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L2 Regularization Model with Removal of Gaussian Noise

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Abstract. In order to remove Gaussian noise from images, a Gaussian noise image restoration method based on the L2 norm regularization model was proposed. The L2 norm is selected as the data fidelity term and the gradient operator and wavelet frame as the regularization term to suppress the image ladder effect and protect the image edge details. Since the objective function of the model is a large convex function, the solving process is very tedious. The split Bregman iterative algorithm and alternate direction multiplier method are combined to restore the image. The experimental results show that show that the alternate direction multiplier method can effectively reduce the difficulty of solving the restoration model, and the image recovered by using this model has a higher peak signal-to-n ratio and better structural similarity and can get a clearer image.

Keywords: image denoising; L2 regularization model; Split Bregman iterative algorithm; Alternate direction multiplier method

AMS Mathematics Subject Classification (2010): 68U10

1. Introduction

In the field of image processing, noise is one of the most common degradation factors in the process of image acquisition, transmission and storage. Sensor defects, channel defects and electromagnetic interference may all produce noisy images. Gaussian noise is one of the basic noises in the field of image noise. It has a small impact on the edge and texture of the image, but has a big impact on the smooth area of the image. Image denoising is to improve the preservation of important details such as edges and textures while removing the noise of digital images. It can be divided into transform domain denoising, spatial domain denoising, partial differential equation (PDE) denoising and variational method denoising. Many classical algorithms have been proposed to remove Gaussian noise in images. [1-4]

In 2010, aiming at the shortage of the traditional Gaussian noise denoising algorithm with large residual noise, Ju et al. [5] proposed a denoising algorithm based on pixel neighborhood correlation according to the impact of noise on image vision. In 2017, in order to improve the image degradation caused by multiple noises, Zhao et al. [6] proposed a Poisson-Gaussian noise image restoration algorithm based on a variational model. In 2019, Xie et al. [7] proposed a second-order total generalized variational image denoising algorithm based on the alternate direction multiplier method (ADMM). In 2020, aiming at the Gaussian noise image restoration problem, Deng et al. [8] proposed an image restoration model based on mixed regularization of TV and Laplacian, and proposed an efficient alternate direction multiplier method to solve it. In 2021, aiming at the problems of fuzzy edge details and low efficiency of the traditional image denoising algorithm when solving the problem of crack image denoising, Fan et al. [9] proposed a crack image denoising algorithm (PCNN-FABF) based on the simplified model of pulsed coupled neural network (PCNN) and fast adaptive bilateral filtering.

Aiming at the deficiency that the total variation regularization model is easy to produce ladder effect and lose the image texture information, this paper takes the regularization model of L2 norm as the basis, selects the gradient operator and the wavelet frame [10] which can eliminate the ladder effect of the image as the regularization term, and proposes a Gaussian noise image restoration method based on L2 norm regularization model. The L2 norm is used as the data fidelity term to suppress noise and prevent data from over-fitting. The texture information of the image is more accurately maintained. The gradient operator and wavelet framework are used as the regularization term to suppress the image ladder effect. In the process of solving the model algorithm, the ADMM algorithm [11] and the split Bregman iterative algorithm [12] were used to decompose the objective function into multiple sub-problems for solving. Finally, this method is compared with other methods based on Peak Signal to Noise Ratio (PSNR) and Structural Similarity Index Measurement (SSIM). Verify the effectiveness of the proposed algorithm and the clarity of the restored image.

2. Regularization image restoration algorithm research

2.1. Proposed restoration model

The restoration model of Gaussian noise image can be reduced to an inverse problem, that is, the unknown real clear image can be recovered from the acquired Gaussian noise image:

$$f = Ku + \varepsilon \tag{1}$$

Among them, ε is Gaussian noise， u is a clear image that is unblurred and unpolluted by noise， f is an observed Gaussian noise image， K represents the fuzzy kernel of the noisy image.

