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Portfolio Optimization under Informationally Asymmetric Markets

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Abstract. Asymmetric information propagation frequently results in distorted financial markets and is generally a feature of informationally inefficient markets. We developed a model of optimal asset allocation using the martingale method to assist an investor in selecting an asset that performs better under the conditions of a market information cascade. In order to confirm that the model satisfies the required conditions, we applied the verification theorem and ascertained that the results produced were optimal. The model outperformed the famous Markowitz mean-variance type model and was shown to produce stable and consistent solutions under such market conditions.

***Keywords*:** Robust optimization; inefficient market; Skew Brownian motion; efficient frontier.

AMS Mathematics Subject Classification (2010): 91G10

1. Introduction

In any investment, investors aim for higher returns with reference to their initial invested capital, but at times, investors face challenges on how to design the best trading strategies to satisfy their desire to gain from a particular invested asset. Return is always accompanied by risk in the financial market (the mantra is that the higher the risk, the higher the return). The ability to manage risk in the financial market is crucial for achieving higher future profits [1, 2, 3, 4]. There is a challenge in determining how to optimally allocate each equity to minimize risk while maximizing the return for a given set of equities and their respective investment returns. This challenge stems primarily from the fact that it is difficult to predict stock price movement in the financial market [5]. In financial markets, stochastic portfolio theory (SPT) introduced by Fernholz [6] is currently playing a great role in analyzing the behaviors of equity markets and portfolio selection than the contributions of modern portfolio theory (MPT) by Markowitz [7]. This is motivated by the currently available and explosive sources of information (mainly unstructured information), especially on social media, which have a direct impact on the existence of market-relative arbitrages.

For a long time, it has been a common practice to assume that market weights consist of continuous covariations in a pathwise sense, and consequently, Itoˆ calculus has been used to construct trading strategies from a given function depending on the market weights without taking care of the corresponding probabilities. Strong relative arbitrage in the market does not rely solely on nondecreasing functions such as Lyapunov and Gamma functions, as proposed by Karatzas and Ruf [8], but rather on a variety of collections of functions depending on the corresponding market weights [9].

It is a normal practice when constructing a market portfolio to take into consideration historical data, such as the previous historical performance of the particular stock or statistical estimates, with the view of better future returns [10]. This argument supports the previously mentioned contribution of social media (as a current source of information) to the performance of the market. In analyzing the portfolio, the improvement of the portfolio's performance does not rely on only the market weights as proposed by Fernholz [6] but also on some other additional input information. The inclusion of other input factors apart from market weights reduces the time bound in which the relative arbitrage of the market is attainable [11].

Most of the financial markets that are existing in today's real-world markets are inefficient markets. An inefficient market is one that does not succeed in incorporating all available information into a true reflection of an asset fair price. Market inefficiencies exist due to information asymmetry, transaction costs, market psychology, and human emotions, among other reasons. In an inefficient market, the returns (profits) depend on the active level of the investors' manager. In a real-world financial market, investors differ in their wealth and financial sophistication, and managers differ in their education and investment strategies. Moreover, investors have different levels of absolute risk aversion and search cost [12]. According to [12], the inefficiency is greater in markets with higher percentage fees (e.g., private equity vs. public) and during times of high-risk aversion (e.g., crisis periods).

In informationally inefficient markets, classical decision theory assumes the value of information to be positive. However, there are some studies that contradict this paradigm. For example, Schredelseker [13] in his study found that badly informed traders could expect higher returns than traders with more information. The results obtained by Schredelseker were based on a small number of traders. Later on, Pfeifer et al. [14] were able to verify the negative value of information on returns for a sufficient large number of traders. Traders with less information seem to be at an advantage because of the part of that information that is unknown to them.

This study proposes a probabilistic robust optimization technique for modeling asset returns under a broad family of stochastic optimization methods in order to address the problem of mispricing of financial instruments in markets that do not conform to the traditional Markowitz portfolio optimdirectly impactization. Stochastic optimization techniques are known to provide a wide range of feasible solutions, and thus, with appropriate constraints, the solutions are likely to be insensitive to sudden changes in parameters.

The following is how the rest of the article is structured: local volatility model with discontinuities under skew Brownian motion, and construction of an asset return optimization model using the martingale method were discussed in Section 2. We presented and discussed some of the results in Section 3. Finally, in Section 4, we conclude our paper with a brief conclusion.

