Journal of Mathematics and Informatics
Vol. 23, 2022, 7-21
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 14 November 2022
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v23a02210

Journal of **Mathematics and Informatics**

Analytical Investigation of Third-Order Time-Fractional Dispersive Partial Differential Equations Using Sumudu Transform Iterative Method

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Received 4 October 2022; accepted 12 November 2022

Abstract. This paper investigates the approximate analytical solutions of third-order time-fractional dispersive partial differential equations in one-and higher-dimensional spaces by employing a newly developed analytical method, the Sumudu transform iterative method. To express fractional derivatives, the Caputo operator is used. Furthermore, the results of this investigation are graphically represented, and the solution graphs reveal that the approximate solutions are closely connected to the exact solutions.

***Keywords*:** Sumudu transform; Iterative Method; Caputo fractional derivative; third-order time-fractional dispersive partial differential equations.

AMS Mathematics Subject Classification (2010): 26A33, 35A22, 35A24, 35R11

1. Introduction

Fractional calculus is a discipline that has seen some exciting progress in recent years. Differential equations with fractional order derivatives are used to model a wide range of systems, with significant applications in the fields of viscoelasticity, electrode-electrolyte polarization, heat conduction, electromagnetic waves, diffusion equations, etc. Due to its wide range of applications, researchers have recently focused a lot of emphasis on the exact and approximate solutions of fractional differential equations. In order to solve the fractional order differential equations, a number of different approaches have been developed. Various analytical and numerical methods have been proposed for the solutions of linear and nonlinear fractional partial differential equations, such as the Adomian decomposition method (ADM) [25], the Laplace Adomian decomposition method (LADM) [4,13,20], the iterative Laplace transform method (ILTM) [5,21,22], the Homotopy analysis Sumudu transform method (HASTM) [16], the homotopy perturbation transform method (HPTM) [14], the fractional differential transform method

(FDTM) [1], the modified fractional differential transform method (MFDTM)[12], the fractional variational iteration method (FVIM) [15], and so on.

In 1993, Watugala suggested the Sumudu transform method (STM) [24] to address engineering problems. Weerakoon [26] utilized this method to solve partial differential equations. The inverse formula of this transform was later discovered by Weerakoon [22]. The Sumudu transform method was employed by Demiray *et al.* [11] to discover exact solutions to fractional differential equations. Recently, Wang and Liu developed the Sumudu transform iterative method (STIM) [23] by combining the Sumudu transform with an iterative technique to find approximate analytical solutions to time-fractional Cauchy reactions-diffusion equations. The Sumudu transform iterative method was used to successfully solve fractional Fokker-Planck equations [2], generalized fractional biological population models [3], solving fractional diffusion equations [6], fractional Schrödinger equations [7], etc.

In the present study, we will investigate the following third-order time-fractional dispersive partial differential equations with different dimensions:

(i) The one-dimensional third-order time-fractional dispersive partial differential equation of the form, is given by [20]

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+a\frac{\partial u(x,t)}{\partial x}+\frac{\partial^{3}u(x,t)}{\partial x^{3}}=g_1(x,t),\ 0<\alpha\le 1,\ \ a>0, t>0, \tag{1}$$

$$u(x,0)=f(x), \tag{2}$$

where $g_1(x,t)$ is a source term.

(ii) The two-dimensional third-order time-fractional dispersive partial differential equation of the form, is given by [20]

$$\frac{\partial^{\alpha}u(x,y,t)}{\partial t^{\alpha}}+c\frac{\partial^{3}u(x,y,t)}{\partial x^{3}}+d\frac{\partial^{3}u(x,y,t)}{\partial y^{3}}=g_2(x,y,t), \quad 0<\alpha\le 1, c>0, d>0, t>0, \tag{3}$$

$$u(x,y,0)=g(x,y), \tag{4}$$

where $g_2(x,y,t)$ is a source term.

The main objective of this work is the expansions of the Sumudu transform iterative technique (STIM) to create an approximate analytical solution for third-order time-fractional dispersive partial differential equations with initial conditions in one-and-higher dimensions.

2. Basic definitions of fractional calculus and Sumudu transform theory

In this section, we give some basic definitions, notations, and properties of fractional calculus and Sumudu transform theory that will be used later in this paper.

