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A Further Study on the Single Axiom Characterization of $(\nabla,\blacktriangle)$-Fuzzy Rough Approximation Operators

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Abstract. The axiomatization is an important work in fuzzy rough set theory, which can provide a concise and direct description for approximation operators of fuzzy rough sets. In this paper, using the inner product and outer product of fuzzy sets, we give a single axiom characterization of some special $(\nabla,\blacktriangle)$- fuzzy rough approximation operators. They include six fuzzy approximation operators generated by reflexive, symmetric, and transitive fuzzy relations and their composition.

Keywords: Single axiomatic characterization; Fuzzy rough set; Inner product; Outer product.

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1. Introduction

Pawlak's rough set theory [9] is a new tool for dealing with uncertain and incomplete knowledge. Nowadays, it has been widely expanded [6,7,8,12,19,22,23]. The upper and lower approximation operator is the most important concept in rough set theory. There are two main methods for the study of approximation operators, the constructive method and the axiomatic method. The constructive method is constructing a pair of upper and lower approximation operators from a binary relation on a nonempty set and studying their properties. The axiomatic method takes abstract operators as the primitive notion and finds some conditions (called axioms) to ensure the existence of binary relation s.t., the approximation operator induced by the binary relation is exactly equal to the given abstract operators [6,21,22].

Nowadays, the axiomatic method has been extended to fuzzy situations by many scholars. At first, scholars used a minimal axiom set (including at least two axioms) to describe various fuzzy rough sets [5,8,10,16,20]. In 2013, Liu [4] first put forward the idea and method of describing (fuzzy) rough sets by a single axiom. The method he gave was based on the outer product and inner product of fuzzy sets; the given characterizations are simple in form. Liu characterized some fuzzy approximation operators based on the maximum triangular norm and the minimum triangular conorm. Along with Liu's research, Wu [17] extended his work to more general $(\triangledown,\blacktriangle)$- fuzzy rough sets determined by any triangular norm $\blacktriangle$ and conorm $\triangledown$, Wu [15] further generalized the corresponding characterizations to intuitionistic fuzzy rough sets. But, the above-mentioned reference only considers the approximation operators generated by symmetric, symmetric and reflexive, symmetric and transitive fuzzy relations, some other important characterizations of the approximation operators generated by reflexive, transitive fuzzy relation and their composition have not been presented. Therefore, in this paper, we shall give the single axiomatic characterization of that $(\triangledown,\blacktriangle)$- fuzzy rough approximation operators; this paper is committed to proving that reflexive, transitive and reflexive+transitive approximation operators generated by reflexive, transitive fuzzy relation and their composition, respectively.

The content of this paper is arranged as follows. Section 2 reviews some basic concepts and symbols used in this paper. Section 3 presents the main results, i.e., gives the single axiomatic characterization of some $(\triangledown,\blacktriangle)$- fuzzy rough approximation operators by using the inner and outer products of fuzzy sets. Finally, we summarize the results and give future work.

2. Preliminaries

We fixed some concepts and symbols for later use.

In this paper, we always use $\blacktriangle:[0,1]^2\to[0,1]$ and $\triangledown:[0,1]^2\to[0,1]$ to represent a left-continuous t-norm and a right-continuous t-conorm [3], respectively.

A decreasing mapping $\mathcal{N}$ from [0, 1] to [0, 1] is referred to be an involutive negation [3] when $\mathcal{N}(1)=0,\mathcal{N}(0)=1$ and $\forall a\in[0,1]$, $\mathcal{N}(\mathcal{N}(a))=a$.

A mapping $A:U\to[0,1]$ is called a fuzzy set in U. The family of all fuzzy sets in U is denoted as $\mathcal{F}(U)$. The operations on $[0,1]$ can be transferred onto $\mathcal{F}(U)$ point-wise. For $A,B\in\mathcal{F}(U)$ and $*\in\{\wedge,\vee,\blacktriangle,\triangledown,\mathcal{N}\}$,we define $(A*B)(x)=A(x)*B(x)$, $A^{\mathcal{N}}(x)=\mathcal{N}(A(x))$. For $a\in[0,1]$, we use $\widehat{a}$ to denote the constant fuzzy set valued a.