2. Methodology

2.1. Local volatility model with discontinuities under skew Brownian motion.

Consider a local volatility model

$$dS(t) = \sigma(S(t))S(t)dW(t)dS(t) \tag{1}$$

where

$$\sigma(S) = \begin{cases} \sigma_1 > 0, \text{if } S \geq 1, \\ \sigma_2 > 0, \text{if } S < 1. \end{cases}$$

Lemma 2.1. *[15, Lemma 1]*

Let $S(t)$ be a solution of (1)*. A stochastic process $Z(t)$ defined by*

$$Z(t) = \frac{\ln S\,(t)}{\sigma\big(S(t)\big)}$$

is a solution of the stochastic differential equation

$$dZ(t) = \mu\big(Z(t)\big)dt + dW(t) + (2p-1)dL_t^0(Z) \tag{2}$$

where

$$\mu(z) = \frac{-\sigma(e^z)}{2} = \begin{cases} \mu_1 = \dfrac{-\sigma_1}{2}, \text{if } z \geq 0, \\ \mu_2 = \dfrac{-\sigma_2}{2}, \text{ if } z < 0, \end{cases}$$

$L_t^0(Z)$ is the symmetric local time of $Z(t)$ and $Z(t)$ is the skew Brownian motion with parameter $p = \frac{\sigma_2}{\sigma_1+\sigma_2}$ and discontinuous $\mu(x)$.

Proof:.

Letting

$$Y(t) = \ln S(t)$$

Applying Itô formula to equation (1),

$$dY(t) = \sigma\big(Y(t)\big)dW(t) - \frac{\sigma^2\big(Y(t)\big)}{2}dt$$

Also, define $Z(t) = h\big(Y(t)\big)$, where

$$h(y) = \begin{cases} \frac{y}{\sigma_1}, \text{ if } y \geq 0 \\ \frac{y}{\sigma_2}, \text{ if } y < 0, \end{cases}$$

and applying Itô-Tanaka, we

$$d\big(Z(t)\big) = dW(t) - \frac{\sigma\big(Y(t)\big)}{2} dt + \frac{1}{2}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) dL_t^0(Y(t)) \tag{3}$$

Again, from Itô Tanaka formula

$$|Z(t)| = \int_0^t sign\big(Z(u)\big) dZ(u) + L_t^0(Z) \tag{4}$$

and

$$\big|h\big(Y(t)\big)\big| = \int_o^t sgn\big(Y(u)\big) h'\big(Y(u)\big) dY(u) + \frac{1}{2}\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right) dL_t^0(Y) \tag{5}$$

where sgn denotes the sign function

$$sgn(x) = \begin{cases} +1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0. \end{cases}$$

From equations (4) and (5), it can be observed that

$$sgn\big(Z(u)\big) dZ(u) = sgn\big(Y(u)\big) h'\big(Y(u)\big) dY(u)$$

and

$$L_t^0(z) = \frac{1}{2}\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right) dL_t^0(Y) \tag{6}$$

substituting (6) into equation (3)

$$\begin{aligned} dZ(t) &= dW(t) - \frac{\sigma\big(Y(t)\big)}{2} dt + \frac{1}{2}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) dL_t^0(Y) \\ &= dW(t) - \frac{\sigma\big(Y(t)\big)}{2} dt + \frac{1}{2}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)\left(\frac{2\sigma_1\sigma_2}{\sigma_1 + \sigma_2}\right) dL_t^0(Z) \\ &= dW(t) - \frac{\sigma\big(Y(t)\big)}{2} dt + \frac{1}{2}\left(\frac{\sigma_2 - \sigma_1}{\sigma_1\sigma_2}\right)\left(\frac{2\sigma_1\sigma_2}{\sigma_1 + \sigma_2}\right) dL_t^0(Z) \\ &= dW(t) - \frac{\sigma\big(e^{Z(t)}\big)}{2} dt + \left(\frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2}\right) dL_t^0(Z) \\ &= -\frac{\sigma\big(e^{Z(t)}\big)}{2} dt + dW(t) + \left(\frac{2\sigma_2}{\sigma_1 + \sigma_2} - 1\right) dL_t^0(Z) \\ dZ(t) &= \mu\big(Z(t)\big) dt + dW(t) + (2p - 1) dL_t^0(Z). \end{aligned}$$

Remark 2.1. *In informationally inefficient markets, the flow and trend of volatility follow the behavior of skew Brownian motion; therefore, the above-stated lemma will be incorporated within our optimization problem constraints to achieve the desired optimal asset returns.*

2.2. The asset return optimization model

Consider a financial market with n risky assets with price $S_i(t)$ given by

$$dS_i(t) = S_i(t)\left(\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t)\right)$$
$$S_i(0) = s_i \qquad \text{for} \quad i = 0,1,2,\dots,n. \tag{7}$$

where $W(t)$ is a d-dimensional Brownian motion, $\mu(t)$ is the stock appreciation rate, and $\sigma(t) = \{\sigma_{ij}\}_{n\times d}$ is the volatility matrix. The market coefficients $\mu(\cdot)$ and $\sigma(\cdot)$ are assumed to be a square integrable functions satisfying the following conditions

(i) $|\mu(p) - \mu(q)| \leq L\cdot|p-q| \qquad |\sigma(p) - \sigma(q)| \leq L\cdot|p-q|$

(ii) $\mu^2(p) \leq L^2\cdot(1+p^2) \qquad \sigma^2(p) \leq L^2\cdot(1+p^2).$

Portfolio optimization emphasizes the diversification of assets, where a common practice for investors is to have a mix of risky and riskless assets. If $S_0(t)$ denotes the price of riskless assets, it can be represented mathematically

$$dS_0(t) = S_0(t)R(t)dt$$
$$S_0(0) = s_0 \tag{8}$$

where $R(t)$ is the nominal interest rate.

