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On the Diophantine Equation $(p+6)^x - p^y = z^2$ where p is a Prime Number with $p \equiv 1 \pmod{28}$

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Abstract. This paper shows that the Diophantine equation $(p+6)^x - p^y = z^2$ where p is a prime number with $p \equiv 1 \pmod{28}$, has a unique non-negative integer solution. The solution is $(x, y, z) = (0,0,0)$.

Keywords: Diophantine equation; integer solution; Mihailescu's theorem

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In 2019, Thongnak, Chuayjan and Kaewong [8] proved that the Diophantine equation $2^x - 3^y = z^2$ has three non-negative integer solutions $(x, y, z) \in \{(0,0,0),(1,0,1),(2,1,1)\}$. In the same year, Burshtein [1] studied the Diophantine equation $(p+1)^x - p^y = z^2$ in which p is a prime number and x, y, z are positive integers with $x + y = 2,3,4$. Burshtein [2] showed that the Diophantine equation $6^x - 11^y = z^2$ has exactly one positive integer solution when $x = 2$, and no positive integer solution when $2 < x \le 16$. Burshtein [3] found all positive integer solutions of the Diophantine equation $p^x - p^y = z^2$, when p is a prime number.

In 2020, Burshtein [4] showed that the Diophantine equation $13^x - 5^y = z^2$ has exactly one positive integer solution $(x, y, z) = (2,2,12)$ and the Diophantine equation $19^x - 5^y = z^2$ has no positive integer solution. Elshahed and Kamarulhaili [5] studied all non-negative integer solutions of the Diophantine equation $(4^n)^x - p^y = z^2$, where p is odd prime and n is a positive integer. In 2021, Thongnak, Chuayjan and Kaewong [9] showed that $(x, y, z) = (0,0,0)$ is the unique non-negative integer solution of the Diophantine equation $7^x - 5^y = z^2$. In 2022, Tadee and Laomalaw [7] found all non-negative integer solutions of the Diophantine equation $2^x - p^y = z^2$, for some prime p.

In this paper, we solve the Diophantine equation of the form $(p+6)^x - p^y = z^2$, where p is a prime number with $p \equiv 1 \pmod{28}$ and x, y, z are non-negative integers.

2. Main results

We begin this section by presenting an important theorem.

Theorem 2.1. (Mihailescu's theorem) [6] The Diophantine equation $a^x - b^y = 1$ has the unique integer solution $(a,b,x,y) = (3,2,2,3)$, where a, b, x and y are integers with $\min\{a,b,x,y\} > 1$.

Lemma 2.1. Let p be an odd prime number. Then the Diophantine equation $1 - p^y = z^2$ has the unique non-negative integer solution $(y,z) = (0,0)$.

Proof: Let y and z be non-negative integers and (y, z) be a solution of the Diophantine equation $1 - p^y = z^2$. Then $(1-z)(1+z) = p^y$. Since p is prime, we have $1 - z = p^v$ and $1 + z = p^{y-v}$, for some non-negative integer v. Therefore $y \geq 2v$ and $2 = p^v(p^{y-2v} + 1)$. Since $p \neq 2$, we have $v = 0$ and so $2 = p^y + 1$. Then $y = 0$. It implies that $z = 0$. Hence, $(y,z) = (0,0)$ is the unique non-negative integer solution.

Lemma 2.2. Let p be a prime number with $p \equiv 1 \pmod{4}$. Then the Diophantine equation $(p+6)^x - 1 = z^2$ has the unique non-negative integer solution $(x,z) = (0,0)$.

Proof: Let x and z be non-negative integers and (x, z) be a solution of the Diophantine equation $(p+6)^x - 1 = z^2$. We consider three following cases.

Case 1. $x = 0$. Then $z^2 = 0$. Hence, $(x,z) = (0,0)$ is a solution.

Case 2. $x = 1$. Then $z^2 = p + 5$. Since $p \equiv 1 \pmod{4}$, we have $z^2 \equiv 2 \pmod{4}$, which contradicts the fact that $z^2 \equiv 0,1 \pmod{4}$.

Case 3. $x > 1$. It is easy to check that $z > 1$. Therefore $\min\{p+6, z, x, 2\} > 1$. Since $(p+6)^x - z^2 = 1$ and Theorem 2.1, we have $p + 6 = 3$, a contradiction.

Theorem 2.2. Let p be a prime number with $p \equiv 1 \pmod{28}$. Then $(x, y, z) = (0,0,0)$ is the unique non-negative integer solution of the Diophantine equation $(p+6)^x - p^y = z^2$.