Definition 1. The fractional derivative of $u(x,t)$ in the Caputo sense is defined as [15, 17]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\eta)^{m-\alpha-1}u^{(m)}(x,\eta)d\eta, \qquad m-1<\alpha\leq m, m\in N, \tag{5}$$

Definition 2. The Sumudu transform is defined over the set of functions

$$\left\{ f(t) \mid \exists\, M,\rho_1>0,\ \rho_2>0,\ |f(t)|<Me^{|t|/\rho_j}\ if\ \ t\in(-1)^j\times[0,\infty) \right\}$$

by the following formula [8, 24]

$$S[f(t)]=F(\omega)=\int_0^\infty e^{-t} f(\omega t)\,dt,\ \ \omega\in(-\rho_1,\rho_2). \tag{6}$$

Definition 3. The Sumudu transform of the Caputo fractional derivative is defined as [11, 23]

$$S\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right]=\omega^{-\alpha}S[u(x,t)]-\sum_{k=o}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0), m-1<\alpha\leq m, m\in N, \tag{7}$$

where $u^{(k)}(x,0)$ is the k-order derivative of $u\,(x,t)$ with respect to t at $t=0$.

3. Basic idea of Sumudu transform iterative method

In order to illustrate the key concept of this method [23], we consider the general fractional partial differential equation with initial conditions of the type

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+Ru(x,t)+N\,u(x,t)=g(x,t),\ m-1<\alpha\leq m,\ m\in N, \tag{8}$$

$$u^{(k)}(x,0)=h_k(x),\ k=0,1,2,\ldots,m-1\ , \tag{9}$$

where $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Caputo fractional derivative of order α, $m-1<\alpha\leq m, m\in N$, defined by equation (5), R is a linear operator and may include other fractional derivatives of order less than α , N is a non-linear operator which may include other fractional derivatives of order less than α and $g\left(x,t\right)$ is a known function.

Applying the Sumudu transform on both sides of equation (8), we have

$$S\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right]+S[Ru\,(x,t)+Nu(x,t)]=S[g(x,t)]. \tag{10}$$

By using the equation (7), we get

$$S[u(x,t)]=\sum_{k=0}^{m-1}\omega^k u^{(k)}(x,0)+\omega^\alpha S[g(x,t)]-\omega^\alpha S[Ru(x,t)+N\,u(x,t)]. \tag{11}$$

On taking inverse Sumudu transform on equation (11), we have

$$u(x,t)=S^{-1}\left[\omega^\alpha\left(\sum_{k=0}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0)+S[g(x,t)]\right)\right] -S^{-1}\left[\omega^\alpha S[Ru(x,t)+Nu(x,t)]\right]. \tag{12}$$

Furthermore, we apply the iterative method proposed by Daftardar-Gejji and Jafari [10], which represents a solution in an infinite series of components as

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t). \tag{13}$$

As R is a linear operator, so we have

$$R\left(\sum_{i=0}^{\infty}u_i(x,t)\right)=\sum_{i=0}^{\infty}R[u_i(x,t)], \tag{14}$$

and the non-linear operator N is decomposed as follows

$$N\left(\sum_{i=0}^{\infty}u_i(x,t)\right)=N[u_0(x,t)]+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j(x,t)\right)-N\left(\sum_{j=0}^{i-1}u_j(x,t)\right)\right\}. \tag{15}$$

Substituting the results given by equations from (13) to (15) in the equation (12), we get

$$\begin{aligned}\sum_{i=0}^{\infty}u_i(x,t)&=S^{-1}\left[\omega^{\alpha}\left(\sum_{k=0}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0)+S[g(x,t)]\right)\right]\\ &-S^{-1}\left[\omega^{\alpha}S\left[\sum_{i=0}^{\infty}R[u_i(x,t)]+N[u_0(x,t)]+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j(x,t)\right)-N\left(\sum_{j=0}^{i-1}u_j(x,t)\right)\right\}\right]\right].\end{aligned} \tag{16}$$