Applying Itô formula to the equation model (7), the price $S_i(t)$ is given by

$$S_i(t) = s_i \exp\left[\int_0^t\left(\mu_i(s) - \tfrac{1}{2}\textstyle\sum_{j=1}^{n}\sigma_{ij}^2(s)\right)ds + \int_0^t \textstyle\sum_{j=1}^{n}\sigma_{ij}(s)dW_j(s)\right]. \tag{9}$$

Definition 2.1. [16, Definition 1.0.2] A pair (π, C) consisting of a portfolio π and a consumption rate C

is said to be self-financing if the corresponding wealth process $X^{\pi,C}(t), t \in [0,T]$ satisfies

$$dX^{\pi,C}(t) = \textstyle\sum_{i=1}^{n}\pi_i(t)X^{\pi,C}(t)\frac{dS_i(t)}{S_i(t)} + (1-\sum_{i=1}^{n}\pi_i(t))X^{\pi,C}(t)\frac{dS_0(t)}{S_0(t)} - C(t)dt. \tag{10}$$

Investors in the financial market at each period can decide what proportion of wealth, $\pi_i(t)$, to invest in the available assets and what the consumption rate, $C(t) \geq 0$ should be. Then, by plugging (7) and (8) into (10), the wealth process equation becomes

$$dX^{\pi,C}(t) = \sum_{i=1}^{n}\pi_i(t)X^{\pi,C}(t)\left(\mu_i(t)dt + \sum_{j=1}^{n}\sigma_{ij}(t)dW_j(t)\right)$$
$$+(1-\textstyle\sum_{i=1}^{n}\pi_i(t))X^{\pi,C}(t)R(t)dt - C(t)dt. \tag{11}$$

In vector and matrix format, (11) can be written as

$$dX^{\pi,C}(t) = X^{\pi,C}(t)[\pi^{\mathcal{T}}(t)(\mu(t) - R(t)1_n)dt + R(t)dt + \pi^{\mathcal{T}}(t)\sigma(t)dW(t)] - C(t)dt, \tag{12}$$

where $\mathcal{T}$ represents the transpose and $\mathbf{1}_n$ is the vector of one's.

Since we are assuming that an investor is investing in an inefficient market, we are expecting the market to exhibit the risk of market price.

Let $\phi(t)$ be risk of market price which is given by

$$\phi(t) = \sigma^{\mathcal{T}}(t)\left(\sigma(t)\sigma^{\mathcal{T}}(t)\right)^{-1}[\mu(t) - R(t)1_n] \tag{13}$$

Inserting (13) into (12), the wealth process can be represented by the equation

$$dX^{\pi,C}(t) = X^{\pi,C}(t)\big[\pi^{\mathcal{T}}(t)\sigma(t)\big(\phi(t)dt + dW(t)\big) + R(t)dt\big] - C(t)dt. \tag{14}$$

Since our aim is to maximize the final wealth for a given utility function, the wealth process described in (14) is one of the constraints for our maximization problem.

Remark 2.2. *Since our aim is to maximize the investor's expected utility final wealth, and since we are looking for optimal asset returns, it is mathematically convenient to assume that there is zero consumption throughout the entire investment period.*

Assuming that the investor lacks complete (exact) information about future prices, an optimal decision will be made by observing stock prices in the past and present. Then, a stochastic optimization problem is given by

$$\max_{\pi\in\mathcal{A}_0(x)E}\big[u\big(X^{\pi}(T)\big)\big]$$
$$s.t$$
$$dX^{\pi}(t) = X^{\pi}(t)\big[R(t)dt + \pi^{\mathcal{T}}(t)\sigma(t)\backslash big\big(\phi(t)dt + dW(t)\big)\big]$$
$$X^{\pi}(0) = x \tag{15}$$

with

$$\mathcal{A}_0(x) = \{(\pi,0) \in \mathcal{A}(x): E\big[u^-\big(X^{\pi}(T)\big)\big] < \infty\}$$

where $\mathcal{A}(x)$ represents the class of admissible pairs, $u(\cdot)$ is the investor's utility function, which is chosen as the constant relative risk aversion (CRRA), and $u_i^-(\cdot) = \max\{-u_i(\cdot),0\}$.

