Journal of Mathematics and Informatics
Vol. 23, 2022, 23-39
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 11 December 2022
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v23a03212

Journal of Mathematics and Informatics

Existence of Solutions for Quasilinear Coupled Systems with Multiple Critical Nonlinearities

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Received 30 October 2022; accepted 10 December 2022

Abstract. This paper is dedicated to investigating a quasilinear elliptic system with p-Laplacian in R^N, which involves critical Hardy-Littlewood-Sobolev nonlinearities and critical Sobolev nonlinearities. Based upon the Hardy-Littlewood-Sobolev inequality and variational methods, we obtain the attainability of the corresponding best constants and the existence of nontrivial solutions.

Keywords: Hardy-Littlewood-Sobolev inequality; quasilinear elliptic system; multiple critical nonlinearities; variational methods

AMS Mathematics Subject Classification (2010): 35B33, 35J50

1. Introduction

In this paper, we study the existence of solutions for the following multiple critical quasilinear coupled system in R^N:

$$\begin{cases} \mathcal{L}_{\nu,p}u = \left(I_\mu * |v|^{p_\mu^*}\right)|u|^{p_\mu^*-2}u + \sum_{i=1}^{m}\frac{\eta_i\alpha_i}{p^*}|u|^{\alpha_i-2}u|v|^{\beta_i}, \\ \mathcal{L}_{\nu,p}v = \left(I_\mu * |u|^{p_\mu^*}\right)|v|^{p_\mu^*-2}v + \sum_{i=1}^{m}\frac{\eta_i\beta_i}{p^*}|u|^{\alpha_i}|v|^{\beta_i-2}v, \end{cases} \tag{1}$$

where $N>3$, $1<p<N$, $\mu\in(0,N)$, $I_\mu(x)=\dfrac{A_\mu}{|x|^{N-\mu}}$ is a Riesz potential for all $x\in R^N\setminus\{0\}$, $A_\mu=\dfrac{\Gamma(\frac{N-\mu}{2})}{\Gamma(\frac{\mu}{2})2^\mu\pi^{N/2}}$; $\mathcal{L}_{\nu,p}\cdot := -\Delta_p\cdot - \nu\dfrac{|\cdot|^{p-2}\cdot}{|x|^p}$, and $\Delta_p=\mathrm{div}(|\nabla\cdot|^{p-2}\nabla\cdot)$ is the classica p-Laplacian; $0\le\nu<\bar{\nu}$ with $\bar{\nu}:=\left(\dfrac{N-p}{p}\right)^p$, $p_\mu^*=\dfrac{(N+\mu)p}{2(N-p)}$ is the Hardy-

Littlewood-Sobolev upper critical exponent; $\eta_i > 0$, $\alpha_i, \beta_i > 1$, and $\alpha_i + \beta_i = p_0^*$ $(i = 1, 2, \cdots, m \in N)$; $p_0^* = p^* := \dfrac{Np}{N-p}$ is the Sobolev critical exponent.

The problem (1) originates from the following nonlinear Choquard equation

$$-\Delta u + V(x)u = \left(I_\mu * |u|^q\right)|u|^{q-2}u, \quad \text{in } R^N, \tag{2}$$

where $\dfrac{N+\mu}{N} \le q \le \dfrac{N+\mu}{N-2}$, $\mu \in (0, N)$. If $N = 3$, $V(x) = 1$, $\mu = 2$ and $q = 2$, the problem (2) is simplified to the Choquard-Pekar equation, which was proposed by Pekar [1] in 1954 to describe the quantum theory of a polaron. In 1976, Choquard applied it to explain an electron trapped in its own hole, as an approximation to Hartree-Fock theory of one-component plasma [2]. For more relevant research on the Choquard equation, please refer to [3-7] and the references therein.

In the last decades, the problem of solutions for differential systems is widely studied [8-10]. In particular, the elliptic partial differential with multiple critical nonlinear have attracted many scholars' attention. For instance, we refer, the Laplacian and fractional Laplacian to [11-14], the *p*-Laplacian to [15-17] and the biharmonic operator to [18]. Compared with only one critical nonlinear term, the asymptotic competition between the energy carried by different critical nonlinear terms makes the problem more difficult. Among these, Filippucci, Pucci and Robert [15] were concerned with the *p*-Laplacian equation with doubly critical nonlinearities as follows

$$-\Delta_p u - \nu \frac{|u|^{p-2}u}{|x|^p} = |u|^{p^*-2}u + \frac{|u|^{p^*(\varsigma)-2}u}{|x|^\varsigma}, \quad \text{in } R^N, \tag{3}$$

where $N \ge 2, \nu \in (-\infty, \bar{\nu}), p \in (1, N), \varsigma \in (0, p), p^* = \dfrac{Np}{N-p},$ and $p^*(\varsigma) = \dfrac{p(N-\varsigma)}{N-p}$ is the Hardy-Sobolev critical exponent. Based on truncation skills, the authors verified that there exists a positive weak solution to the problem (3) in R^N via Mountain Pass Lemma and some analytical techniques. Later on, Ghoussoub and Shakerian [14] extended the problem (3) to the case of fractional Laplacian, and studied the existence of solutions through the *s*-harmonic extension and concentration compactness principle.

To the best of our knowledge, the classical methods mentioned above for solving doubly critical problems are no longer directly applicable to the Choquard equation because the convolution terms are nonlocal. For example, Lei and Zhang [19] considered the doubly critical nonlinearities with the upper and lower Hardy-Littlewood-Sobolev critical exponents, and overcame the lack of compactness caused by the doubly critical nonlinearities with the help of Pohozǎev-type identity. When the Sobolev-Hardy term of (3) is replaced by $\left(I_\mu * |u|^{p_\mu^*}\right)|u|^{p_\mu^*-2}u$, Shen [20] proved the existence of nontrivial solutions through the variational method together with the refinement of Hardy-Littlewood-Sobolev inequality in [21]. While in [22], Su, Chen, Liu and Che considered the following *p*-Laplacian equation with multiple critical nonlinearities

$$-\Delta_p u - \nu \frac{|u|^{p-2}u}{|x|^p} = \sum_{i=1}^{m}\left(I_{\mu_i} * |u|^{p_{\mu_i}^*}\right)|u|^{p_{\mu_i}^*-2}u + |u|^{p^*-2}u, \quad \text{in } R^N, \tag{4}$$

where $N=3,4,5,\ \nu\in[0,\bar{\nu}),\ p\in(1,2],\ p^*=\frac{Np}{N-p},\ p^*_{\mu_i}=\frac{(N+\mu_i)p}{2(N-p)}$, The authors established the refined Sobolev inequality with the Coulomb norm. Under some conditions about the parameters μ_i, they verified that the problem (4) admits a nonnegative ground state solution by variational methods. After that, for $N\geq 3$, $p\in(1,N)$, Xia and Su [23] obtained the same conclusion as [22] while relaxing the conditions required in [22].