The research shows that the total variation regularization (TV) image restoration model can not only effectively remove the noise, but also retain the edge features and sharpness of the image. Therefore, it has been well applied in the field of image restoration. According to the type of noise in the image, this paper selects the L2 norm suitable for removing Gaussian noise as the data fidelity term. For the image ladder effect caused by TV model, this paper takes the total variational function as the regular term or the L2 norm of wavelet coefficient as the regular term to effectively suppress the image ladder effect. The linear combination of the two is used as the regular term for better image restoration. In this regard, we propose the following Gaussian noise image restoration model:

$$\min_{u,K} \frac{\lambda}{2}\|Ku - f\|_2^2 + \alpha\|\nabla u\|_2 + \beta\|Wu\|_2 + \gamma\|u\|_2 \tag{2}$$

In equation (2) α,β,γ is the regularized adjustment parameter, $\|\nabla u\|_2=\sqrt{u_x^2+u_y^2}$, u_x and u_y respectively represent clear images u partial derivatives in the horizontal and vertical directions, W represents the wavelet analysis operator matrix, in order to facilitate the fast operation of the algorithm, take the W tight frame transformation, mean $W^TW=I$, W^T is the wavelet synthesis operator matrix, I is the identity matrix.

The first term in Equation (2) is the data fidelity term, keep the details of the real clear image. The remaining items are regular items to improve the clarity of the restored image. The restoration model proposed in this paper is a multi-variable solution model. The split Bregman iterative algorithm is used to split the global problem into several local sub-problems, and the ADMM algorithm is used to solve the problem, which simplifies the calculation process and improves the accuracy of the solution.

2.2. Restoration model solution

2.2.1. Real image solution

The real image is solved, in order to simplify the calculation process, and auxiliary variables are introduced $Ku-f=m,\nabla u=n,Wu=t$, Transform the objective function (2) into:

$$\min_{u,m,n,t}\frac{\lambda}{2}\|m\|_2^2+\alpha\|n\|_2+\beta\|t\|_2+\gamma\|u\|_2 \tag{3}$$

The restoration problem model is:

$$\begin{cases}\min\limits_{u,K}\frac{\lambda}{2}\|Ku-f\|_2^2+\alpha\|\nabla u\|_2+\beta\|Wu\|_2+\gamma\|u\|_2\\ s.t. Ku-f=m,\nabla u=n,Wu=t\end{cases}, \tag{4}$$

The augmented Lagrange function [13] corresponding to equation (3) is

$$\psi(u,m,n,t,a,b,c)=\min_{u,m,n,t}\frac{\lambda}{2}\|m\|_2^2+\alpha\|n\|_2+\beta\|t\|_2+\gamma\|u\|_2+\eta_1\langle Ku-f-m,a\rangle+ \\ \frac{\eta_1}{2}\|Ku-f-m\|_2^2+\eta_2\langle\nabla u-n,b\rangle+\frac{\eta_2}{2}\|\nabla u-n\|_2^2+\eta_3\langle Wu-t,c\rangle+\frac{\eta_3}{2}\|Wu-t\|_2^2 \tag{5}$$

In Equation (5) $\eta_1,\eta_2,\eta_3>0$ is the three penalty parameters, a,b,c is three Lagrange multipliers. For equation (5), the split Bregman iterative algorithm is used to decompose into the following sub-problems for solving:

$$u^{k+1}=\arg\min_u\gamma\|u\|_2+\eta_1\left\langle Ku-f-m^k,a^k\right\rangle \\ +\frac{\eta_1}{2}\left\|Ku-f-m^k\right\|_2^2+\eta_2\left\langle\nabla u-n^k,b^k\right\rangle \\ +\frac{\eta_2}{2}\left\|\nabla u-n^k\right\|_2^2+\eta_3\left\langle Wu-t^k,c^k\right\rangle+\frac{\eta_3}{2}\left\|Wu-t^k\right\|_2^2 \tag{6}$$

$$m^{k+1}=\arg\min_{m}\frac{\lambda}{2}\|m\|_2^2-\eta_1\left\langle m-\left(Ku^{k+1}-f\right),a^k\right\rangle$$
$$+\frac{\eta_1}{2}\left\|m-\left(Ku^{k+1}-f\right)\right\|_2^2 \tag{7}$$

$$a^{k+1}=a^k+Ku^{k+1}-f-m^{k+1} \tag{8}$$

$$n^{k+1}=\arg\min_{n}\alpha\|n\|_2-\eta_2\left\langle n-\nabla u^{k+1},b^k\right\rangle+\frac{\eta_2}{2}\left\|n-\nabla u^{k+1}\right\|_2^2 \tag{9}$$