Prominent financial stock pricing models are built on the assumption that asset returns follow a normal Gaussian distribution (normal Brownian motion). However, many authors (for example, [17, 18, 19]) argue that in practice, stock returns are often characterized by skewness and kurtosis. In particular, asset returns in inefficient markets do not follow the normal standard Brownian motion but are instead characterized by skewness and kurtosis, hence following skew Brownian motion. Through incorporating skew Brownian motion in the constraints of maximization equation (15), the desired maximization equation becomes

$$\max_{\pi\in\mathcal{A}_0(x)} E\big[u\big(X^{\pi}(T)\big)\big] \tag{16}$$
$$\text{s.t}$$
$$dX^{\pi}(t) = X^{\pi}(t)\Big[R(t)dt + \pi^{T}(t)\sigma(t)\Big(\phi(t)dt + \mu\big(Z(t)\big)dt + dW(t) + (2p-1)dL_t^0(Z)\Big)\Big]$$
$$X^{\pi}(0) = x$$

with

$$\mathcal{A}_0(x) = \{(\pi,0) \in \mathcal{A}(x): E\big[u^-\big(X^{\pi}(T)\big)\big] < \infty\}$$

and

$$u(x) = \frac{x^{1+\gamma}}{1+\gamma}, \qquad \gamma \neq -1 \tag{17}$$

in which γ represents the constant coefficient of risk aversion of an investor.
The solution of the optimal problem (16), will be obtained by using martingale method as used by [20]. in which the maximization equation (16) is equivalent to

$$\max_G E\left(u(G)\right)$$
$$\text{s.t}$$
$$E(P(T)G) = x \tag{18}$$

where G denotes all possible $\mathcal{F}(T)$-measurable contingent claims, given by

$$G = X^{\pi}(T)$$

and $P(t)$ is the stochastic discount factor defined by

$$P(t) = \exp\{-\int_0^t R(s)ds - \frac{1}{2}\int_0^t ||\phi(s)||^2\, ds - \int_0^t \phi^{\mathcal{T}}(s)dW(s)\} \tag{19}$$

$$\Psi(t) = e^{-\int_0^t R(s)ds},$$

or, simply

in which $\Psi(t)$ is given by

$$P(t) = \Psi(t)Y_0(t),$$

and $Y_0(t)$ is given by

$$Y_0(t) = e^{-\frac{1}{2}\int_0^t |\phi(s)|^2 ds - \int_0^t \phi^{\mathcal{T}}(s)dW(s)}. \tag{20}$$

The Lagrangian function of equation (18) is given by

$$L(G,\lambda) = E[u(G) + \lambda(x - P(T)G)],$$

where $\lambda(>0)$ is the Lagrangian multiplier whose value is obtained from the budget constraint of the maximization equation (18). From the first-order condition, the derivative of Lagrangian function with respect to G is equal to zero. Mathematically, first-order condition can be represented as

$$\frac{\partial L}{\partial G} = E[u'(G) - \lambda P(T)] = 0. \tag{21}$$

Solving for G from equation (21), the optimum contingent claim, denoted by G^* is given by

$$G^* = (u')^{-1}\left(\lambda P(T)\right). \tag{22}$$

For the constant relative risk aversion (CRRA) utility function (17), its inverse is given by

$$(u')^{-1}(x) = x^{\frac{1}{\gamma}}. \tag{23}$$

Using equation (23) in equation (22), the optimum can be written as

$$G^* = \left(\lambda P(T)\right)^{\frac{1}{\gamma}} = \lambda^{\frac{1}{\gamma}}\left(P(T)\right)^{\frac{1}{\gamma}}. \tag{24}$$

Recall that the Lagrangian multiplier $\lambda(>0)$ is obtained from the budget constraint

$$E[P(T)G] = x$$

$$E\left[P(T)\lambda^{\frac{1}{\gamma}}\left(P(T)\right)^{\frac{1}{\gamma}}\right] = x$$

$$E\left[\lambda^{\frac{1}{\gamma}}\left(P(T)\right)^{\frac{\gamma+1}{\gamma}}\right] = x$$

$$\lambda^{\frac{1}{\gamma}} = \frac{x}{E\left[\left(P(T)\right)^{\frac{\gamma+1}{\gamma}}\right]}. \quad (25)$$

Substituting equation (25) into equation (24), we obtain

$$G^* = \frac{x\left(P(T)\right)^{\frac{1}{\gamma}}}{E\left[\left(P(T)\right)^{\frac{\gamma+1}{\gamma}}\right]} \quad (26)$$

Corollary 1. If G^* is the optimum of the maximization equation (18), then there exists a portfolio $\pi^* \in \mathcal{A}(x)$ and π^* is the optimal for the terminal wealth optimization equation (16). The corresponding value of the optimal portfolio process satisfies

$$X^{\pi^*}(t) = \frac{1}{P(t)}E_t[P(T)G^*], \qquad t \in [0,T] \quad (27)$$

where E_t is the conditional expectation with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.
Substituting equation (26) in equation (27), the optimal wealth process becomes

$$X^{\pi^*}(t) = \frac{1}{P(t)}E_t[P(T)G^*]$$

$$= \frac{1}{P(t)}E_t\left[P(T)\frac{x\left(P(T)\right)^{\frac{1}{\gamma}}}{E\left[\left(P(T)\right)^{\frac{\gamma+1}{\gamma}}\right]}\right]$$

$$= \frac{x}{P(t)}\frac{E_t\left[\left(P(T)\right)^{\frac{\gamma+1}{\gamma}}\right]}{E\left[\left(P(T)\right)^{\frac{\gamma+1}{\gamma}}\right]}.$$