A fuzzy set $\Gamma\in\mathcal{F}(U\times U)$ is called a fuzzy relation (FR for short) on U. The pair

(U,Γ) is referred a fuzzy approximation space (FAS). Furthermore, Γ is called

(1) reflexive if $\Gamma(x,x)=1,\forall x\in U$. (2) symmetric $\Gamma(x,y)=\Gamma(y,x),\forall x,y\in U$.

(3) $\blacktriangle$-transitive if $\Gamma(x,z)\geq \vee_{y\in U}\Gamma(x,y)\blacktriangle\Gamma(y,z),\forall x,y,z\in U$.

Γ is termed a tolerance provided it fulfils (1) + (2), a preorder provided (1) + (3), and an equivalence provided (1) +(2) +(3).

Definition 2.1. [17] Let $A,B\in\mathcal{F}(U)$, we define the inner product $I(A,B)$, outer product $O[A,B]$ of A and B as

$$I(A,B)=\vee_{x\in U}(A(x)\blacktriangle B(x)),\ \mathrm{O}[A,B]=\wedge_{x\in U}(A(x)\nabla B(x)).$$

The following proposition collects some properties of the inner product and outer produce.

Proposition 2.1. [17] Let $A,B,C,A_j(j\in J)\in\mathcal{F}(U),\ a\in[0,1]$ then

(I1) $I(A,B)=I(B,A)$. (I2) $A\subseteq B\Leftrightarrow I(A,C)\leq I(B,C)$. (I3) $I(\widehat{a}\blacktriangle A,B)=a\blacktriangle I(A,B)$.

(I4) $I(\vee_{j\in J}A_j,B)=\vee_{j\in J}I(A_j,B)$. (O1) $O[A,B]=O[B,A]$. (O2) $A\subseteq B\Leftrightarrow O[A,C]\leq O[B,C]$.

(O3) $O[\widehat{a}\nabla A,B]=a\nabla O[A,B]$. (O4) $O[\wedge_{j\in J}A_j,B]=\wedge_{j\in J}O[A_j,B]$.

Definition 2.2. [7,17] Let (U,Γ) be a FAS. For each $A\in\mathcal{F}(U)$, the pair $(\underline{\Gamma}(A),\overline{\Gamma}(A))$ is said to be $(\nabla,\blacktriangle)$-fuzzy rough sets of A, where $\overline{\Gamma}(A)$ and $\underline{\Gamma}(A)$ are defined by: $\forall x\in U$,

$$\underline{\Gamma}(A)(x)=\wedge_{y\in U}(\Gamma^{\mathcal{N}}(x,y)\nabla A(y)),\ \ \overline{\Gamma}(A)(x)=\vee_{y\in U}(\Gamma(x,y)\blacktriangle A(y)).$$

$\underline{\Gamma}$ and $\overline{\Gamma}$ are called ∇-fuzzy rough lower approximation operator and $\blacktriangle$-fuzzy rough upper approximation operator, respectively.

3. The main results

In this section, we shall give the single axiom characterizations of some fuzzy rough approximation operators by using inner product and outer product of fuzzy sets.

In this following, we always assume that $H,L:\mathcal{F}(U)\to\mathcal{F}(U)$ be a pair of operators.

In [17], Wu has given the single axiom characterizations of approximation operators generated by fuzzy relation, symmetric fuzzy relation, symmetric+transitive fuzzy relation, symmetric+reflexive (i.e., tolerance) fuzzy relation, and equivalent fuzzy relation, respectively. But, the axiomatic of the approximation operators generated by reflexive fuzzy relation, transitive fuzzy relation, and reflexive+transitive (i.e., preordered) fuzzy relation have not been presented, respectively.