Recently, the doubly critical coupled systems have also been studied by scholars (see [24-26]). Wang and Zhang [25] generalized the equation in [13] to the following doubly critical fractional elliptic system in R^N

$$\begin{cases}(-\Delta)^s u-\zeta\frac{u}{|x|^{2s}}=(I_\mu*|u|^{2^\sharp_{s,\mu}})|u|^{2^\sharp_{s,\mu}-2}u+\frac{|u|^{2^*_s(b)-2}u}{|x|^b}+\frac{\eta\alpha}{\alpha+\beta}\frac{|u|^{\alpha-2}u|v|^\beta}{|x|^b},\\ (-\Delta)^s v-\zeta\frac{v}{|x|^{2s}}=(I_\mu*|v|^{2^\sharp_{s,\mu}})|v|^{2^\sharp_{s,\mu}-2}v+\frac{|v|^{2^*_s(b)-2}v}{|x|^b}+\frac{\eta\beta}{\alpha+\beta}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^b},\end{cases} \tag{5}$$

where $s\in(0,1)$, $0\leq\zeta<4^s\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$, $\eta\in[0,+\infty)$, $0<\mu,b<2s<N$, $\alpha,\beta>1$, $\alpha+\beta=2^*_s(b)=\frac{2(N-b)}{N-2s}$ is the fractional Hardy-Sobolev critical exponent, and $2^\sharp_{s,\mu}=\frac{N+\mu}{N-2s}$ is the fractional Hardy-LittlewoodSobolev critical upper exponent. By utilizing the variational methods, the authors established the extremal function of the best constants and proved the existence of solutions for the problem (5). The main idea comes from the discussion of a single equation by Yang and Wu [13]. Subsequently, Yang [26] established the improved Sobolev inequality with partial singular weight and extended the results in [27] to the doubly critical coupled systems. Li and Yang [24] established two new improved Sobolev inequalities and overcame the difficulties caused by the strongly coupled terms in the doubly critical elliptic systems they studied. Finally, they proved the existence of a nontrivial weak solution with the same idea as [11].

Based on the above researches, we naturally raise a question: whether the corresponding best constant and nontrivial solution exist for the elliptic system with multiple critical nonlinearities and strongly coupled terms like (1)? We emphasize that in addition to the difficulty caused by the nonlocality of convolution terms, the strong coupling makes the previous works in [13, 20, 22, 23, 25] no longer fully applicable to the problem (1). For this reason, we introduce some crucial tools in Section 2, which are useful to prove the existence of minimizers for the corresponding best constant and are also important to exclude the case where the (PS) sequence is zero.

This paper is organized as follows. The appropriate functional framework for the system (1) and the main results are presented in Section 2. The extremals of the corresponding best constants are achieved in Section 3, and the existence of solutions to (1) is proved in Section 4.

2. Preliminaries and main results

Throughout this article, we will utilize $C, \tilde{C}, C_i$ $(i=1,2,\cdots)$ to represent the positive constants, which vary from line to line. We employ $B_r(x)$ to stand for the ball with radius r and center x in R^N. And for convenience, let $I_\mu(x)=|x|^{\mu-N}$.

Consider the usual Sobolev space $D^{1,p}(R^N)$, which is the completion of $C_0^\infty(R^N)$ with the norm $\left(\int_{R^N}|\nabla u|^p dx\right)^{\frac{1}{p}}$. We recall the standard Hardy inequality (see [28, Lemma 2.1])

$$\bar{\nu}\int_{R^N}\frac{|u|^p}{|x|^p}dx \le \int_{R^N}|\nabla u|^p dx, \quad \forall u\in D^{1,p}(R^N), \tag{6}$$

where $\bar{\nu}=\left(\frac{N-p}{p}\right)^p$. Thanks to (6), the embedding $D^{1,p}(R^N) \subsetneq L^p(R^N,|x|^{-p})$ is continuous. For $\nu\in[0,\bar{\nu})$, the space $D^{1,p}(R^N)$ is endowed with the norm

$$\|u\|_\nu := \left(\int_{R^N}|\nabla u|^p dx - \nu\int_{R^N}\frac{|u|^p}{|x|^p}dx\right)^{\frac{1}{p}}, \quad \forall u\in D^{1,p}(R^N). \tag{7}$$

According to the Hardy inequality (6), $\|\cdot\|_\nu$ is equivalent to the usual norm $\left(\int_{R^N}|\nabla\cdot|^p\, dx\right)^{\frac{1}{p}}$.

In the current paper, we work in the space $E=\left(D^{1,p}(R^N)\right)^2 := D^{1,p}(R^N)\times D^{1,p}(R^N)$, E is a reflexive separable Banach space with respect to the norm

$$\|(u,v)\|_E = \left(\|u\|_\nu^p + \|v\|_\nu^p\right)^{\frac{1}{p}}, \quad \forall (u,v)\in E.$$

The energy functional associated with the system (1) is defined on E by

$$\mathcal{J}(u,v)=\frac{1}{p}\|(u,v)\|_E^p - \frac{2}{2p_\mu^*}\int_{R^N}\int_{R^N}\frac{|u(x)|^{p_\mu^*}|v(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy - \frac{1}{p^*}\int_{R^N}\sum_{i=1}^m \eta_i |u|^{\alpha_i}|v|^{\beta_i}dx. \tag{8}$$

It is trivial to verify that $\mathcal{J}\in C^1(E,R)$. Consequently, we see that the solutions of (1) are in fact critical points of $\mathcal{J}(u,v)$. More precisely, we call that $(u,v)\in E$ is a weak solution to (1), if for any $(\phi_1,\phi_2)\in E$, there holds

$$\begin{aligned} 0=\langle \mathcal{J}'(u,v),(\phi_1,\phi_2)\rangle &= \int_{R^N}\left(|\nabla u|^{p-2}\nabla u\nabla\phi_1 + |\nabla v|^{p-2}\nabla v\nabla\phi_2 - \nu\,\frac{|u|^{p-2}u\phi_1 + |v|^{p-2}v\phi_2}{|x|^p}\right)dx \\ &-\int_{R^N}\int_{R^N}\frac{|u(x)|^{p_\mu^*-2}u(x)\phi_1(x)|v(y)|^{p_\mu^*} + |v(x)|^{p_\mu^*-2}v(x)\phi_2(x)|u(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy \\ &-\int_{R^N}\sum_{i=1}^m\frac{1}{p^*}\left(\eta_i\alpha_i|u|^{\alpha_i-2}u|v|^{\beta_i}\phi_1 + \eta_i\beta_i|u|^{\alpha_i}|v|^{\beta_i-2}v\phi_2\right)dx. \end{aligned} \tag{9}$$

In the following, the Morrey space will be introduced, which is indispensable to the refinement of Hardy-Littlewood-Sobolev inequality and the refinement of Sobolev inequality.

Definition 2.1. (See [29]) Let $1\le r<\infty$ and $\varpi\in(0,N]$. A measurable $u: R^N\to R$ is said to belong to the Morrey space $\mathcal{L}^{r,\varpi}(R^N)$ if and only if

$$\|u\|_{\mathcal{L}^{r,\varpi}(R^N)} := \Big(\sup_{\mathcal{R}>0, x\in R^N} \mathcal{R}^{\varpi-N}\int_{B_{\mathcal{R}}(x)}|u(y)|^r dy\Big)^{\frac{1}{r}} < \infty. \tag{10}$$

The following refinement of Sobolev inequality has been proved in [30].

Lemma 2.1. ([30, Theorem 1.2]) For any $1<p<N$, there exists $C_1 = C_1(N,p)>0$ such that for any θ and γ satisfying $1\le \gamma < p^* = \dfrac{Np}{N-p}$, $\dfrac{p}{p^*}\le \theta < 1$, we have

$$\Big(\int_{R^N}|u|^{p^*}dx\Big)^{\frac{1}{p^*}} \le C_1 \|u\|^{\theta}_{D^{1,p}(R^N)} \|u\|^{1-\theta}_{\mathcal{L}^{\gamma,\frac{\gamma(N-p)}{p}}(R^N)} \tag{11}$$

for all $u\in D^{1,p}(R^N)$.