Multiplying by $P(t)$ both sides, we have

$$P(t)X^{\pi^*}(t) = x\frac{E_t\left[\left(P(T)\right)^{\frac{\gamma+1}{\gamma}}\right]}{E\left[\left(P(T)\right)^{\frac{\gamma+1}{\gamma}}\right]}. \quad (28)$$

Introducing the exponential martingale

$$Y(t) = e^{-\left(\frac{\gamma+1}{\gamma}\right)\int_0^t \phi^{\mathcal{T}}(s)dW(s) - \frac{1}{2}\left(\frac{\gamma+1}{\gamma}\right)^2\int_0^t ||\phi(s)||^2 ds}. \quad (29)$$

If we define a function $g(t)$ by

$$g(t) = e^{-\left(\frac{\gamma+1}{\gamma}\right)\int_0^t\left(R(s)ds - \frac{1}{2\gamma}||\phi(s)||^2\right)ds} \quad (30)$$

then

$$(P(t))^{\frac{\gamma+1}{\gamma}} = \exp\{\left(\frac{\gamma+1}{\gamma}\right)\left(-\int_0^t R(s)ds - \frac{1}{2}\int_0^t ||\phi(s)||^2\, ds - \int_0^t \phi^{\mathcal{T}}(s)dW(s)\right)\}$$

$$= \left(e^{-\left(\frac{\gamma+1}{\gamma}\right)\int_0^t\left(R(s)ds - \frac{1}{2\gamma}||\phi(s)||^2\right)ds}\right)\left(e^{-\left(\frac{\gamma+1}{\gamma}\right)\int_0^t \phi^{\mathcal{T}}(s)dW(s) - \frac{1}{2}\left(\frac{\gamma+1}{\gamma}\right)^2\int_0^t ||\phi(s)||^2 ds}\right)$$

$$= g(t)Y(t). \tag{31}$$

Noting that the martingale $Y(t)$ has an expectation of one, the fraction on the left-hand side of equation (28) can be written as

$$\frac{E_t\left[(P(T))^{\frac{\gamma+1}{\gamma}}\right]}{E\left[(P(T))^{\frac{\gamma+1}{\gamma}}\right]} = \frac{E_t[g(T)Y(T)]}{E[g(T)Y(T)]} = \frac{g(T)E_t[Y(T)]}{g(T)E[Y(T)]}$$

$$= Y(t). \tag{32}$$

Substituting equation (32) into equation (28), we have

$$P(t)X^{\pi^*}(t) = xY(t). \tag{33}$$

Differentiating, equation (33) becomes

$$d\left(P(t)X^{\pi^*}(t)\right) = xd\big(Y(t)\big),$$

$$= -xY(t)\left(\frac{\gamma+1}{\gamma}\right)\phi^{\mathcal{T}}(t)dW(t). \tag{34}$$

Substituting equation (33) into equation (34)

$$d\left(P(t)X^{\pi^*}(t)\right) = -P(t)X^{\pi^*}(t)\left(\frac{\gamma+1}{\gamma}\right)\phi^{\mathcal{T}}(t)dW(t). \tag{35}$$

Applying Itô formula on the stochastic discount factor $P(t)$, equation (19) can be written as

$$d\big(P(t)\big) = -P(t)[R(t)dt + \phi^{\mathcal{T}}(t)dW(t)]. \tag{36}$$

Using equation (36) and $d\big(X^{\pi}(t)\big)$ found in the maximization equation (16), then we can also compute $d\big(P(t)X^{\pi}(t)\big)$ using the product rule for the stochastic processes $P(t)$ and $X^{\pi}(t)$, to have

$$\begin{aligned} d\big(P(t)X^{\pi}(t)\big) &= X^{\pi}(t)d\big(P(t)\big) + P(t)d\big(X^{\pi}(t)\big) + d\big(P(t)\big)d\big(X^{\pi}(t)\big) \\ &= -P(t)X^{\pi}(t)[R(t)dt + \phi^{\mathcal{T}}(t)dW(t)] \\ &\quad + P(t)X^{\pi}(t)\Big[R(t)dt \\ &\quad + \pi^{\mathcal{T}}(t)\sigma(t)\left(\phi(t)dt + \mu\big(Z(t)\big)dt + dW(t)\right. \\ &\quad \left. + (2p-1)dL_t^0(Z)\right)\Big] - P(t)X^{\pi}(t)\phi^{\mathcal{T}}(t)\pi(t)\sigma(t)dt \end{aligned}$$

$$
\begin{aligned}
= -P(t)X^{\pi}(t)R(t)dt - P(t)X^{\pi}(t)\phi^{\mathcal{T}}(t)dW(t) + P(t)X^{\pi}(t)R(t)dt \\
+ P(t)X^{\pi}(t)\phi^{\mathcal{T}}(t)\pi(t)\sigma(t)dt \\
+ P(t)X^{\pi}(t)\pi^{\mathcal{T}}(t)\sigma(t)\mu\big(Z(t)\big)dt \\
+ P(t)X^{\pi}(t)\pi(t)\sigma^{\mathcal{T}}(t)dW(t) \\
+ P(t)X^{\pi}(t)\pi^{\mathcal{T}}(t)(2p-1)dL_t^0(Z) \\
- P(t)X^{\pi}(t)\phi^{\mathcal{T}}(t)\pi(t)\sigma(t)dt.
\end{aligned}
$$