$$b^{k+1}=b^k+\nabla u^{k+1}-n^{k+1} \tag{10}$$

$$t^{k+1}=\arg\min_{t}\beta\|t\|_2-\eta_3\left\langle t-Wu^{k+1},c^k\right\rangle+\frac{\eta_3}{2}\left\|t-Wu^{k+1}\right\|_2^2 \tag{11}$$

$$c^{k+1}=c^k+Wu^{k+1}-t^{k+1} \tag{12}$$

To (6) for the gradient and make it to 0, then

$$\eta_1K^T\left(Ku-f-m^k+a^k\right)+\eta_2\nabla^T\left(\nabla u-n^k+b^k\right)+$$
$$\eta_3W^T\left(Wu-t^k+c^k\right)-\gamma\frac{u}{\left\|u^k\right\|_2}=0 \tag{13}$$

Be available by $W^TW=I$

$$\eta_1K^T\left(f+m^k-a^k\right)+\eta_2\nabla^T\left(n^k-b^k\right)+\eta_3W^T\left(t^k-c^k\right)$$
$$=\left(\eta_1K^TK+\eta_2\nabla^T\nabla+\eta_3-\frac{\gamma}{\left\|u^k\right\|_2}\right)u \tag{14}$$

The fast Fourier transform of equation (16) is obtained:

$$u^{k+1}=F^{-1}\left(\frac{U}{V}\right) \tag{15}$$

$$U=\eta_1F(K)^T*F\left(f+m^k-a^k\right)+\eta_2F(\nabla)^T*F\left(n^k-b^k\right)$$
$$+\eta_3F(W)^T*F\left(t^k-c^k\right) \tag{16}$$

$$V=\eta_1F(K)^T*F(K)+\eta_2F(\nabla)^T*F(\nabla)-\eta_3\frac{\gamma}{\left\|u^k\right\|_2} \tag{17}$$

Equation (15) F represents the two-dimensional Fourier transform, F^{-1} represents the inverse Fourier transform of two dimensions, * represents component multiplication, Similarly, the division of the above equation is component division.

For sub-problem equations (7), (9) and (11), two-dimensional shrinkage algorithm is used to solve, then

$$m^{k+1}=\max\left(\left\|a^k+Ku^{k+1}\right\|_2-\frac{1}{\eta_1},0\right)\cdot\frac{a^k+Ku^{k+1}}{\left\|a^k+Ku^{k+1}\right\|_2} \tag{18}$$

$$n^{k+1}=\max\left(\left\|b^k+\nabla u^{k+1}\right\|_2-\frac{1}{\eta_2},0\right)\cdot\frac{b^k+\nabla u^{k+1}}{\left\|b^k+\nabla u^{k+1}\right\|_2} \tag{19}$$

$$t^{k+1}=\max\left(\left\|c^k+Wu^{k+1}\right\|_2-\frac{1}{\eta_3},0\right)\cdot\frac{c^k+Wu^{k+1}}{\left\|c^k+Wu^{k+1}\right\|_2} \tag{20}$$

2.2.2. Fuzzy kernel solution

For the solution of fuzzy kernel, the regularization term of real image u in restoration model (2) is ignored, Keeping the regularization term of the fuzzy kernel A the following solution model is obtained

$$\min_K \frac{\lambda}{2}\left\|Ku-f\right\|_2^2 \tag{21}$$

Take the derivative of equation (21) and set it to 0, and then apply the fast Fourier transform to it

$$K=F^{-1}\left(\frac{F(u)^T*F(f)}{F(u)^T*F(u)}\right) \tag{22}$$

In equation (22), F denotes the two-dimensional Fourier transform， F^{-1} denotes the inverse two-dimensional Fourier transform,* represents component multiplication， Similarly, the division of the above equation is component division.

2.2.3. Image restoration algorithm

Algorithm: Regularized Gaussian noise image restoration algorithm

input： $\eta_1,\eta_2,\eta_3,\alpha,\beta,\lambda$

Output: u

Step 1 Set the parameters $k=0$, the maximum number of iterations is s；

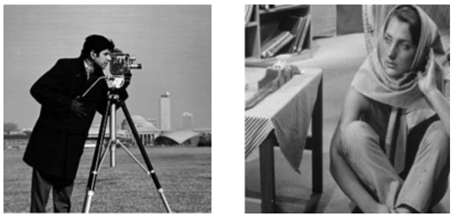
Step 2 Calculate u^{k+1} by Equation (15)； Calculate A by Equation (22)；
Calculate m^{k+1} by Equation (18)； Calculate a^{k+1} by Equation (8)；
Calculate n^{k+1} by Equation (19)； Calculate b^{k+1} by Equation (10)；
Calculate t^{k+1} by Equation (20)； Calculate c^{k+1} by Equation (12)；