Lemma 2.2. (Hardy-Littlewood-Sobolev Inequality, see [31, Theorem 4.3]) Let $r,t>1$, $\mu\in(0,N)$ with $1/r+(N-\mu)/N+1/t=2$, $f\in L^r(R^N)$ and $h\in L^t(R^N)$. There exists a constant $C(r,t,N,\mu)$ independent of f,h such that

$$\int_{R^N}\int_{R^N}\frac{f(x)h(y)}{|x-y|^{N-\mu}}dxdy \le C(r,t,N,\mu)\|f\|_{L^r(R^N)}\|h\|_{L^t(R^N)}. \tag{12}$$

If $r=t=\dfrac{2N}{N+\mu}$, then (12) be an equality if and only if $f\equiv(constant)h$ and

$$h(x) = A(\delta^2+|x-x_0|^2)^{-\frac{(N+\mu)}{2}}$$

for some $A\in\mathbb{C}$, $0\ne\delta\in R$ and $x_0\in R^N$.

Thanks to Lemma 2.2, we have the following inequality

$$\int_{R^N}\int_{R^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy \le C_2\|u\|^{2p_\mu^*}_{L^{p^*}(R^N)}. \tag{13}$$

Moreover, by virtue of (11) and (13) Su and Chen [21] verified the refinement of Hardy-Littlewood-Sobolev inequality as follows.

Lemma 2.3. ([21, Lemma 3.1]) For any $\mu\in(0,N)$, $1<p<N$, there exists $C_3>0$ such that for γ and θ satisfying $1\le\gamma<p^*=\dfrac{Np}{N-p}$, $\dfrac{p}{p^*}\le\theta<1$, we have

$$\Big(\int_{R^N}\int_{R^N}\frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy\Big)^{\frac{1}{p_\mu^*}} \le C_3\|u\|^{2\theta}_{D^{1,p}(R^N)}\|u\|^{2(1-\theta)}_{\mathcal{L}^{\gamma,\frac{\gamma(N-p)}{p}}(R^N)} \tag{14}$$

for all $u\in D^{1,p}(R^N)$.

Next, we introduce a basic but crucial inequality, which proof is similar to that of [31, Theorem 9.8] and [32, Lemma 2.3].

Lemma 2.4. Let $N \geq 3$, $\mu \in (0,N)$, $q \in [p, p_\mu^*]$, then for any $|u|^q, |v|^q \in L^{\frac{2N}{N+\mu}}(R^N)$,

$$\int_{R^N}\int_{R^N} \frac{|u(x)|^q |v(y)|^q}{|x-y|^{N-\mu}} dxdy \leq \Big(\int_{R^N}\int_{R^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^{N-\mu}} dxdy\Big)^{\frac{1}{2}} \Big(\int_{R^N}\int_{R^N} \frac{|v(x)|^q |v(y)|^q}{|x-y|^{N-\mu}} dxdy\Big)^{\frac{1}{2}}.$$

Proof: According to the semigroup property of the Riesz potential in [7], we have that

$$I_{\mu/2} * I_{\mu/2} = I_{\mu/2+\mu/2} = I_\mu, \quad \forall \mu \in (0,N).$$

By the Cauchy-Schwarz inequality, we derive

$$\begin{aligned}
&\int_{R^N}\int_{R^N} \frac{|u(x)|^q |v(y)|^q}{|x-y|^{N-\mu}} dxdy \\
&= \frac{1}{A_\mu}\int_{R^N} (I_\mu * |u|^q)|v|^q\, dx = \frac{1}{A_\mu}\int_{R^N} (I_{\mu/2} * |u|^q)(I_{\mu/2} * |v|^q) dx \\
&\leq \frac{1}{A_\mu}\Big(\int_{R^N} (I_{\mu/2} * |u|^q)^2 dx\Big)^{\frac{1}{2}} \Big(\int_{R^N} (I_{\mu/2} * |v|^q)^2 dx\Big)^{\frac{1}{2}} \\
&= \Big(\int_{R^N}\int_{R^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^{N-\mu}} dxdy\Big)^{\frac{1}{2}} \Big(\int_{R^N}\int_{R^N} \frac{|v(x)|^q |v(y)|^q}{|x-y|^{N-\mu}} dxdy\Big)^{\frac{1}{2}}.
\end{aligned}$$

Hence, the conclusion is proved. □

For $\nu \in [0, \bar{\nu})$, we now define the following best constants:

$$\mathcal{S}_\nu := \inf_{u \in D^{1,p}(R^N)\setminus\{0\}} \frac{\int_{R^N} |\nabla u|^p dx - \nu \int_{R^N} \frac{|u|^p}{|x|^p} dx}{\Big(\int_{R^N} |u|^{p^*} dx\Big)^{\frac{p}{p^*}}}, \tag{15}$$

$$\mathcal{S}_{\nu,\eta_i} := \inf_{u,v \in D^{1,p}(R^N)\setminus\{0\}} \frac{\int_{R^N} (|\nabla u|^p - \nu \frac{|u|^p}{|x|^p} + |\nabla v|^p - \nu \frac{|v|^p}{|x|^p}) dx}{\Big(\int_{R^N} \sum_{i=1}^m \eta_i |u|^{\alpha_i} |v|^{\beta_i}\, dx\Big)^{\frac{p}{p^*}}}, \tag{16}$$

$$\mathcal{A}_{\nu,\mu} := \inf_{u \in D^{1,p}(R^N)\setminus\{0\}} \frac{\int_{R^N} |\nabla u|^p dx - \nu \int_{R^N} \frac{|u|^p}{|x|^p} dx}{\Big(\int_{R^N}\int_{R^N} \frac{|u(x)|^{p_\mu^*} |u(y)|^{p_\mu^*}}{|x-y|^{N-\mu}} dxdy\Big)^{\frac{p}{2p_\mu^*}}}, \tag{17}$$

$$\widetilde{\mathcal{A}}_{\nu,\mu} := \inf_{u,v \in D^{1,p}(R^N)\setminus\{0\}} \frac{\int_{R^N} (|\nabla u|^p - \nu \frac{|u|^p}{|x|^p} + |\nabla v|^p - \nu \frac{|v|^p}{|x|^p}) dx}{\Big(\int_{R^N}\int_{R^N} \frac{|u(x)|^{p_\mu^*} |v(y)|^{p_\mu^*}}{|x-y|^{N-\mu}} dxdy\Big)^{\frac{p}{2p_\mu^*}}}. \tag{18}$$

Considering [33], $\mathcal{S}_\nu$ is attained in R^N by the extremal functions as follows

$$\tilde{U}_{\nu,\varepsilon}(x) := \varepsilon^{-\frac{N-p}{p}} U_\nu\Big(\frac{x}{\varepsilon}\Big), \quad \forall \varepsilon > 0, \tag{19}$$

which satisfy

$$\int_{R^N} \Big(|\nabla \tilde{U}_{\nu,\varepsilon}(x)|^p - \nu \frac{|\tilde{U}_{\nu,\varepsilon}(x)|^p}{|x|^p}\Big) dx = \int_{R^N} |\tilde{U}_{\nu,\varepsilon}(x)|^{p^*} dx = \mathcal{S}_\nu^{\frac{N}{p}},$$

where $U_{\nu}(x)$ is a radially symmetric, decreasing function and possesses some asymptotic conditions (see [33] for details).

