algebra induced by $(\c{S}, \check{T})$-fuzzy rough set.

Definition 3.5. A t-conorms $\c{S}$ is said to be weakly distributed to t-norms $\check{T}$, if $\forall a, b, c \in [0,1]$, $a\c{S}(b\check{T}c) \geq b\check{T}(a\c{S}c)$.

Lemma 3.3. If $\check{T} = \check{T}_M, \c{S} = \c{S}_M$ or $\check{T} = \check{T}_P, \c{S} = \c{S}_P$ then $\c{S}, \check{T}$ satisfies weak distributivity.

Proof: (1) Let $\check{T} = \check{T}_M$ and $\c{S} = \c{S}_M$, then $\forall a, b, c \in [0,1]$

$$a\c{S}_M(b\check{T}_M c) = a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \geq b \wedge (a \vee c) = b\check{T}_M(a\c{S}_M c),$$

i.e., weak distributivity is satisfied.

(2) Let $\check{T} = \check{T}_P$ and $\c{S} = \c{S}_P$, then $\forall a, b, c \in [0,1]$

$$a\c{S}_P(b\check{T}_P c) = a\c{S}_P(b * c) = a + b * c - a * b * c,$$

$$b\check{T}_P(a\c{S}_P c) = b * (a + c - a * c) = b * a + b * c - a * b * c,$$

by $a \geq b * a$ we have

$$a\c{S}_P(b\check{T}_P c) \geq b\check{T}_P(a\c{S}_P c),$$

i.e., weak distributivity is satisfied.

Theorem 3.3. Let $(U, Я)$ be a reflexive FAPS, and $\c{S}, \check{T}$ satisfy weak distributivity. Then the family

$$\Im_{Я} = \{A \in F(U) \mid \underline{\c{S}Я}(A) = A = \overline{\check{T}Я}(A)\},$$

forms an enriched fuzzy σ-algebra over U.

Proof: (1) By Я is reflexive

$$\bigvee_{y\in R_s(x)} Я(x,y)=1,$$

we have

$$\underline{ŞЯ}(U)(x)=\bigwedge_{y\in U}(\sim Я(x,y)ŞU(y))=\bigwedge_{y\in U}U(y)=1=U(x),$$

$$\overline{ŤЯ}(U)(x)=\bigvee_{y\in U}(Я(x,y)ŤU(y))=\bigvee_{y\in U}Я(x,y)=1=U(x),$$

so $\underline{ŞЯ}(U)=U=\overline{ŤЯ}(U)$, hence $U\in\Im_Я$.

(2) For $A\in\Im_Я$, $\alpha\in[0,1]$, there is $\underline{ŞЯ}(A)=A=\overline{ŤЯ}(A)$. Since Я is reflexive, given by Theorem 2.3, we have

$$\underline{ŞЯ}(\alpha ŤA)\le\alpha ŤA, \qquad ①$$

on the other hand

$$\begin{aligned}\underline{ŞЯ}(\hat{\alpha}ŤA)(x)&=\bigwedge_{y\in U}[\sim Я(x,y)Ş(\alpha ŤA(y))]\\&\ge\bigwedge_{y\in U}\alpha Ť[\sim Я(x,y)ŞA(y)]\\&=\alpha Ť\bigwedge_{y\in U}[\sim Я(x,y)ŞA(y)]\\&=\alpha Ť(\underline{ŞЯ}(A)(x))\\&=(\hat{\alpha}ŤA)(x). \qquad ②\end{aligned}$$

It can be obtained by ① and ②,

$$\underline{ŞЯ}(\hat{\alpha}ŤA)=\hat{\alpha}ŤA.$$

Furthermore,

$$\begin{aligned}\overline{ŤЯ}(\hat{\alpha}ŤA)(x)&=\bigvee_{y\in U}[Я(x,y)Ť(\alpha ŤA(y))]\\&=\alpha Ť\bigvee_{y\in U}[Я(x,y)ŤA(y)]\\&=\alpha Ť(\overline{ŤЯ}(A)(x))\\&=(\hat{\alpha}ŤA)(x).\end{aligned}$$

So have $\underline{ŞЯ}(\hat{\alpha}ŤA)=\hat{\alpha}ŤA=\overline{ŤЯ}(\hat{\alpha}ŤA)$.

A combination of the above we have $\hat{\alpha}ŤA\in\Im_Я$.