Collecting like terms together we have

$$
d\big(P(t)X^{\pi}(t)\big) = P(t)X^{\pi}(t)[\sigma^{\mathcal{T}}(t)\pi(t) - \phi(t)]^{\mathcal{T}}dW(t) + P(t)X^{\pi}(t)\pi^{\mathcal{T}}(t)\sigma(t)\big[\mu\big(Z(t)\big)dt + (2p-1)dL_t^0(Z)\big]. \tag{37}
$$

Replacing $X^{\pi}(t)$ by $X^{\pi^*}(t)$, equation (37) becomes

$$
d\left(P(t)X^{\pi^*}(t)\right) = P(t)X^{\pi^*}(t)[\sigma^{\mathcal{T}}(t)\pi^*(t) - \phi(t)]^{\mathcal{T}}dW(t) + P(t)X^{\pi^*}(t)\pi^{\mathcal{T}}(t)\sigma(t)\big[\mu\big(Z(t)\big)dt + (2p-1)dL_t^0(Z)\big]. \tag{38}
$$

Comparing $dW(t)$ terms of equation (35) and equation (38), the optimal portfolio $\pi^*(t)$ is given by

$$
\sigma^{\mathcal{T}}(t)\pi^*(t) - \phi(t) = -\left(\frac{\gamma+1}{\gamma}\right)\phi(t) \quad = -\frac{1}{\gamma}\phi(t) - \phi(t)
$$

$$
\pi^*(t) = -\frac{1}{\gamma}\big(\sigma^{-1}(t)\big)^{\mathcal{T}}\phi(t). \tag{39}
$$

To obtain the optimal expected utility of the final wealth of an investor, we substitute G^* of equation (26) into the objective function of the maximization equation (16)

$$
\max_{\pi\in\mathcal{A}_0(x)} E\left[u\big(X^{\pi}(T)\big)\right] = E\big(u(G^*)\big)
$$

$$
= E\left[u\left(\frac{x\big(P(T)\big)^{\frac{1}{\gamma}}}{E\left[\big(P(T)\big)^{\frac{\gamma+1}{\gamma}}\right]}\right)\right]
$$

$$
= E\left[\frac{1}{1+\gamma}\left(\frac{x\big(P(T)\big)^{\frac{1}{\gamma}}}{E\left[\big(P(T)\big)^{\frac{\gamma+1}{\gamma}}\right]}\right)^{1+\gamma}\right]
$$

$$
= \frac{x^{1+\gamma}}{1+\gamma}E\left[\left(\frac{\big(P(T)\big)^{\frac{1}{\gamma}}}{E\left[\big(P(T)\big)^{\frac{\gamma+1}{\gamma}}\right]}\right)^{1+\gamma}\right]
$$

$$= \frac{x^{1+\gamma}}{1+\gamma}\left(E\left[\left(P(T)\right)^{\frac{\gamma+1}{\gamma}}\right]\right)^{-\gamma}. \tag{40}$$

Using equation (31) with $E[Y(T)] = 1$, we have

$$\max_{\pi\in\mathcal{A}_0(x)} E\left[u\left(X^\pi(T)\right)\right] = \frac{x^{1+\gamma}}{1+\gamma}\exp\{(\gamma+1)\int_0^T \left(R(t) - \frac{1}{2\gamma}||\phi(t)||^2\right)dt\} \tag{41}$$

Equation (41) is the desired model for optimal asset returns in informationally inefficient markets, whereby its corresponding optimal portfolio is defined in equation (39).

Verification theorem

The main task under this section is to check whether or not our constructed model (41) satisfies the Hamilton Jacobi Bellman (HJB) equation, and if it does, the verification theorem tells us that our obtained value function (model 41) is optimal.

Theorem 2.2. *(Verification theorem) [21, Theorem 19.6] Suppose that we have two functions*
$Q(t,x)$ *and* $f(t,x)$, *such that*

- Q *is sufficiently integrable, and solve the HJB equation*
$$\begin{cases} \dfrac{\partial Q}{\partial t}(t,x) + \sup\limits_{b\in B} \{F(t,x,b) + \mathcal{D}Q(t,x)\} = 0 \quad \forall(t,x) \in (0,T)\times\mathcal{R}^n \\ Q(T,x) = \Phi(x) \quad \forall x \in \mathcal{R}^n. \end{cases}$$
- *The function* $f(t,x)$ *is an admissible control law.*
- *For each fixed point* (t,x), *the*

supremum in the expression

$$\sup_{b\in B} \{F(x,t,b) + \mathcal{D}Q(t,x)$$

is attained by the choice $b = f(t,x)$

Then, the following holds:

1. *The optimal value function* $V(t,x)$ *to the control problem is given by*
$$V(t,x) = Q(t,x).$$
2. *There exists an optimal control law* $\hat{b}(t,x)$, *and in fact* $\hat{b}(t,x) = f(t,x)$.