Moreover, for $\eta_i > 0,\ \alpha_i, \beta_i > 1$ with $\alpha_i + \beta_i = p^*$ $(i = 1, 2, \cdots, m)$, we define

$$h(\sigma) := \frac{1+\sigma^p}{(\sum_{i=1}^{m} \eta_i \sigma^{\beta_i})^{\frac{p}{p^*}}}, \quad \sigma > 0, \tag{20}$$

$$h(\sigma_{\min}) := \min_{\sigma>0} h(\sigma) > 0, \tag{21}$$

where $\sigma_{\min} > 0$ is a minimal point of $h(\sigma)$. By direct computation, $\sigma_{\min}$ satisfies the following equation

$$\sum_{i=1}^{m} (\eta_i \alpha_i \sigma^{p+\beta_i-1} - \eta_i \beta_i \sigma^{\beta_i-1}) = 0, \quad \sigma > 0. \tag{22}$$

The main results of the current paper are as follows.

Theorem 2.1. Suppose $\nu \in [0, \bar{\nu}),\ \mu \in (0, N)$.

(i) $\mathcal{S}_{\nu,\eta_i} = h(\sigma_{\min}) \mathcal{S}_{\nu}$.

(ii) $\mathcal{S}_{\nu,\eta_i}$ has the minimizers $\big(\tilde{U}_{\nu,\varepsilon}(x), \sigma_{\min} \tilde{U}_{\nu,\varepsilon}(x)\big),\ \forall \varepsilon > 0,$ where $\tilde{U}_{\nu,\varepsilon}(x)$ are the extremal functions of $\mathcal{S}_{\nu}$ defined as in (19).

Theorem 2.2. If $\nu \in [0, \bar{\nu})$, $1 < p < N$, $\mu \in (0, N)$, $\alpha_i, \beta_i > 1$ with $\alpha_i + \beta_i = p^*$, then system (1) has at least one nontrivial solution in E.

Remark 2.1. As we know, our paper improves and generalizes the result of the doubly critical *p*-Laplacian equation in [20] to the multiple critical coupled system. In addition, it is worth noting that the result on the existence of solutions is still new even if $\eta_i = 0$.

3. Minimizers of $\widetilde{\mathcal{A}}_{\nu,\mu}$ and $\mathcal{S}_{\nu,\eta_i}$

In this part, we demonstrate that the best constants $\widetilde{\mathcal{A}}_{\nu,\mu}$ and $\mathcal{S}_{\nu,\eta_i}$ can be achieved. Firstly, we show the minimizer of $\widetilde{\mathcal{A}}_{\nu,\mu}$. Our method is similar to [32, Lemma 2.6]. In order to convince the readers, we complete it here.

Lemma 3.1. If $\nu \in [0, \bar{\nu}),\ \mu \in (0, N)$, then $\widetilde{\mathcal{A}}_{\nu,\mu} = 2\mathcal{A}_{\nu,\mu}$.

Proof: Let $\{\tau_n\}$ be a minimizing sequence of $\mathcal{A}_{\nu,\mu}$, and let $u_n = r_1 \tau_n$, $v_n = r_2 \tau_n$, $r_1, r_2 > 0$ to be chosen later. Then, by (18) that

$$\begin{aligned}\widetilde{\mathcal{A}}_{\nu,\mu} &\le \frac{\int_{R^N}\left(r_1^p|\nabla\tau_n|^p+r_2^p|\nabla\tau_n|^p-\nu\frac{r_1^p|\tau_n|^p+r_2^p|\tau_n|^p}{|x|^p}\right)dx}{\left(\int_{R^N}\int_{R^N}\frac{r_1^{p_\mu^*}r_2^{p_\mu^*}|\tau_n(x)|^{p_\mu^*}|\tau_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy\right)^{\frac{p}{2p_\mu^*}}}\\ &=\left[\left(\frac{r_1}{r_2}\right)^{\frac{p}{2}}+\left(\frac{r_2}{r_1}\right)^{\frac{p}{2}}\right]\frac{\int_{R^N}\left(|\nabla\tau_n|^p-\nu\frac{|\tau_n|^p}{|x|^p}\right)dx}{\left(\int_{R^N}\int_{R^N}\frac{|\tau_n(x)|^{p_\mu^*}|\tau_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy\right)^{\frac{p}{2p_\mu^*}}}\end{aligned} \tag{23}$$

Setting the function $\psi(x)=x^{\frac{p}{2}}+x^{-\frac{p}{2}}$. It is easy to calculate that $\min_{x>0}\psi(x)=\psi(1)=2$. Then, we choose r_1, r_2 in (23) such that $r_1=r_2$. Therefore, we infer from (23) that

$$\widetilde{\mathcal{A}}_{\nu,\mu}\le 2\mathcal{A}_{\nu,\mu}\quad \text{as } n\to\infty. \tag{24}$$

Then, suppose $\{(u_n,v_n)\}$ is a minimizing sequence of $\widetilde{\mathcal{A}}_{\nu,\mu}$. Let $z_n=r_nv_n$ for some $r_n>0$ such that

$$\int_{R^N}\int_{R^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy=\int_{R^N}\int_{R^N}\frac{|z_n(x)|^{p_\mu^*}|z_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy.$$

This along with Lemma 2.4 gives

$$\begin{aligned}\int_{R^N}\int_{R^N}\frac{|u_n(x)|^{p_\mu^*}|z_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy &\le\int_{R^N}\int_{R^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy\\ &=\int_{R^N}\int_{R^N}\frac{|z_n(x)|^{p_\mu^*}|z_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy.\end{aligned}$$

Then, we derive

$$\begin{aligned}&\frac{\int_{R^N}(|\nabla u_n|^p-\nu\frac{|u_n|^p}{|x|^p}+|\nabla v_n|^p-\nu\frac{|v_n|^p}{|x|^p})dx}{\left(\int_{R^N}\int_{R^N}\frac{|u_n(x)|^{p_\mu^*}|v_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy\right)^{\frac{p}{2p_\mu^*}}}=\frac{r_n^{\frac{p}{2}}\int_{R^N}(|\nabla u_n|^p-\nu\frac{|u_n|^p}{|x|^p}+|\nabla v_n|^p-\nu\frac{|v_n|^p}{|x|^p})dx}{\left(\int_{R^N}\int_{R^N}\frac{|u_n(x)|^{p_\mu^*}|z_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy\right)^{\frac{p}{2p_\mu^*}}}\\ &\ge\frac{r_n^{\frac{p}{2}}\|u_n\|_\nu^p}{\left(\int_{R^N}\int_{R^N}\frac{|u_n(x)|^{p_\mu^*}|u_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy\right)^{\frac{p}{2p_\mu^*}}}+\frac{r_n^{\frac{p}{2}}r_n^{-p}\|z_n\|_\nu^p}{\left(\int_{R^N}\int_{R^N}\frac{|z_n(x)|^{p_\mu^*}|z_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy\right)^{\frac{p}{2p_\mu^*}}}\\ &\ge\left(r_n^{\frac{p}{2}}+r_n^{-\frac{p}{2}}\right)\mathcal{A}_{\nu,\mu}=\psi(r_n)\mathcal{A}_{\nu,\mu}\ge 2\mathcal{A}_{\nu,\mu}.\end{aligned} \tag{25}$$

As $n\to\infty$ in (25), we deduce

$$\widetilde{\mathcal{A}}_{\nu,\mu}\ge 2\mathcal{A}_{\nu,\mu}. \tag{26}$$

From (24) and (26), we get the conclusion desired. □

Lemma 3.2. If $\nu\in[0,\bar{\nu})$, $\mu\in(0,N)$, then $\widetilde{\mathcal{A}}_{\nu,\mu}$ is achieved by a radially symmetric, nonnegative and nonincreasing function in R^N.

