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A brief note

A Short Note on Two Diophantine Equations $9^x - 3^y = z^2$ and $13^x - 7^y = z^2$

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Abstract. In this short note, we show that the Diophantine equation $9^x - 3^y = z^2$ has all non-negative integer solutions $(x, y, z) \in \{(r, 2r, 0): r \in \mathbb{N} \cup \{0\}\}$ and the Diophantine equation $13^x - 7^y = z^2$ have the unique non-negative integer solution $(x, y, z) = (0,0,0)$.

Keywords: Diophantine equation; integer solution; Mihailescu's theorem

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In recent articles, Diophantine equations of the form $a^x - b^y = z^2$, where a, b are positive integers and x, y, z are non-negative integers, have been studied (see for instance [1, 2, 3, 4, 5, 7, 9, 10, 11, 12]). In 2022, Tadee [8] proved that the Diophantine equation $(p+6)^x - p^y = z^2$, where p is a prime number with $p \equiv 1 \pmod{28}$, has the unique non-negative integer solution $(x, y, z) = (0, 0, 0)$. In this paper, we will solve the Diophantine equation, for some prime p with $p \not\equiv 1 \pmod{28}$, which are $p = 3$ and $p = 7$.

2. Preliminary

Theorem 2.1. (Mihailescu's theorem) [6] The Diophantine equation $a^x - b^y = 1$ has the unique integer solution $(a, b, x, y) = (3, 2, 2, 3)$, where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.

3. Main results

Theorem 3.1. The Diophantine equation $9^x - 3^y = z^2$ has all non-negative integer solutions $(x, y, z) \in \{(r, 2r, 0): r \in \mathbb{N} \cup \{0\}\}$.

Proof: Let x, y and z be non-negative integers such that $9^x - 3^y = z^2$. Then $3^{2x} - z^2 = 3^y$ or $(3^x - z)(3^x + z) = 3^y$. There exists a non-negative integer r such that $3^x - z = 3^r$ and

$3^x + z = 3^{y-r}$. Then $y \geq 2r$ and $2 \cdot 3^x = 3^r(3^{y-2r}+1)$. This implies that $x = r$ and so $3^{y-2r} = 1$. Thus $y = 2r$. Consequently, $z = 0$. Hence, $(x, y, z) \in \{(r, 2r, 0) : r \in \mathbb{N} \cup \{0\}\}$ are all non-negative integer solutions of the Diophantine equation $9^x - 3^y = z^2$.

Theorem 3.2. The Diophantine equation $13^x - 7^y = z^2$ has only one non-negative integer solution $(x, y, z) = (0,0,0)$.

Proof: Let x, y and z be non-negative integers such that $13^x - 7^y = z^2$.

Case 1. $y = 0$. Therefore $13^x - z^2 = 1$. If $x = 1$, then $z^2 = 12$. This is impossible since z is an integer. If $x > 1$, then $z > 1$ and so $\min\{13, z, x, 2\} > 1$. By Theorem 2.1, we have $13 = 3$, a contradiction. Thus $x = 0$ and so $z = 0$. Then $(x, y, z) = (0,0,0)$ is a solution.

Case 2. $y > 0$. Then $7^y \equiv 0 \pmod 7$. This implies that $z^2 = 13^x - 7^y \equiv (-1)^x \pmod 7$. If x is odd, then $z^2 \equiv -1 \pmod 7$, which contradicts the fact that $z^2 \equiv 0, 1, 2, 4 \pmod 7$. Thus, x is even. There exists a non-negative integer k such that $x = 2k$. Then $13^{2k} - z^2 = 7^y$ or $(13^k - z)(13^k + z) = 7^y$. Thus $13^k - z = 7^u$ and $13^k + z = 7^{y-u}$, for some non-negative integer u. It follows that $y \geq 2u$ and $2 \cdot 13^k = 7^u(7^{y-2u}+1)$. If $u > 0$, then $7 \mid 2 \cdot 13^k$, a contradiction. Thus $u = 0$ and so $2 \cdot 13^k = 7^y + 1$. This is impossible since $7^y + 1 \equiv 1 \pmod 7$ and $2 \cdot 13^k \equiv 2(-1)^k \equiv -2, 2 \pmod 7$.

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