Definition 3.1. [17] Let $\mathrm{K}:\mathcal{F}(U)\to\mathcal{F}(U)$ be an arbitrary operator and $A\in\mathcal{F}(U)$, define

$$\mathrm{K}_{\triangledown}^{-1}(A)(y)=\bigwedge_{x\in U}\left(\mathrm{K}(1_{U-\{y\}})(x)\nabla A(x)\right),\mathrm{K}_{\blacktriangle}^{-1}(A)(y)=\bigvee_{x\in U}\left(\mathrm{K}(1_y)(x)\blacktriangle A(x)\right),y\in U.$$

Then $\mathrm{K}_{\triangledown}^{-1},\mathrm{K}_{\blacktriangle}^{-1}:\mathcal{F}(U)\to\mathcal{F}(U)$ are called ∇ - lower inverse operator and $\blacktriangle$ - upper inverse operator of K, respectively. Obviously,

$$\mathrm{K}_{\triangledown}^{-1}(A)(y)=O[\mathrm{K}(1_{U-\{y\}}),A],\mathrm{K}_{\blacktriangle}^{-1}(A)(y)=I(\mathrm{K}(1_y),A).$$

The next theorems summarize the main results in [17,18].

Theorem 3.1. (1) There is an unique FR Γ on U with $H=\overline{\Gamma}$ iff H fulfills (AFU1) and (AFU2) iff H fulfills (AFU), where

(AFU1): $\forall A\in\mathcal{F}(U),\forall a\in[0,1],H(A\blacktriangle\widehat{a})=H(A)\blacktriangle\widehat{a}$.

(AFU2): $\forall A_j\in\mathcal{F}(U),j\in J,H(\bigvee_{j\in J}A_j)=\bigvee_{j\in J}H(A_j)$.

(AFU): $\forall A_j\in\mathcal{F}(U),\forall a_j\in[0,1],j\in J,H(\bigvee_{j\in J}(\widehat{a_j}\blacktriangle A_j))=\bigvee_{j\in J}H(\widehat{a_j}\blacktriangle A_j)$.

(2) There is an unique reflexive FR Γ on U with $H=\overline{\Gamma}$ iff H fulfills (AFU1) , (AFU2) and (AFU3): $\forall A\in\mathcal{F}(U),\ A\subseteq H(A)$.

(3) There is an unique symmetric FR Γ on U with $H=\overline{\Gamma}$ iff H fulfills (AFU1) , (AFU2) and (AFU4): $\forall x,y\in U,H(1_x)(y)=H(1_y)(x)$.

(4) There is an unique $\blacktriangle$ -transitive FR Γ on U with $H=\overline{\Gamma}$ iff H fulfills (AFU1) , (AFU2) and (AFU5): $\forall A\in\mathcal{F}(U),\ HH(A)\subseteq H(A)$.

Theorem 3.2. (1) There is an unique FR Γ on U with $L=\underline{\Gamma}$ iff L fulfills (AFL1) and

(AFL2) iff L fulfills (AFL), where

(AFL1): $\forall A \in \mathcal{F}(U), \forall a \in [0,1], L(A\nabla \widehat{a}) = H(A)\nabla \widehat{a}$.

(AFL2): $\forall A_j \in \mathcal{F}(U), j \in J, H(\vee_{j\in J} A_j) = \vee_{j\in J} H(A_j)$.

(AFL): $\forall A_j \in \mathcal{F}(U), \forall a_j \in [0,1], j \in J, L(\wedge_{j\in J}(\widehat{a_j}\nabla A_j)) = \wedge_{j\in J} L(\widehat{a_j}\nabla A_j)$.

(2) There is an unique reflexive FR Γ on U with $L = \underline{\Gamma}$ iff L fulfills (AFL1) , (AFL2) and (AFL3): $\forall A \in \mathcal{F}(U),\ A \supseteq L(A)$.

(3) There is an unique symmetric FR Γ on U with $L = \underline{\Gamma}$ iff L fulfills (AFL1) , (AFL2) and (AFL4): $\forall x, y \in U, L(1_{U-\{x\}})(y) = L(1_{U-\{y\}})(x)$.

(4)There is an unique $\blacktriangle$-transitive FR Γ on U with $L = \underline{\Gamma}$ iff L fulfills (AFL1) , (AFL2) and (AFL5): $\forall A \in \mathcal{F}(U),\ LL(A) \supseteq L(A)$.

Theorem 3.3. (1) There is an unique FR Γ on U with $H = \overline{\Gamma}$ iff H fulfills (SAFU):

$\forall A, B \in \mathcal{F}(U), I(A, H(B)) = I(B, H_{\blacktriangle}^{-1}(A))$.