Proof: From [21], we know that the extremal of $\mathcal{A}_{\nu,\mu}$ is radially symmetric, nonnegative and nonincreasing. Combining with Lemma 3.1, we complete this proof. □

To prove Theorem 2.1, we just need to find the relationship between the best constants $\mathcal{S}_{\nu,\eta_i}$ and $\mathcal{S}_\nu$ by the similar method in [34, Theorem 2.2].

Proof of Theorem 2.1. (i) Let $\{\omega_n\}$ be a minimizing sequence of $\mathcal{S}_\nu$, and let $\hat{u}_n = \sigma_1\omega_n, \hat{v}_n = \sigma_2\omega_n, \sigma_1, \sigma_2 > 0$ to be chosen later. Then, it follows from (16) that

$$\begin{aligned}\mathcal{S}_{\nu,\eta_i} &\le \frac{\int_{R^N}\Big(|\nabla(\sigma_1\omega_n)|^p - \nu\frac{|\sigma_1\omega_n|^p}{|x|^p} + |\nabla(\sigma_2\omega_n)|^p - \nu\frac{|\sigma_2\omega_n|^p}{|x|^p}\Big)dx}{\Big(\int_{R^N}\sum_{i=1}^m \eta_i|\sigma_1\omega_n|^{\alpha_i}|\sigma_2\omega_n|^{\beta_i}\,dx\Big)^{\frac{p}{p^*}}} \\ &= \frac{\sigma_1^p + \sigma_2^p}{\Big(\sum_{i=1}^m \eta_i\sigma_1^{\alpha_i}\sigma_2^{\beta_i}\Big)^{\frac{p}{p^*}}}\cdot\frac{\int_{R^N}(|\nabla\omega_n|^p - \nu\frac{|\omega_n|^p}{|x|^p})dx}{\Big(\int_{R^N}|\omega_n|^{p^*}dx\Big)^{\frac{p}{p^*}}}.\end{aligned} \tag{27}$$

It is worth noting that from (20)-(22), $\min_{\sigma>0} h(\sigma)$ is achieved at finite $\sigma_{\min} > 0$. Then, choosing $\sigma_1, \sigma_2 > 0$ in (27) such that $\frac{\sigma_2}{\sigma_1} = \sigma_{\min}$ with the minimum value

$$h\Big(\frac{\sigma_2}{\sigma_1}\Big) = h(\sigma_{\min}) = \frac{\sigma_1^p + \sigma_2^p}{\Big(\sum_{i=1}^m \eta_i\sigma_1^{\alpha_i}\sigma_2^{\beta_i}\Big)^{\frac{p}{p^*}}}.$$

Therefore, we infer from (27) that

$$\mathcal{S}_{\nu,\eta_i} \le h(\sigma_{\min})\mathcal{S}_\nu \quad \text{as } n\to\infty. \tag{28}$$

For another, suppose $\{(\hat{u}_n, \hat{v}_n)\}$ is a minimizing sequence for $\mathcal{S}_{\nu,\eta_i}$ and let $\hat{z}_n = \sigma_n\hat{v}_n$ for some $\sigma_n > 0$ such that

$$\sigma_n^{\alpha_i+\beta_i} = \frac{\int_{R^N}|\hat{u}_n|^{\alpha_i+\beta_i}dx}{\int_{R^N}|\hat{v}_n|^{\alpha_i+\beta_i}dx}.$$

Then,

$$\int_{R^N}|\hat{z}_n|^{\alpha_i+\beta_i}dx = \int_{R^N}|\hat{u}_n|^{\alpha_i+\beta_i}dx. \tag{29}$$

According to the Young inequality, we have

$$\begin{aligned}\int_{R^N}|\hat{u}_n|^{\alpha_i}|\hat{z}_n|^{\beta_i}\,dx &\le \frac{\alpha_i}{\alpha_i+\beta_i}\int_{R^N}|\hat{u}_n|^{\alpha_i+\beta_i}dx + \frac{\beta_i}{\alpha_i+\beta_i}\int_{R^N}|\hat{u}_n|^{\alpha_i+\beta_i}dx \\ &= \int_{R^N}|\hat{u}_n|^{\alpha_i+\beta_i}dx = \int_{R^N}|\hat{z}_n|^{\alpha_i+\beta_i}dx.\end{aligned} \tag{30}$$

Consequently, we infer from (29) and (30) that

$$\frac{\int_{R^N}(|\nabla\hat{u}_n|^p-\nu\frac{|\hat{u}_n|^p}{|x|^p}+|\nabla\hat{v}_n|^p-\nu\frac{|\hat{v}_n|^p}{|x|^p})dx}{\left(\int_{R^N}\sum_{i=1}^m\eta_i|\hat{u}_n|^{\alpha_i}|\hat{v}_n|^{\beta_i}dx\right)^{\frac{p}{p^*}}}=\frac{\int_{R^N}(|\nabla\hat{u}_n|^p-\nu\frac{|\hat{u}_n|^p}{|x|^p}+\sigma_n^{-p}|\nabla\hat{z}_n|^p-\nu\frac{\sigma_n^{-p}|\hat{z}_n|^p}{|x|^p})dx}{\left(\int_{R^N}\sum_{i=1}^m\eta_i\sigma_n^{-\beta_i}|\hat{u}_n|^{\alpha_i}|\hat{z}_n|^{\beta_i}dx\right)^{\frac{p}{p^*}}}$$

$$\geq\frac{1}{(\sum_{i=1}^m\eta_i\sigma_n^{-\beta_i})^{\frac{p}{p^*}}}\cdot\frac{\int_{R^N}(|\nabla\hat{u}_n|^p-\nu\frac{|\hat{u}_n|^p}{|x|^p})dx}{\left(\int_{R^N}|\hat{u}_n|^{\alpha+\beta}dx\right)^{\frac{p}{p^*}}}+\frac{\sigma_n^{-p}}{(\sum_{i=1}^m\eta_i\sigma_n^{-\beta_i})^{\frac{p}{p^*}}}\cdot\frac{\int_{R^N}(|\nabla\hat{z}_n|^p-\nu\frac{|\hat{z}_n|^p}{|x|^p})dx}{\left(\int_{R^N}|\hat{z}_n|^{\alpha+\beta}dx\right)^{\frac{p}{p^*}}}$$

$$\geq\frac{1+\sigma_n^{-p}}{(\sum_{i=1}^m\eta_i\sigma_n^{-\beta_i})^{\frac{p}{p^*}}}\mathcal{S}_\nu=h(\sigma_n^{-1})\mathcal{S}_\nu\geq h(\sigma_{\min})\mathcal{S}_\nu.$$

As $n\to\infty$ in the above inequality, we derive

$$\mathcal{S}_{\nu,\eta_i}\geq h(\sigma_{\min})\mathcal{S}_\nu. \tag{31}$$

It follows from (28) and (31) that

$$\mathcal{S}_{\nu,\eta_i}=h(\sigma_{\min})\mathcal{S}_\nu. \tag{32}$$

(ii) By (16), (20) and (32), the desired result follows. □

4. The nontrivial solution of system (1)

In this part, we demonstrate the existence of solutions for the problem (1) and seek critical points of $\mathcal{J}$ by some technical lemmas and Mountain Pass Lemma.