Step 3 Remember $k=k+1$， if $\frac{\left\|u^{k+1}-u^k\right\|_2^2}{\left\|u^k\right\|_2^2}<r$ or $k>s$， proceed to the next step, otherwise proceed to (2)；

Step 4 Obtain a clear image u

3. Experimental results and analysis

The hardware equipment of the experiment is processor：Intel Core(TM)i5-4210U CPU@1.70GHz2.40GHz memory is 4.00GB； The system is Windows10 64-bit operating system. The running environment is MATLAB R2021a. In order to verify the effectiveness of the image restoration model proposed in this paper based on L2 regularization Gaussian noise, this paper uses two standard images to add different types of Gaussian noise respectively, and then uses the literature algorithm [14], literature algorithm [15], literature algorithm [16] and this algorithm to denoise different fuzzy images. The test image in this paper is shown in Figure 1.



(a) cameraman (b) barbara

Figure 1: Standard test image

3.1. Comparison of subjective effects

By comparing the restoration effect through intuitive visual sense, the restoration results of the literature algorithm [14], the literature algorithm [15], the literature algorithm [16] and the algorithm in this paper under different variances are presented. Figure 2 and 3 respectively show the restoration results of Gaussian noise image with added mean 0 and variance 0.06 under different algorithms; Figure 4 and Figure 5 respectively show the restoration results of Gaussian noise image with added mean 0 and variance 0.08 under different algorithms.

(a) (b) (c) (d) (e) (f)

Figure 2: Restoration results of the "cameraman" image with a variance of 0.06
(a) Standard image (b) Noise image (c) Reference [14] algorithm (d) Reference [15] algorithm (e) Reference [16] algorithm (f) proposed algorithm

(a) (b) (c) (d) (e) (f)

Figure 3: Restoration results of the " barbara" image with a variance of 0.06
(a) Standard image (b) Noise image (c) Reference [14] algorithm (d) Reference [15] algorithm (e) Reference [16] algorithm (f) proposed algorithm

(a) (b) (c) (d) (e) (f)

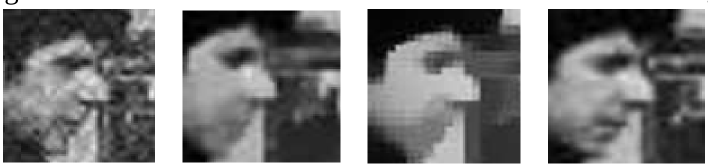
Figure 4: Restoration results of the "cameraman" image with a variance of 0.08
(a) Standard image (b) Noise image (c) Reference [14] algorithm (d) Reference [15] algorithm (e) Reference [16] algorithm (f) proposed algorithm

(a) (b) (c) (d) (e) (f)

Figure 5: Restoration results of the " barbara" image with a variance of 0.08
(a) Standard image (b) Noise image (c) Reference [14] algorithm (d) Reference [15] algorithm (e) Reference [16] algorithm (f) proposed algorithm

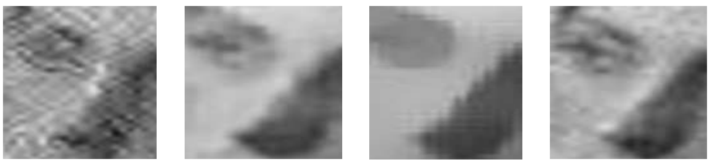
Through the comparison of the above experimental results, it can be seen that the restoration results of the method in literature [16] are too smooth, and many details such as edges and textures are seriously lost. Compared with literature [16], the denoising effect of the algorithm in literature [15] has been improved. Although the restored image retains the details well, the ladder effect is not effectively suppressed in the image. Literature [14] cannot completely remove the noise in the image, and the image brightness is low, which makes the restored image tend to be fuzzy. However, the algorithm in this paper has the best restoration quality. The edge contour of the restored image is clearer, more edge information and texture details are maintained, and the result is closer to the original clear image.