(2) There is an unique symmetric FR Γ on U with $H = \overline{\Gamma}$ iff H fulfills (SAFUS): $\forall A, B \in \mathcal{F}(U), I(A, H(B)) = I(H(A), B)$.

(3) There is an unique tolerance FR Γ on U with $H = \overline{\Gamma}$ iff H fulfills (SAFUSR): $\forall A, B \in \mathcal{F}(U), I(A, H(B)) = I(B, A \vee H(A))$.

(4) There is an unique symmetric and $\blacktriangle$-transitive FR Γ on U with $H = \overline{\Gamma}$ iff H fulfills (SAFUST): $\forall A, B \in \mathcal{F}(U), I(A, H(B)) = I(B, H(A) \vee HH(A))$.

(5) There is an unique equivalent FR Γ on U with $H = \overline{\Gamma}$ iff H fulfills (SAFUE): $\forall A, B \in \mathcal{F}(U), I(A, H(B)) = I(B, A \vee H(A) \vee HH(A))$.

Theorem 3.4. (1) There is an unique FR Γ on U with $L = \underline{\Gamma}$ iff L fulfills (SAFL):

$\forall A, B \in \mathcal{F}(U), O[A, L(B)] = O[B, L_{\nabla}^{-1}(A)]$.

(2) There is an unique symmetric FR Γ on U with $L = \underline{\Gamma}$ iff L fulfills (SAFLS): $\forall A, B \in \mathcal{F}(U), O[A, L(B)] = O[B, L(A)]$.

(3) There is an unique tolerance FR Γ on U with $L = \underline{\Gamma}$ iff L fulfills (SAFLSR): $\forall A, B \in \mathcal{F}(U), O[A, L(B)] = O[B, A \wedge L(A)]$.

(4) There is an unique symmetric and $\blacktriangle$-transitive FR Γ on U with $L = \underline{\Gamma}$ iff L fulfills (SAFLST): $\forall A, B \in \mathcal{F}(U), O[A, L(B)] = O[B, L(A) \wedge LL(A)]$.

(5) There is an unique equivalent FR Γ on U with $L = \underline{\Gamma}$ iff L fulfills (SAFLE): $\forall A, B \in \mathcal{F}(U), O[A, L(B)] = O[B, A \wedge L(A) \wedge LL(A)]$.

Next, we give the single axiom characterizations on the six fuzzy rough approximation operators mentioned above.

Theorem 3.5. There is an unique reflexive FR Γ on U with $H = \overline{\Gamma}$ iff H fulfills (SAFUR): $\forall A, B \in \mathcal{F}(U), I(A, H(B)) = I(B, A \vee H_{\blacktriangle}^{-1}(A))$.

Proof: From Theorem 3.3(1) we need only check that (AFU3)+(SAFU) $\Leftrightarrow$ (SAFUR).
$\Rightarrow$. $\forall A, B \in \mathcal{F}(U)$, by (AFU3), then for any $y \in U$,

$$H_{\blacktriangle}^{-1}(A)(y) = \vee_{x \in U} (H(1_y)(x) \blacktriangle A(x)) \geq \vee_{x \in U} (1_y(x) \blacktriangle A(x)) = A(y),$$

which means $H_{\blacktriangle}^{-1}(A) \supseteq A$, then $A \vee H_{\blacktriangle}^{-1}(A) = H_{\blacktriangle}^{-1}(A)$, by (SAFU),

$$I(A, H(B)) = I(B, H_{\blacktriangle}^{-1}(A)) = I(B, A \vee H_{\blacktriangle}^{-1}(A)),$$

i.e., (SAFUR) holds.
$\Leftarrow$. $\forall A, B \in \mathcal{F}(U)$, by (SAFUR),

$$I(B, A) = I(A, B) \leq I(A, B \vee H_{\blacktriangle}^{-1}) = I(B, H(A)),$$

it follows by Proposistion2.1 (I2) we have $A \subseteq H(A)$, i.e., (AFU3) holds. Then form (AFU3) we further obtain $A \subseteq H_{\blacktriangle}^{-1}(A)$, so $A \vee H_{\blacktriangle}^{-1}(A) = H_{\blacktriangle}^{-1}(A)$, hence

$$I(A, H(B)) = I(B, A \vee H_{\blacktriangle}^{-1}(A)) = I(B, H_{\blacktriangle}^{-1}(A)),$$

i.e., (SAFU) holds.