Proof: Let x, y, z be non-negative integers and (x, y, z) be a solution of the Diophantine equation $(p+6)^x - p^y = z^2$. If $x = 0$ or $y = 0$, then $(x, y, z) = (0,0,0)$, by Lemma 2.1 and 2.2, respectively. Now, we consider case $x > 0$ and $y > 0$. Since $p \equiv 1 \pmod{28}$, it implies that $p \equiv 1 \pmod{4}$ and $p \equiv 1 \pmod{7}$. Therefore $(p+6)^x - p^y \equiv (-1)^x - 1 \pmod{4}$. Since $p \equiv 1 \pmod{4}$, we obtain that p and $p+6$ are odd. Thus, z^2 is even and so $z^2 \equiv 0 \pmod{4}$. Since $(p+6)^x - p^y = z^2$, it follows that $(-1)^x - 1 \equiv 0 \pmod{4}$. We see that $x = 2k$, for some positive integer k. Therefore $((p+6)^k - z)((p+6)^k + z) = p^y$. Since p is prime, there exists a non-negative integer u such that $(p+6)^k - z = p^u$ and $(p+6)^k + z = p^{y-u}$. Then $y \geq 2u$ and $2(p+6)^k = p^u(p^{y-2u} + 1)$.

Assume that $u > 0$. Then $p \mid 2(p+6)^k$ and so $p = 2$ or $p = 3$. This is impossible since $p \equiv 1 \pmod{28}$. Thus $u = 0$. Consequently, $2(p+6)^k = p^y + 1$. Since $p \equiv 1 \pmod 7$, we get $2(p+6)^k \equiv 0 \pmod 7$ and $p^y + 1 \equiv 2 \pmod 7$. Thus $0 \equiv 2 \pmod 7$, a contradiction.

Corollary 2.1. The Diophantine equation $35^x - 29^y = z^2$ has the unique non-negative integer solution $(x, y, z) = (0,0,0)$.

Proof: This corollary follows directly from Theorem 2.2.

Corollary 2.2. Let n be a positive integer and p be a prime number with $p \equiv 1 \pmod{28}$. Then the Diophantine equation $(p+6)^x - p^y = z^{2n}$ has the unique non-negative integer solution $(x, y, z) = (0,0,0)$.

Proof: Let a, b, c be non-negative integers such that $(p+6)^a - p^b = c^{2n}$. Then (a, b, c^n) is a non-negative integer solution (x, y, z) of the Diophantine equation $(p+6)^x - p^y = z^2$. By Theorem 2.2, we obtain that $(a, b, c^n) = (0,0,0)$. Then $a = b = c = 0$. Hence, $(0,0,0)$ is the unique non-negative integer solution of the equation $(p+6)^x - p^y = z^{2n}$.

3. Conclusion

In this article, the Diophantine equation $(p+6)^x - p^y = z^2$, when p is a prime number and x, y, z are non-negative integers, is investigated. We found that $(x, y, z) = (0,0,0)$ is the unique non-negative integer solution of the equation in the following cases: 1) $x = 0$ and $p \neq 2$, 2) $y = 0$ and $p \equiv 1 \pmod 4$, and 3) $p \equiv 1 \pmod{28}$. For example, if $p = 29$, then the Diophantine equation $35^x - 29^y = z^2$ has the unique non-negative integer solution $(x, y, z) = (0,0,0)$.

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Conflict of interest. The paper is written by single author so there is no conflict of interest.

Authors' Contributions. It is a single author paper. So, full credit goes to the author.

REFERENCES

1. N.Burshtein, All the solutions of the Diophantine equations $(p+1)^x - p^y = z^2$ and $p^x - (p+1)^y = z^2$ when p is prime and $x + y = 2,3,4$, *Annals of Pure and Applied Mathematics*, 19(1) (2019) 53-57.
2. N.Burshtein, A short note on solutions of the Diophantine equations $6^x + 11^y = z^2$ and $6^x - 11^y = z^2$ in positive integers x, y, z, *Annals of Pure and Applied Mathematics*, 19(2) (2019) 55-56.

3. N.Burshtein, All the solutions of the Diophantine equations $p^x + p^y = z^2$ and $p^x - p^y = z^2$ when $p \geq 2$ is prime, *Annals of Pure and Applied Mathematics*, 19(2) (2019) 111-119.
4. N.Burshtein, All the solutions of the Diophantine equations $13^x - 5^y = z^2$, $19^x - 5^y = z^2$ in positive integers x, y, z, *Annals of Pure and Applied Mathematics*, 22(2) (2020) 93-96.
5. A.Elshahed and H.Kamarulhaili, On the Diophantine equation $(4^n)^x - p^y = z^2$, *WSEAS Transactions on Mathematics*, 19 (2020) 349-352.
6. P.Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, *Journal für die Reine und Angewandte Mathematik*, 572 (2004) 167-195.
7. S.Tadee and N.Laomalaw, On the Diophantine equations $n^x - n^y = z^2$ and $2^x - p^y = z^2$, *Phranakhon Rajabhat Research Journal (Science and Technology)*, 17(1) (2022) 10-16.
8. S.Thongnak, W.Chuayjan and T.Kaewong, On the exponential Diophantine equation $2^x - 3^y = z^2$, *Southeast-Asian Journal of Sciences*, 7(1) (2019) 1-4.
9. S.Thongnak, W.Chuayjan, T.Kaewong, The solution of the exponential Diophantine equation $7^x - 5^y = z^2$, *Mathematical Journal by The Mathematical Association of Thailand Under The Patronage of His Majesty The King*, 66(703) (2021) 62-67.