Proof: See [21] □

Remark 2.3. $\mathcal{D}$ *in Theorem 2.2 is the partial differential operator defined by*

$$\mathcal{D} = \mu(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2}$$

in which $\mu(\cdot)$ *and* $\sigma(\cdot)$ *stand for the drift and diffusion coefficients for the corresponding dynamic equation respectively.*

Proposition 2.3. *If the optimal portfolio π^* is given by* (39) *with the market price of risk $\phi = \frac{\mu - R}{\sigma}$*
and $\mu = R + \sigma\sqrt{2R\gamma}$, then the value function

$$V(t,x) = \frac{x^{1+\gamma}}{1+\gamma} e^{(1+\gamma)\left(R - \frac{1}{2\gamma}\phi^2\right)t}$$

satisfies the HJB

$$\frac{\partial V}{\partial t}(t,x) + \sup_{\pi\in\mathcal{R},C=0} \{F(t,C) + \mathcal{D}V(t,x)\} = 0, \qquad (42)$$

and hence (from the verification theorem) $V(t,x)$ is an optimal value function.

Proof: The main aim is to show that the value function $V(t,x)$ satisfies the corresponding HJB. Now, assuming that the parameters γ, R, ϕ and μ are deterministic, then the value function (41) can be written as

$$V(t,x) = \frac{x^{1+\gamma}}{1+\gamma} e^{(1+\gamma)\left(R - \frac{1}{2\gamma}\phi^2\right)t},$$

from which we obtain

$$\frac{\partial V}{\partial t} = x^{1+\gamma}\left(R - \frac{1}{2\gamma}\phi^2\right) e^{(1+\gamma)\left(R - \frac{1}{2\gamma}\phi^2\right)t}.$$

$$\frac{\partial V}{\partial x} = x^{\gamma} e^{(1+\gamma)\left(R - \frac{1}{2\gamma}\phi^2\right)t}.$$

$$\frac{\partial^2 V}{\partial x^2} = \gamma x^{\gamma-1} e^{(1+\gamma)\left(R - \frac{1}{2\gamma}\phi^2\right)t}.$$

By using the dynamics given in equation (15), the corresponding HJB becomes

$$\frac{\partial V}{\partial t} + \sup_{\pi\in\mathcal{R},C=0} \{F(t,C) + (xR)\frac{\partial V}{\partial x} + (x\pi\sigma\phi)\frac{\partial V}{\partial x} + \left(\frac{1}{2}x^2\pi^2\sigma^2\right)\frac{\partial^2 V}{\partial x^2}\} = 0.$$

Substituting the optimal portfolio π^* obtained in equation (39) and letting the instantaneous utility function for consumption be given by

$$F(t,C) = e^{-\rho t}C^{\gamma}$$

where $\rho > 0$ is the rate of time preference, then we have

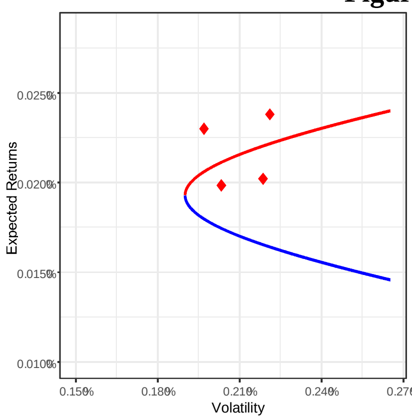
$$\frac{\partial V}{\partial t}(t,x) + \sup_{\pi\in\mathcal{R},C=0} \{F(t,C) + \mathcal{D}V(t,x)\}$$