Firstly, we define the Nehari manifold with respect to $\mathcal{J}$ as follows

$$\mathcal{N}=\left\{(u,v)\in\left(D^{1,p}(R^N)\setminus\{0\}\right)^2:\langle\mathcal{J}'(u,v),(u,v)\rangle=0\right\}.$$

Hence, a minimizer of the minimization problem

$$c_0=\inf_{(u,v)\in\mathcal{N}}\mathcal{J}(u,v)$$

is a solution to the problem (1). Meanwhile, we set

$$c_\gamma=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{J}(\gamma(t)),$$

where $\Gamma=\left\{\gamma\in C\left([0,1],E\right):\gamma(0)=0,\gamma(1)=e\right\}$, and set $c_m=\inf_{(u,v)\in E}\max_{t\geq0}\mathcal{J}(tu,tv)$. Using the argument of [35, Theorem 4.2] under the condition $\mu\in(0,N)$, we obtain

$$c_0=c_\gamma=c_m. \tag{33}$$

Lemma 4.1. If $\nu\in[0,\bar{\nu})$ and $\mu\in(0,N)$, then

$$c_m<c^*:=\min\left\{\frac{p+\mu}{(N+\mu)p}\left(\frac{\widetilde{\mathcal{A}}_{\nu,\mu}}{2^{\frac{N-p}{N+\mu}}}\right)^{\frac{N+\mu}{p+\mu}},\frac{1}{N}\mathcal{S}_{\nu,\eta_i}^{\frac{N}{p}}\right\}.$$

Proof: Lemma 3.2 and Theorem 2.1 imply that $\widetilde{\mathcal{A}}_{\nu,\mu}$ and $\mathcal{S}_{\nu,\eta_i}$ are achieved in E. Generally, we suppose that $(u_1,v_1)\in\left(D^{1,p}(R^N)\setminus\{0\}\right)^2$ and $(u_2,v_2)\in\left(D^{1,p}(R^N)\setminus\{0\}\right)^2$ are the minimizers of $\widetilde{\mathcal{A}}_{\nu,\mu}$ and $\mathcal{S}_{\nu,\eta_i}$ respectively. Define the following for any $t\geq0$,

$$g_1(t)=\frac{t^p}{p}\int_{R^N}\Big(|\nabla u_1|^p+|\nabla v_1|^p-\nu\frac{|u_1|^p+|v_1|^p}{|x|^p}\Big)dx-\frac{2t^{2p_\mu^*}}{2p_\mu^*}\int_{R^N}\int_{R^N}\frac{|u_1(x)|^{p_\mu^*}|v_1(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy$$

and

$$g_2(t)=\frac{t^p}{p}\int_{R^N}\Big(|\nabla u_2|^p+|\nabla v_2|^p-\nu\frac{|u_2|^p+|v_2|^p}{|x|^p}\Big)dx-\frac{t^{p^*}}{p^*}\int_{R^N}\sum_{i=1}^m\eta_i|u_2|^{\alpha_i}|v_2|^{\beta_i}dx.$$

We infer from (8) that

$$\max_{t\geq 0}\mathcal{J}(tu_1,tv_1)\leq\max_{t\geq 0}g_1(t)\quad\text{and}\quad\max_{t\geq 0}\mathcal{J}(tu_2,tv_2)\leq\max_{t\geq 0}g_2(t).$$

By direct arithmetic, the function $g_1(t)$ takes its maximum at

$$t_1=\left(\frac{\int_{R^N}\Big(|\nabla u_1|^p+|\nabla v_1|^p-\nu\frac{|u_1|^p+|v_1|^p}{|x|^p}\Big)dx}{2\int_{R^N}\int_{R^N}\frac{|u_1(x)|^{p_\mu^*}|v_1(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy}\right)^{\frac{1}{2p_\mu^*-p}},$$

and $g_2(t)$ achieves its maximum at

$$t_2=\left(\frac{\int_{R^N}\Big(|\nabla u_2|^p+|\nabla v_2|^p-\nu\frac{|u_2|^p+|v_2|^p}{|x|^p}\Big)dx}{\int_{R^N}\sum_{i=1}^m\eta_i|u_2|^{\alpha_i}|v_2|^{\beta_i}\,dx}\right)^{\frac{1}{p^*-p}}.$$

Thus, it implies that

$$\max_{t\geq 0}g_1(t)=g_1(t_1)=\frac{p+\mu}{(N+\mu)p}\Big(\frac{\widetilde{\mathcal{A}}_{\nu,\mu}}{2^{\frac{N-p}{N+\mu}}}\Big)^{\frac{N+\mu}{p+\mu}},$$

and

$$\max_{t\geq 0}g_2(t)=g_2(t_2)=\frac{1}{N}\mathcal{S}_{\nu,\eta_i}^{\frac{N}{p}}.$$

Then, we show that

$$\max_{t\geq 0}\mathcal{J}(tu_1,tv_1)<g_1(t_1),\ \max_{t\geq 0}\mathcal{J}(tu_2,tv_2)<g_2(t_2).$$

Assuming that there exist $\tilde{t}_1>0,\tilde{t}_2>0$ satisfying $\mathcal{J}(\tilde{t}_1u_1,\tilde{t}_1v_1)=g_1(t_1)$, $\mathcal{J}(\tilde{t}_2u_2,\tilde{t}_2v_2)=g_2(t_2)$, that is,

$$g_1(\tilde{t}_1)-\frac{\tilde{t}_1^{p^*}}{p^*}\int_{R^N}\sum_{i=1}^m\eta_i|u_1|^{\alpha_i}|v_1|^{\beta_i}dx=g_1(t_1)$$

and

$$g_2(\tilde{t}_2)-\frac{2\tilde{t}_2^{2p_\mu^*}}{2p_\mu^*}\int_{R^N}\int_{R^N}\frac{|u_2(x)|^{p_\mu^*}|v_2(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy=g_2(t_2).$$

Above two equalities imply that $g_1(t_1)<g_1(\tilde{t}_1)$ and $g_2(t_2)<g_2(\tilde{t}_2)$, which yields a contradiction. Thus, we get the desired result. □

Then, we demonstrate that the functional $\mathcal{J}$ satisfies the geometry structure of Mountain Pass Lemma without the (PS) condition.

Lemma 4.2. Suppose $\nu\in[0,\bar{\nu}),\ \mu\in(0,N)$.

(i) There exist $\rho, \tilde{R} > 0$ such that $\mathcal{J}(u,v)|_{\|(u,v)\|_E = \tilde{R}} \geq \rho$ for all $(u,v) \in E$.

(ii) There exists $e \in E$ with $\|e\|_E > \tilde{R}$ such that $\mathcal{J}(e) < 0$.

Proof: (i) By (16) and (18), we see that

$$\mathcal{J}(u,v) = \frac{1}{p}\|(u,v)\|_E^p - \frac{1}{p^*}\int_{R^N}\sum_{i=1}^m \eta_i |u|^{\alpha_i}|v|^{\beta_i}dx - \frac{2}{2p_\mu^*}\int_{R^N}\int_{R^N}\frac{|u(x)|^{p_\mu^*}|v(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy$$

$$\geq \frac{1}{p}\|(u,v)\|_E^p - \frac{\mathcal{S}_{v,\eta_i}^{-\frac{p^*}{p}}}{p^*}\|(u,v)\|_E^{p^*} - \frac{2\widetilde{\mathcal{A}}_{v,\mu}^{-\frac{2p_\mu^*}{p}}}{2p_\mu^*}\|(u,v)\|_E^{2p_\mu^*} \geq \rho > 0,$$

for $\|(u,v)\|_E = \tilde{R} > 0$ small enough. We complete the proof of the first assertion.