Theorem 3.6. There is an unique $\blacktriangle$-transitive FR Γ on U with $H=\overline{\Gamma}$ iff H fulfills

(SAFUT): $\forall A,B\in\mathcal{F}(U), I(A,H(B))=I(B,H_{\blacktriangle}^{-1}(A)\vee H_{\blacktriangle}^{-1}H_{\blacktriangle}^{-1}(A))$.

Proof: Form Theorem 3.3(1) and Theorem 3.1(1), we need only check that (AFU5)+(SAFU) $\Rightarrow$ (SAFUT) and (AFU5)+(AFU)$\Leftarrow$(SAFUT).

$\Rightarrow$. $\forall A\in\mathcal{F}(U),\forall y\in U$, by (AFU5),

$$\begin{aligned}H_{\blacktriangle}^{-1}H_{\blacktriangle}^{-1}(A)(y)&=\vee_{x\in U}(H(1_y)(x)\blacktriangle H_{\blacktriangle}^{-1}(A)(x))=\vee_{x\in U}\{H(1_y)(x)\blacktriangle\vee_{z\in U}[H(1_x)(z)\blacktriangle A(z)]\}\\&=\vee_{x\in U}\vee_{z\in U}\{H(1_y)(x)\blacktriangle H(1_x)(z)\blacktriangle A(z)\}=\vee_{x\in U}\vee_{z\in U}\{H[1_x\blacktriangle H(1_y)(x)](z)\blacktriangle A(z)\}\\&=\vee_{z\in U}H\{\vee_{x\in U}[1_x\blacktriangle H(1_y)(x)](z)\blacktriangle A(z)\}=\vee_{z\in U}(HH(1_y)(z)\blacktriangle A(z))\\&\le\vee_{z\in U}(H(1_y)(z)\blacktriangle A(z))=H_{\blacktriangle}^{-1}A(y),\end{aligned}$$

so $H_{\blacktriangle}^{-1}H_{\blacktriangle}^{-1}(A)\subseteq H_{\blacktriangle}^{-1}A$, then $H_{\blacktriangle}^{-1}A=H_{\blacktriangle}^{-1}(A)\vee H_{\blacktriangle}^{-1}H_{\blacktriangle}^{-1}(A)$, by (SAFU), we have

$$I(A,H(B))=I(B,H_{\blacktriangle}^{-1}(A))=I(B,H_{\blacktriangle}^{-1}(A)\vee H_{\blacktriangle}^{-1}H_{\blacktriangle}^{-1}(A)),$$

i.e., (SAFUT) holds.

$\Leftarrow$. $\forall A,B\in\mathcal{F}(U)$, by (SAFUT),

$$\begin{aligned}I(B,H(\vee_{j\in J}(\widehat{a}_j\blacktriangle A_j)))&=I(\vee_{j\in J}(\widehat{a}_j\blacktriangle A_j),H_{\blacktriangle}^{-1}(B)\vee H_{\blacktriangle}^{-1}H_{\blacktriangle}^{-1}(B))\\&=\vee_{j\in J}I((\widehat{a}_j\blacktriangle A_j),H_{\blacktriangle}^{-1}(B)\vee H_{\blacktriangle}^{-1}H_{\blacktriangle}^{-1}(B))\\&=\vee_{j\in J}\widehat{a}_j\blacktriangle I(A_j,H_{\blacktriangle}^{-1}(B)\vee H_{\blacktriangle}^{-1}H_{\blacktriangle}^{-1}(B))\\&=\vee_{j\in J}\widehat{a}_j\blacktriangle I(B,H(A_j))=I(B,\vee_{j\in J}\widehat{a}_j\blacktriangle H(A_j)),\end{aligned}$$

then $H(\vee_{j\in J}(\widehat{a}_j\blacktriangle A_j))=\vee_{j\in J}\widehat{a}_j\blacktriangle H(A_j)$, so (AFU) is true.