In order to make subjective visual evaluation more clearly, partial experimental images were enlarged to further compare the denoising effect of the images. The local amplification of the restoration results of the cameraman image with Gaussian noise with a variance of 0.06 is shown in Figure 6. The local amplification of the restoration results of barbara image with Gaussian noise with a variance of 0.08 is shown in Figure 7.



(a) (b) (c) (d)

Figure 6: Local magnification of the cameraman image with Gaussian noise of 0.06
(a) Reference [14] algorithm (b) Reference [15] algorithm (c) Reference [16] algorithm (d) Proposed algorithm



(a) (b) (c) (d)

Figure 7: Local magnification of the barbara image with Gaussian noise of 0.08
(a) Reference [14] algorithm (b) Reference [15] algorithm (c) Reference [16] algorithm (d) Proposed algorithm

By observing the local magnification comparison restored by different algorithms in Figure 6 and Figure 7, it can be seen that Figure 6(c) is difficult to see the outline of the portrait and the surrounding details, resulting in a ladder effect. Although figure 6(a) and figure6 (b) can clearly see the outline of the portrait, figure6 (a) can not completely remove the noise, figure6 (b) is too smooth; Figure 6(d) can not only see the outline and details of the portrait, but also suppress the ladder effect. In FIG. 7(a) and FIG. 7(c), it is difficult to see the contour of the eyes, eyebrows and image details; FIG. 7(b) Although the outline of the eyes can be clearly seen, the image still has some blur; In Figure 7(d), not only the Gaussian noise is removed, but also the edge information of the enlarged image is well preserved.

3.2. Objective evaluation indicators

To further objectively verify the effectiveness of the proposed method, two indexes of peak signal-to-noise ratio (PSNR) and structural similarity ratio (SSIM) were used to evaluate the restored images. The larger the peak signal-to-noise ratio (PSNR) is, the closer the structural similarity ratio (SSIM) is to 1, indicating that the closer the restored image is to the real clear image, the better the restoration effect will be. The opposite effect is worse [17].

Table 1: Objective evaluation of the restoration results of tcameraman image

Method	Variance	PSNR	SSIM
Methods in Literature [14]	0.06	28.3999	0.8639
	0.08	27.2688	0.9151
Methods in Literature [15]	0.06	21.8217	0.9074
	0.08	20.0729	0.8777
Methods in Literature [16]	0.06	21.4273	0.9222
	0.08	21.3016	0.8192
In this paper	0.06	30.2666	0.9631
	0.08	29.4638	0.9205

Table 2: Objective evaluation of restoration results for barbara image

Method	Variance	PSNR	SSIM
Methods in Literature [14]	0.06	23.9412	0.8814
	0.08	21.6383	0.8311
Methods in Literature [15]	0.06	27.7610	0.9085
	0.08	26.0719	0.8243
Methods in Literature [16]	0.06	22.0143	0.8459
	0.08	22.0123	0.8261
In this paper	0.06	29.2823	0.9520
	0.08	28.1663	0.9113

Table 1 and Table 2 respectively show the experimental data of denoising with reference [14], reference [15], reference [16] and the model proposed in this paper after adding Gaussian noise with mean of 0 and variance of 0.06 and 0.08 for the images of cameraman and barbara respectively. The peak signal to noise ratio and structural similarity were used to evaluate the denoised image objectively. By analyzing the data in Table 1 and

Table 2, it can be seen that compared with the other three models, the model proposed in this paper has improved the peak signal to noise ratio (PSNR) and structural similarity ratio (SSIM) values after image denoising. Therefore, the Gaussian noise image restoration model proposed in this paper has the best denoising performance, can obtain a better clear image, effectively suppress the ladder effect, and obtain more useful details.

4. Conclusion

In this paper, a Gaussian noise image restoration model based on L2 norm regularization is proposed to solve the problem that the TV model can save the edge features of the image well, but it can produce the step effect in processing the relatively smooth area of the image. The split Bregman iterative algorithm and ADMM algorithm are combined to achieve algorithm optimization, fuzzy kernel estimation and original clear image restoration. The comparison data show that the proposed algorithm has a more ideal restoration effect, can effectively suppress the ladder effect, and better preserve the image edge and texture information. This method not only improves the subjective visual effect of the restored images, but also has better performance in the aspects of peak signal-to-noise ratio and structural similarity.

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