(ii) For a fixed $(u_0, v_0) \in E$,

$$\mathcal{J}(tu_0, tv_0) = \frac{t^p}{p}\|(u_0,v_0)\|_E^p - \frac{t^{p^*}}{p^*}\int_{R^N}\sum_{i=1}^m \eta_i |u_0|^{\alpha_i}|v_0|^{\beta_i}dx$$

$$-\frac{2t^{2p_\mu^*}}{2p_\mu^*}\int_{R^N}\int_{R^N}\frac{|u_0(x)|^{p_\mu^*}|v_0(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy \to -\infty$$

as $t \to +\infty$. Choosing t_0 large enough such that $\|(t_0u_0, t_0v_0)\|_E > \tilde{R}$ and letting $e = (t_0u_0, t_0v_0)$, the conclusion (ii) follows. □

Next, we display the nonzero $(PS)_c$ sequence with the help of the following.

Lemma 4.3. Suppose $\{(u_n, v_n)\} \subset E$ is a $(PS)_c$ sequence of $\mathcal{J}$ with $c \in (0, c^*)$, where c^* is given in Lemma 4.1. If $\mu \in (0, N)$ and $\nu \in [0, \bar{\nu})$, then

$$\limsup_{n\to\infty}\int_{R^N}\sum_{i=1}^m \eta_i |u_n|^{\alpha_i}|v_n|^{\beta_i}\,dx > 0 \tag{34}$$

and

$$\limsup_{n\to\infty}\int_{R^N}\int_{R^N}\frac{|u_n(x)|^{p_\mu^*}|v_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy > 0. \tag{35}$$

Proof: By $\mathcal{J}(u_n, v_n) \to c$ and $\mathcal{J}'(u_n, v_n) \to 0$ as $n \to \infty$, there holds

$$c + o(1)\|(u_n,v_n)\|_E = \mathcal{J}(u_n,v_n) - \frac{1}{p^*}\langle \mathcal{J}'(u_n,v_n),(u_n,v_n)\rangle$$

$$= \Big(\frac{1}{p} - \frac{1}{p^*}\Big)\|(u_n,v_n)\|_E^p + 2\Big(\frac{1}{p^*} - \frac{1}{2p_\mu^*}\Big)\int_{R^N}\int_{R^N}\frac{|u_n(x)|^{p_\mu^*}|v_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy$$

$$\geq \Big(\frac{1}{p} - \frac{1}{p^*}\Big)\|(u_n,v_n)\|_E^p,$$

which means $\{(u_n, v_n)\}$ is uniformly bounded in E. Let us now assume that

$$\limsup_{n\to\infty}\int_{R^N}\sum_{i=1}^m \eta_i |u_n|^{\alpha_i}|v_n|^{\beta_i}\,dx = 0.$$

It follows from $\mathcal{J}(u_n, v_n) \to c$, $\mathcal{J}'(u_n, v_n) \to 0$ that

$$c+o(1)=\frac{1}{p}\|(u_n,v_n)\|_E^p-\frac{2}{2p_\mu^*}\int_{R^N}\int_{R^N}\frac{|u_n(x)|^{p_\mu^*}|v_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy$$

and

$$o(1)=\|(u_n,v_n)\|_E^p-2\int_{R^N}\int_{R^N}\frac{|u_n(x)|^{p_\mu^*}|v_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy. \tag{36}$$

Consequently,

$$c+o(1)=\Big(\frac{1}{p}-\frac{1}{2p_\mu^*}\Big)\|(u_n,v_n)\|_E^p. \tag{37}$$

By (18) and (36), we obtain

$$\|(u_n,v_n)\|_E^p\geq\Big(\frac{\widetilde{\mathcal{A}}_{\nu,\mu}}{2^{\frac{N-p}{N+\mu}}}\Big)^{\frac{N+\mu}{p+\mu}}.$$

Then, combined with (37), implies that

$$c\geq\frac{p+\mu}{(N+\mu)p}\Big(\frac{\widetilde{\mathcal{A}}_{\nu,\mu}}{2^{\frac{N-p}{N+\mu}}}\Big)^{\frac{N+\mu}{p+\mu}},$$

which contradicts $c\in(0,c^*)$. Similarly, we can complete the proof of (35). □

Finally, we will show the existence of solutions for (1).

Proof of Theorem 2.2. We infer from Lemma 4.2 and Lemma 4.3 that there exists a $(PS)_{c_m}$ sequence $\{(u_n,v_n)\}$ of $\mathcal{J}$ which is bounded in E. Combining Lemma 4.1, 4.3 and Lemma 2.3, 2.4, one can find $C>0$ such that

$$\|u_n\|_{\mathcal{L}^{p,N-p}(R^N)}\geq C,\quad \|v_n\|_{\mathcal{L}^{p,N-p}(R^N)}\geq C.$$

Thus, there exist $\lambda_n>0$ and $x_n\in R^N$ such that

$$\lambda_n^{-p}\int_{B_{\lambda_n}(x_n)}|u_n(y)|^p dy\geq\|u_n\|_{\mathcal{L}^{p,N-p}(R^N)}-\frac{C}{2n}\geq\tilde{C}>0, \tag{38}$$

and

$$\lambda_n^{-p}\int_{B_{\lambda_n}(x_n)}|v_n(y)|^p dy\geq\|v_n\|_{\mathcal{L}^{p,N-p}(R^N)}-\frac{C}{2n}\geq\tilde{C}>0. \tag{39}$$

Setting $\bar{u}_n=\lambda_n^{\frac{N-p}{p}}u_n(\lambda_n x),\bar{v}_n=\lambda_n^{\frac{N-p}{p}}v_n(\lambda_n x)$. By direct computation, we derive

$$\mathcal{J}(\bar{u}_n,\bar{v}_n)=\mathcal{J}(u_n,v_n)\to c_m,\quad \langle\mathcal{J}'(\bar{u}_n,\bar{v}_n),(\varphi_1,\varphi_2)\rangle\to0\quad\text{as}\quad n\to\infty.$$

Proceeding as in Lemma 4.3, we obtain that $\{(\bar{u}_n,\bar{v}_n)\}$ is bounded in E uniformly. Therefore, there exists $\{(\bar{u},\bar{v})\}$ such that

$$\bar{u}_n\rightharpoonup\bar{u},\bar{v}_n\rightharpoonup\bar{v}\ \text{ in } D^{1,p}(R^N),$$
$$\bar{u}_n\to\bar{u},\bar{v}_n\to\bar{v}\ \text{ in } L^q_{\text{loc}}(R^N) \text{ for any } q\in[p,p^*),$$
$$\bar{u}_n\to\bar{u},\bar{v}_n\to\bar{v}\ \text{ a.e. in } R^N,$$

and $(\bar{u},\bar{v})$ is a weak solution of (1). From (38) and (39), we have

$$\int_{B_1(\frac{x_n}{\lambda_n})} |\overline{u}_n(y)|^p dy \geq \tilde{C} > 0, \qquad \int_{B_1(\frac{x_n}{\lambda_n})} |\overline{v}_n(y)|^p dy \geq \tilde{C} > 0. \tag{40}$$