Furthermore, take $A=1_x, B=1_z$ in (SAFUT), then

$$\begin{aligned}H(1_z)(x)&=H_{\blacktriangle}^{-1}(1_x)(z)\vee H_{\blacktriangle}^{-1}H_{\blacktriangle}^{-1}(1_x)(z)=H_{\blacktriangle}^{-1}(1_x)(z)\vee\vee_{y\in U}(H(1_z)(y)\blacktriangle H_{\blacktriangle}^{-1}(1_x)(y))\\&=H(1_z)(x)\vee\vee_{y\in U}(H(1_z)(y)\blacktriangle H(1_y)(x))=H(1_z)(x)\vee H(\vee_{y\in U}[H(1_z)(y)\blacktriangle 1_y))(x)\\&=H(1_z)(x)\vee HH(1_z)(x),\end{aligned}$$

which means $H(1_z)\supseteq HH(1_z)$, then

$$HH(A) = HH(\vee_{y\in U} 1_y \blacktriangle A(y)) = \vee_{y\in U} (A(y) \blacktriangle HH(1_y))$$
$$\leq \vee_{y\in U} (A(y) \blacktriangle H(1_y)) = H(\vee_{y\in U} (A(y) \blacktriangle 1_y)) = H(A),$$

which means $HH(A) \subseteq H(A)$, i.e., (AFU5) holds.

Theorem 3.7. There is an unique preorder FR Γ on U with $H = \overline{\Gamma}$ iff H fulfills

(SAFURT): $\forall A, B \in \mathcal{F}(U), I(A, H(B)) = I(B, A \vee H_{\blacktriangle}^{-1}(A) \vee H_{\blacktriangle}^{-1} H_{\blacktriangle}^{-1}(A)).$

Proof: From Theorem 3.6 we need only check that (AFU3)+(SAFUT) $\Leftrightarrow$ (SAFURT).

$\Rightarrow$. $\forall A, B \in \mathcal{F}(U)$, by (AFU3) we obtain $A \subseteq H(A)$, which means $A \subseteq H_{\blacktriangle}^{-1}(A)$,then by (SAFUT), it follows that

$$I(B, A \vee H_{\blacktriangle}^{-1}(A) \vee H_{\blacktriangle}^{-1} H_{\blacktriangle}^{-1}(A)) = I(B, H_{\blacktriangle}^{-1}(A) \vee H_{\blacktriangle}^{-1} H_{\blacktriangle}^{-1}(A)) = I(A, H(B)),$$

i.e., (SAFURT) holds.

$\Leftarrow$. $\forall A, B \in \mathcal{F}(U)$, by (SAFURT), we have

$$I(A, H(B)) = I(B, A \vee H_{\blacktriangle}^{-1}(A) \vee H_{\blacktriangle}^{-1} H_{\blacktriangle}^{-1}(A)) \geq I(B, A) = I(A, B),$$

which means $H(B) \supseteq B$, i.e., (AFU3) holds.

Furthermore, from (AFU3) we can conclude that $A \subseteq H_{\blacktriangle}^{-1}(A)$, then

$$A \vee H_{\blacktriangle}^{-1}(A) \vee H_{\blacktriangle}^{-1} H_{\blacktriangle}^{-1}(A) = H_{\blacktriangle}^{-1}(A) \vee H_{\blacktriangle}^{-1} H_{\blacktriangle}^{-1}(A),$$

applying it in (SAFURT), then

$$I(A, H(B)) = I(B, A \vee H_{\blacktriangle}^{-1}(A) \vee H_{\blacktriangle}^{-1} H_{\blacktriangle}^{-1}(A)) = I(B, H_{\blacktriangle}^{-1}(A) \vee H_{\blacktriangle}^{-1} H_{\blacktriangle}^{-1}(A)),$$

i.e., (SAFUT) holds.

In the following, we consider the lower approximation.