$$= x^{1+\gamma}\left(R - \frac{1}{2\gamma}\phi^2\right)e^{(1+\gamma)\left(R-\frac{1}{2\gamma}\phi^2\right)t} + (xR)x^{\gamma}e^{(1+\gamma)\left(R-\frac{1}{2\gamma}\phi^2\right)t}$$
$$+ (x\pi\sigma\phi)x^{\gamma}e^{(1+\gamma)\left(R-\frac{1}{2\gamma}\phi^2\right)t} + \left(\frac{1}{2}x^2\pi^2\sigma^2\right)\gamma x^{\gamma-1}e^{(1+\gamma)\left(R-\frac{1}{2\gamma}\phi^2\right)t}.$$
$$= \left(2R - \frac{1}{2\gamma}\phi^2 + \pi\sigma\phi + \frac{1}{2}\pi^2\sigma^2\gamma\right)\left(x^{1+\gamma}e^{(1+\gamma)\left(R-\frac{1}{2\gamma}\phi^2\right)t}\right)$$
$$= \left(2R - \frac{1}{2\gamma}\pi^2\sigma^2\gamma^2 + \pi\sigma\phi + \frac{1}{2}\pi^2\sigma^2\gamma\right)\left(x^{1+\gamma}e^{(1+\gamma)\left(R-\frac{1}{2\gamma}\phi^2\right)t}\right) \quad \text{since } \pi^* = -\frac{\phi}{\gamma\sigma}$$
$$= (2R + \pi\sigma\phi)\left(x^{1+\gamma}e^{(1+\gamma)\left(R-\frac{1}{2\gamma}\phi^2\right)t}\right)$$
$$= \left(2R + \left(\frac{R-\mu}{\gamma\sigma^2}\right)\sigma\left(\frac{\mu-R}{\sigma}\right)\right)\left(x^{1+\gamma}e^{(1+\gamma)\left(R-\frac{1}{2\gamma}\phi^2\right)t}\right) \qquad \text{since } \phi = \frac{\mu-R}{\sigma}$$
$$= \left(2R - \frac{(R-\mu)^2}{\gamma\sigma^2}\right)\left(x^{1+\gamma}e^{(1+\gamma)\left(R-\frac{1}{2\gamma}\phi^2\right)t}\right)$$
$$= 0\cdot\left(x^{1+\gamma}e^{(1+\gamma)\left(R-\frac{1}{2\gamma}\phi^2\right)t}\right) \qquad \text{since } \mu = R + \sigma\sqrt{2R\gamma}$$
$$= 0.$$

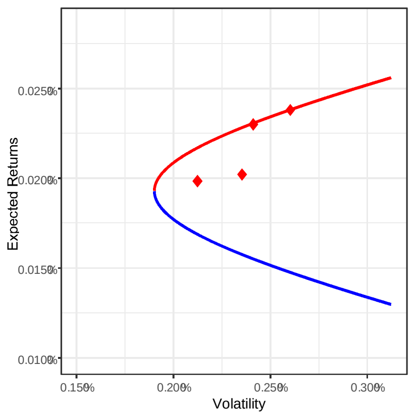
Since the value function $V(t,x)$ satisfies the HJB, it implies (from the verification theorem) that $V(t,x)$ is an optimal value function.

□

Figure 1: Efficient frontiers.



(a) Based on Markowitz model



(b) Based on the constructed model

3. Results and discussion

The comparison was made between the efficient frontier of the mean-variance Markowitz model [7] and our constructed model (41). The results were presented in Figure 1. Our constructed model (41) (Figure 1(b)) clearly outperforms the Markowitz mean-variance model (Figure 1(a)) because it has a higher efficient frontier (portfolios with higher expected returns or portfolios with lower standard deviation of return). Only two assets in Markowitz's model are within the frontier curve, whereas all four considered assets are within the efficient curve in our constructed model. Additionally, it can be clearly observed from the two frontier curves that Markowitz's mean variance model restricts the number of possible portfolios.

Table 1: Stability and consistency of the model when input parameters are perturbed.

Initial parameters γ = 1.00, R = 0.900, ϕ=-0.7100		
Perturbed parameters	Number of assets	Mean Squared Error (MSE)
	20	0.000236228
γ = 1.01, R = 0.901, ϕ=-0.7101	15	0.000240055
	10	0.000244013
	5	0.000248097
	20	0.000252327
γ = 1.02, R = 0.902, ϕ=-0.7102	15	0.000256651
	10	0.000261149
	5	0.000265772
	20	0.000275467
γ = 1.03, R = 0.903, ϕ=-0.7103	15	0.000280496
	10	0.000285698
	5	0.000291047
	20	0.000296556
γ = 1.04, R = 0.904, ϕ=-0.7104	15	0.000302217
	10	0.000308045
	5	0.000314034

According to [22], stability describes the situation where small variations of the output results correspond to small variations of the input parameters. Through small perturbations of the parameters in the constructed model (41), the stability has been tested, and the results were presented in Table 1. The closeness of the return from the original input parameters and the return from the perturbed parameters were tested using Mean Squared Error (MSE). MSE was found to be relatively small in each case, indicating that the difference between the original and perturbed parameter returns was also relatively small. It was discovered that MSE grew as the number of assets declined, supporting the theory that the more asset diversification, the better the returns. This finding indicates the consistency of the model.
It can be clearly seen from the table that small changes in parameters do not give rise to large changes in the return. This situation indicates that our constructed model (41), does not suffer from error maximization, hence the stability and robustness of the model.

4. Conclusions

In this paper, we have adapted the martingale method to construct the asset return model in informationally inefficient markets. As the constituent number of assets increases, the impact on return and variance increases, indicating that the constructed model was efficient. The performance of the model was tested by comparing the corresponding efficient frontier with that of the Markowitz mean-variance model, and it was found that our constructed model performs better than that of the Markowitz mean-variance model. Our constructed model was found to be stable as the small change in the input parameters did not result in a large change in the output.

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Conflicts of Interest. The authors declare no conflicts of interest.

Authors' contributions. All authors contributed equally to this work.

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