Now, we verify that $\{\frac{x_n}{\lambda_n}\}$ is bounded. For any $0 < \xi < p$, it follows from Hölder inequality that

$$0 < \tilde{C} \leq \int_{B_1(\frac{x_n}{\lambda_n})} |\overline{u}_n|^p dy = \int_{B_1(\frac{x_n}{\lambda_n})} |y|^{\frac{p\xi}{\frac{p(N-\xi)}{N-p}}} \frac{|\overline{u}_n|^p}{|y|^{\frac{p\xi}{\frac{p(N-\xi)}{N-p}}}} dy$$

$$\leq \Big(\int_{B_1(\frac{x_n}{\lambda_n})} |y|^{\frac{\xi(N-p)}{p-\xi}} dy\Big)^{1-\frac{N-p}{N-\xi}} \Big(\int_{B_1(\frac{x_n}{\lambda_n})} \frac{|\overline{u}_n|^{\frac{p(N-\xi)}{N-p}}}{|y|^{\xi}} dy\Big)^{\frac{N-p}{N-\xi}}.$$

Using the rearrangement inequality [31, Theorem 3.4], we derive

$$\int_{B_1(\frac{x_n}{\lambda_n})} |y|^{\frac{\xi(N-p)}{p-\xi}} dy \leq \int_{B_1(0)} |y|^{\frac{\xi(N-p)}{p-\xi}} dy \leq C.$$

Thus,

$$0 < C \leq \int_{B_1(\frac{x_n}{\lambda_n})} \frac{|\overline{u}_n|^{\frac{p(N-\xi)}{N-p}}}{|y|^{\xi}} dy \tag{41}$$

Then, suppose $|\frac{x_n}{\lambda_n}| \to \infty \quad (n \to \infty)$. For any $y \in B_1(\frac{x_n}{\lambda_n})$, if n is large enough, we obtain $|y| \geq |\frac{x_n}{\lambda_n}| - 1$. Hence, we infer from Hölder inequality that

$$\int_{B_1(\frac{x_n}{\lambda_n})} \frac{|\overline{u}_n|^{\frac{p(N-\xi)}{N-p}}}{|y|^{\xi}} dy \leq \frac{1}{\big(|\frac{x_n}{\lambda_n}| - 1\big)^{\xi}} \int_{B_1(\frac{x_n}{\lambda_n})} |\overline{u}_n|^{\frac{p(N-\xi)}{N-p}} dy \leq \frac{|B_1(\frac{x_n}{\lambda_n})|^{\frac{\xi}{N}}}{\big(|\frac{x_n}{\lambda_n}| - 1\big)^{\xi}} \Big(\int_{B_1(\frac{x_n}{\lambda_n})} |\overline{u}_n|^{\frac{Np}{N-p}} dy\Big)^{\frac{N-\xi}{N}}$$

$$\leq \frac{|B_1(\frac{x_n}{\lambda_n})|^{\frac{\xi}{N}}}{\big(|\frac{x_n}{\lambda_n}| - 1\big)^{\xi}} \|\overline{u}_n\|_{v}^{\frac{N-\xi}{N}} \leq \frac{C}{\big(|\frac{x_n}{\lambda_n}| - 1\big)^{\xi}} \to 0 \qquad (n \to \infty),$$

which contradicts (41). Therefore, $\{\frac{x_n}{\lambda_n}\}$ is bounded. By (40), we derive that there exists $\overline{R} > 0$ such that

$$\int_{B_{\overline{R}}(0)} |\overline{u}_n(y)|^p dy \geq \int_{B_1(\frac{x_n}{\lambda_n})} |\overline{u}_n(y)|^p dy \geq \tilde{C} > 0,$$

and

$$\int_{B_{\overline{R}}(0)} |\overline{v}_n(y)|^p dy \geq \int_{B_1(\frac{x_n}{\lambda_n})} |\overline{v}_n(y)|^p dy \geq \tilde{C} > 0.$$

Through the compact embedding $D^{1,p}(R^N) \subseteq L^p_{\text{loc}}(R^N)$, we observe

$$\int_{B_{\overline{R}}(0)} |\overline{u}(y)|^p \, dy \geq \tilde{C} > 0, \qquad \int_{B_{\overline{R}}(0)} |\overline{v}(y)|^p \, dy \geq \tilde{C} > 0.$$

Thus, $\bar{u} \neq 0, \bar{v} \neq 0$. Applying the arguments in [36, Lemma 2.1], there holds

$$\int_{R^N} |\bar{u}_n|^{\alpha_i} |\bar{v}_n|^{\beta_i} dx - \int_{R^N} |\bar{u}_n - \bar{u}|^{\alpha_i} |\bar{v}_n - \bar{v}|^{\beta_i} dx = \int_{R^N} |\bar{u}|^{\alpha_i} |\bar{v}|^{\beta_i} dx + o(1). \tag{42}$$

By the similar argument of claim (2) in [32, Theorem 4.13], we derive

$$\begin{aligned}&\int_{R^N}\int_{R^N} \frac{|\bar{u}_n(x)|^{p_\mu^*}|\bar{v}_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy - \int_{R^N}\int_{R^N} \frac{|\bar{u}_n(x)-\bar{u}(x)|^{p_\mu^*}|\bar{v}_n(y)-\bar{v}(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy \\ &= \int_{R^N}\int_{R^N} \frac{|\bar{u}(x)|^{p_\mu^*}|\bar{v}(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy + o(1).\end{aligned} \tag{43}$$

Finally, combining (42) with (43), we deduce that

$$\begin{aligned} c_0 = c_m &= \mathcal{J}(\bar{u}_n,\bar{v}_n) - \frac{1}{p}\langle \mathcal{J}'(\bar{u}_n,\bar{v}_n),(\bar{u}_n,\bar{v}_n)\rangle + o(1) \\ &= \Big(\frac{2}{p} - \frac{2}{2p_\mu^*}\Big)\int_{R^N}\int_{R^N} \frac{|\bar{u}_n(x)|^{p_\mu^*}|\bar{v}_n(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy + \Big(\frac{1}{p} - \frac{1}{p^*}\Big)\int_{R^N}\sum_{i=1}^{m}\eta_i |\bar{u}_n|^{\alpha_i}|\bar{v}_n|^{\beta_i} dx + o(1) \\ &\geq \Big(\frac{2}{p} - \frac{2}{2p_\mu^*}\Big)\int_{R^N}\int_{R^N} \frac{|\bar{u}(x)|^{p_\mu^*}|\bar{v}(y)|^{p_\mu^*}}{|x-y|^{N-\mu}}dxdy + \Big(\frac{1}{p} - \frac{1}{p^*}\Big)\int_{R^N}\sum_{i=1}^{m}\eta_i |\bar{u}|^{\alpha_i}|\bar{v}|^{\beta_i} dx + o(1) \\ &= \mathcal{J}(\bar{u},\bar{v}) - \frac{1}{p}\langle \mathcal{J}'(\bar{u},\bar{v}),(\bar{u},\bar{v})\rangle = \mathcal{J}(\bar{u},\bar{v}) \geq c_0. \end{aligned}$$

Therefore, $(\bar{u},\bar{v})$ is a nontrivial solution of (1) with $\mathcal{J}(\bar{u},\bar{v}) = c_0$. We finish the proof. □

5. Conclusion

Choquard equation with multiple critical exponents is widely studied, and this kind of equation can be extended to coupled systems. In this paper, we use the variational method to discuss the existence of the best constants and solutions for strongly coupled Choquard systems with multiple critical exponents in R^N. Next, we will continue to study the best constant of the system in the case of fractional and discuss the existence of its solution on this basis.

Acknowledgements. This work is supported by the National Natural Science Foundation of China (No.11971339) and the Natural Science Foundation of Chongqing, China (No.cstc2021jcyj-msxmX0412). The authors are very grateful to the reviewers for their efforts to improve the manuscript.

Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. All the authors contributed equally to this work.

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