Theorem 3.8. There is an unique reflexive FR Γ on U with $L = \underline{\Gamma}$ iff L fulfills (SAFLR):

$\forall A, B \in \mathcal{F}(U), O[A, L(B)] = O[B, A \wedge L_{\triangledown}^{-1}(A)].$

Proof: From Theorem 3.4(1) we need only check that (AFL3)+(SAFL) $\Leftrightarrow$ (SAFLR).
$\Rightarrow$. $\forall A, B \in \mathcal{F}(U)$, by (AFL3), then for any $y \in U$,

$$L_{\triangledown}^{-1}(A)(y) = \wedge_{x\in U} (L(1_{U\text{-}\{y\}})(x) \triangledown A(x)) \leq \wedge_{x\in U} ((1_{U\text{-}\{y\}})(x) \triangledown A(x)) = A(y),$$

which means $A \supseteq L_{\triangledown}^{-1}(A)$, then $L_{\triangledown}^{-1}(A) = A \wedge L_{\triangledown}^{-1}(A)$, by (SAFL)

$$O[A,L(B)]=O[B,L_{\nabla}^{-1}(A)]=O[B,A\wedge L_{\nabla}^{-1}(A)],$$

i.e., (SAFLR) holds.

$\Leftarrow$. $\forall A,B\in\mathcal{F}(U)$, by (SAFLR), then $O[B,A]=O[A,B]\geq O[A,B\wedge L_{\nabla}^{-1}(B)]=O[B,L(A)]$, it follows by Proposistion2.1 (O2) we have $A\supseteq L(A)$, i.e., (AFL3) holds. Then from (AFL3) we further obtain $L_{\nabla}^{-1}(A)\subseteq A$, so $A\wedge L_{\nabla}^{-1}(A)=L_{\nabla}^{-1}(A)$, hence

$$O[A,L(B)]=O[B,A\wedge L_{\nabla}^{-1}(A)]=O[B,L_{\nabla}^{-1}(A)],$$

i.e., (SAFL) holds.

Theorem 3.9. There is an unique $\blacktriangle$-transitive FR Γ on U with $L=\underline{\Gamma}$ iff L fulfills

(SAFLT): $\forall A,B\in\mathcal{F}(U),O[A,L(B)]=O[B,L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)].$

Proof: Form Theorem 3.4(1) and Theorem 3.2(1), We need only check that (AFL5)+(SAFL) $\Rightarrow$ (SAFLT) and (AFL5)+(AFL)$\Leftarrow$(SAFUT).

$\Rightarrow$. $\forall A\in\mathcal{F}(U),\forall x\in A,\forall y\in U$, by (AFL5),

$$\begin{aligned}L_{\nabla}^{-1}L_{\nabla}^{-1}(A)(y)&=\underset{x\in U}{\wedge}(L(1_{\text{U-\{y\}}})(x)\nabla L_{\nabla}^{-1}A(x))=\underset{x\in U}{\wedge}(L(1_{\text{U-\{y\}}})(x)\nabla\underset{z\in U}{\wedge}(L(1_{U-\{x\}})(z)\nabla A(z))\\&=\underset{x\in U}{\wedge}\underset{z\in U}{\wedge}(L(1_{\text{U-\{y\}}})(x)\nabla(L(1_{U-\{x\}})(z)\nabla A(z))\\&=\underset{z\in U}{\wedge}(L(\underset{x\in U}{\wedge}(1_{U-\{x\}})\nabla L(1_{\text{U-\{y\}}})(x))(z)\nabla A(z))\\&=\underset{z\in U}{\wedge}(LL(1_{\text{U-\{y\}}})(z)\nabla A(z))\geq\underset{z\in U}{\wedge}(L(1_{\text{U-\{y\}}})(z)\nabla A(z))=L_{\nabla}^{-1}(A)(y),\end{aligned}$$

which means $L_{\nabla}^{-1}(A)\subseteq L_{\nabla}^{-1}L_{\nabla}^{-1}(A)$, then $L_{\nabla}^{-1}(A)=L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)$, by (SAFL), we have

$O[A,L(B)]=O[B,L_{\nabla}^{-1}(A)]=O[B,L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)]$, i.e., (SAFLT) holds.

$\Leftarrow$. $\forall A,B\in\mathcal{F}(U)$, by (SAFLT),

$$\begin{aligned}O[B,L(\underset{j\in J}{\wedge}(\widehat{a}_j\nabla A_j))]&=O[\underset{j\in J}{\wedge}(\widehat{a}_j\nabla A_j),L_{\nabla}^{-1}(B)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(B)]\\&=\underset{j\in J}{\wedge}\widehat{a}_j\nabla O[A_j,L_{\nabla}^{-1}(B)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(B)]\\&=\underset{j\in J}{\wedge}\widehat{a}_j\nabla O[B,L(A_j)]=O[B,\underset{j\in J}{\wedge}\widehat{a}_j\nabla L(A_j)],\end{aligned}$$

then $L(\underset{j\in J}{\wedge}(\widehat{a}_j\nabla A_j))=\underset{j\in J}{\wedge}\widehat{a}_j\nabla L(A_j)$, so (AFL)is true.