(3) If $A_n\in\Im_Я$, $n\in N$, then

$$\underline{ŞЯ}(A_n)=A_n=\overline{ŤЯ}(A_n),$$

consequently

$$\underline{ŞЯ}(\bigcup_{n\in N}A_n)\subseteq\overline{ŤЯ}(\bigcup_{n\in N}A_n)=\bigcup_{n\in N}\overline{ŤЯ}(A_n)=\bigcup_{n\in N}A_n$$

$$\bigcup_{n\in N} A_n = \bigcup_{n\in N} \underline{\text{ŞЯ}}(A_n) \subseteq \underline{\text{ŞЯ}}(\bigcup_{n\in N} A_n)$$

It can be seen that,

$$\underline{\text{ŞЯ}}(\bigcup_{n\in N} A_n) = \bigcup_{n\in N} A_n = \bigcup_{n\in N} \overline{\text{ŤЯ}}(A_n),$$

hence $\bigcup_{n\in N} A_n \in \Im_{\text{Я}}$.

(4) If $A \in \Im_{\text{Я}}$, then $\underline{\text{ŞЯ}}(A) = A = \overline{\text{ŤЯ}}(A)$, from the duality of $\overline{\text{ŤЯ}}$ and $\underline{\text{ŞЯ}}$,

$$\underline{\text{ŞЯ}}(\sim A) = \sim \overline{\text{ŤЯ}}(A) = \sim A = \sim \underline{\text{ŞЯ}}(A) = \overline{\text{ŤЯ}}(\sim A),$$

so $\sim A \in \Im_{\text{Я}}$.

To sum up, $\Im_{\text{Я}}$ forms an enriched fuzzy σ-algebra.

Theorem 3.4. Let $(U, \Im)$ be an enriched fuzzy σ-algebra, then there is a reflexive fuzzy relation Я on U that makes $\Im_{\text{Я}} \subseteq \Im$.

Proof: For $x, y \in U$, define

$$\text{Я}(x,y) = \vee_{A\in\Im} (A(x)\text{Ť}A(y)).$$

Then it follows by $U \in \Im$, we know

$$\text{Я}(x,x) = \vee_{A\in\Im} (A(x)\text{Ť}A(x)) = 1,$$

i.e., Я is reflexive.

Let $A \in \Im_{\text{Я}}$, then $\underline{\text{ŞЯ}}(A) = A = \overline{\text{ŤЯ}}(A)$, and so $\forall x \in U$

$$\begin{aligned} A(x) &= \vee_{y\in U} \text{Я}(x,y)\text{Ť}A(y) \\ &= \vee_{y\in U} \vee_{B\in\Im} [B(x)\text{Ť}(A(y)\text{Ť}B(y))] \\ &= \vee_{B\in\Im} [B(x)\text{Ť}(\vee_{y\in U} B(y)\text{Ť}A(y))], \end{aligned}$$

denote $\vee_{y\in U} B(y)\text{Ť}A(y) = \beta$, then $A(x) = (\vee_{B\in\Im} \hat{\beta}\,\text{Ť}B)(x)$. That means $A = \bigcup_{B\in\Im} \hat{\beta}\,\text{Ť}B \in \Im$.

So, $\Im_{\text{Я}} \subseteq \Im$ is proved.

From Theorem 3.2 it is known that when Я is a reflexive binary relation we have $\Im_{\text{Я}} = \Im$. But for the fuzzy case, we only prove that $\Im_{\text{Я}} \subseteq \Im$ in Theorem 3.4. Hence, the following problem arise naturally.

Problem. Under what conditions can we get $\Im_{\text{Я}} = \Im$?

4. Concluding remarks

In this paper, we propose the concept of enriched fuzzy σ-algebras and examine their relationship with rough fuzzy sets and (Ş,Ť)-fuzzy rough sets. We have proven the following: (1) each σ-algebra can induce an enriched fuzzy σ-algebra; (2) each serial generalized rough fuzzy approximation space can induce an enriched fuzzy σ-algebra; (3)

each reflexive (Ş,Ť) -fuzzy rough approximation space can induce an enriched fuzzy σ-algebra; (4) each enriched fuzzy σ-algebra generated by a σ-algebra can be induced by a serial rough fuzzy approximation space. In [2] and [5], the researchers discussed more general (L,M) -fuzzy σ -algebra and L-intuitionistic fuzzy [10] σ -algebra, both of which can be regarded as the extensions of fuzzy σ-algebra. In the future, we will shall consider the corresponding enriched concepts within that framework.

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