Furthermore, take $A=1_{U\text{-}\{x\}}, B=1_{U-\{z\}}$ in (SAFLT), then

$$\begin{aligned} L(1_{U-\{z\}})(x) &= L_{\nabla}^{-1}(1_{U-\{x\}})(z)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(1_{U-\{x\}})(z) \\ &= L_{\nabla}^{-1}(1_{U-\{x\}})(z)\wedge \bigwedge_{y\in U}(L(1_{U-\{z\}})(y)\nabla L_{\nabla}^{-1}(1_{U-\{x\}})(y)) \\ &= L(1_{U-\{z\}})(x)\wedge \bigwedge_{y\in U}(L(1_{U-\{z\}})(y)\nabla L(1_{U-\{y\}})(x)) \\ &= L(1_{U-\{z\}})(x)\wedge L(\bigwedge_{y\in U}(\underline{\Gamma}(1_{U-\{z\}})(y)\nabla 1_{U-\{y\}})(x) \\ &= L(1_{U-\{z\}})(x)\wedge LL(1_{U-\{z\}})(x), \end{aligned}$$

which means $L(1_{U-\{z\}})\subseteq LL(1_{U-\{z\}})$, then

$$\begin{aligned} LL(A) &= LL(\bigwedge_{y\in U}1_{U\text{-}\{y\}}\triangledown A(y)) = \bigwedge_{y\in U}(A(y)\triangledown LL(1_{U\text{-}\{y\}})) \\ &\geq \bigwedge_{y\in U}(A(y)\triangledown L(1_{U\text{-}\{y\}})) = L(\bigwedge_{y\in U}A(y)\triangledown(1_{U\text{-}\{y\}})) = L(A), \end{aligned}$$

which means $L(A)\subseteq LL(A)$, i.e., (AFL5) holds.

Theorem 3.10. There is an unique preorder FR Γ on U with $L=\underline{\Gamma}$ iff L fulfills

(SAFLRT): $\forall A,B\in\mathcal{F}(U), O[A,L(B)]=O[B,A\wedge L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)].$

Proof: From Theorem 3.9 we need only check that (AFL3)+(SAFLT) $\Leftrightarrow$ (SAFLRT).

$\Rightarrow$. $\forall A,B\in\mathcal{F}(U)$, by (AFL3) we obtain, $A\supseteq L(A)$, which further means $A\supseteq L_{\nabla}^{-1}(A)$, then by (SAFLT),

$$O[B,A\wedge L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)]=O[B,L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)]=O[A,L(B)],$$

i.e., (SAFLRT) holds.

$\Leftarrow$. $\forall A,B\in\mathcal{F}(U)$, by (SAFLRT),

$$O[A,L(B)]=O[B,A\wedge L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)]\leq O[B,A]=O[A,B],$$

which means $L(B)\subseteq B$, i.e., (AFL3) holds.

Furthermore, from (AFL3) we can conclude that $A\supseteq L_{\nabla}^{-1}(A)$, then

$A\wedge L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)=L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)$, applying it in (SAFURT), then

$$O[A,L(B)]=O[B,A\wedge L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)]=O[B,L_{\nabla}^{-1}(A)\wedge L_{\nabla}^{-1}L_{\nabla}^{-1}(A)],$$

i.e., (SAFUT) holds.

4. Concluding remarks

The axiomatic characterization of approximate operators is an important problem in the development of rough set theory. In [17], Wu characterized 10 (∇,▲)- fuzzy rough approximation operators by single axiom respectively. In this paper, we characterized another 6 (∇,▲)- fuzzy rough approximation operators, thus perfecting Wu's work.

In [1,2,11,13,14], the scholars studied the axiom characterization of more general complete residuated lattice-valued (which is an extension of t-norm) fuzzy rough sets. In future work, we shall consider the single axiom characterizations of the lattice-valued fuzzy rough sets mentioned.

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