We have formulated the recurrence relations as

$$\left.\begin{aligned}u_0(x,t)&=S^{-1}\left[\omega^{\alpha}\left(\sum_{k=0}^{m-1}\omega^{-\alpha+k}u^{(k)}(x,0)+S\big(g(x,t)\big)\right)\right]\\ u_1(x,t)&=-S^{-1}\left[\omega^{\alpha}S\left[R\big(u_0(x,t)\big)+N\big(u_0(x,t)\big)\right]\right]\\ u_{m+1}(x,t)&=-S^{-1}\left[\omega^{\alpha}S\left[R\big(u_m(x,t)\big)-\left\{N\left(\sum_{j=0}^{m}u_j(x,t)\right)-N\left(\sum_{j=0}^{m-1}u_j(x,t)\right)\right\}\right]\right],m\geq 1\end{aligned}\right\}. \tag{17}$$

Therefore, the approximate analytical solution for equations (8) and (9) in truncated series form is determined by

$$u(x,t)\cong\lim_{N\to\infty}\sum_{m=0}^{N}u_m(x,t). \tag{18}$$

In general, the solutions in the preceding series quickly converge. The classical approach to the convergence of this type of series was provided by Bhalekar and Daftardar-Gejji [9] and Daftardar-Gejji and Jafari [10].

4. Solution of third-order time-fractional dispersive partial differential equations

In this part, the reliable method described above is used to solve homogenous and non-homogenous third-order time-fractional dispersive partial differential equations with initial conditions in one and higher-dimensional spaces.

Example 1. Consider the following homogenous third-order time-fractional dispersive partial differential equation in one dimensional space [16,19,20]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+2\frac{\partial u(x,t)}{\partial x}+\frac{\partial^3 u(x,t)}{\partial x^3}=0,\ \ 0<\alpha\le 1, t>0, \tag{19}$$

with the initial condition

$$u(x,0)=\sin x. \tag{20}$$

Taking the Sumudu transform of the above equation (19), we have

$$S\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right]=-S\left[2\frac{\partial u(x,t)}{\partial x}+\frac{\partial^3 u(x,t)}{\partial x^3}\right]. \tag{21}$$

By using equation (7), we have

$$S\left[u(x,t)\right]=u(x,0)-\omega^\alpha S\left(2\frac{\partial u(x,t)}{\partial x}+\frac{\partial^3 u(x,t)}{\partial x^3}\right). \tag{22}$$

Applying inverse Sumudu transform to the equation (22), we obtain

$$u(x,t)=S^{-1}\left[u(x,0)\right]-S^{-1}\left(\omega^\alpha S\left(2\frac{\partial u(x,t)}{\partial x}+\frac{\partial^3 u(x,t)}{\partial x^3}\right)\right). \tag{23}$$

Substituting the results from equations (13) to (15) in the equation (23) and making use of the result given by the equation (17), we determine the components of the STIM solution as follows

$$u_0(x,t)=S^{-1}\left[u(x,0)\right]=\sin x, \tag{24}$$

$$u_1(x,t)=-S^{-1}\left(\omega^\alpha S\left(2\frac{\partial u_0}{\partial x}+\frac{\partial^3 u_0}{\partial x^3}\right)\right)=-\cos x\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{25}$$

$$\begin{aligned}u_2(x,t)&=-S^{-1}\left(\omega^\alpha S\left(2\frac{\partial (u_0+u_1)}{\partial x}+\frac{\partial^3 (u_0+u_1)}{\partial x^3}\right)\right)+S^{-1}\left(\omega^\alpha S\left(2\frac{\partial u_0}{\partial x}+\frac{\partial^3 u_0}{\partial x^3}\right)\right)\\&=-\sin x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\end{aligned} \tag{26}$$

$$\begin{aligned}u_3(x,t)&=-S^{-1}\left(\omega^\alpha S\left(2\frac{\partial (u_0+u_1+u_2)}{\partial x}+\frac{\partial^3 (u_0+u_1+u_2)}{\partial x^3}\right)\right)\\&\quad+S^{-1}\left(\omega^\alpha S\left(2\frac{\partial (u_0+u_1)}{\partial x}+\frac{\partial^3 (u_0+u_1)}{\partial x^3}\right)\right)\\&=\cos x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\end{aligned} \tag{27}$$

and so on. The remaining components may be obtained similarly.

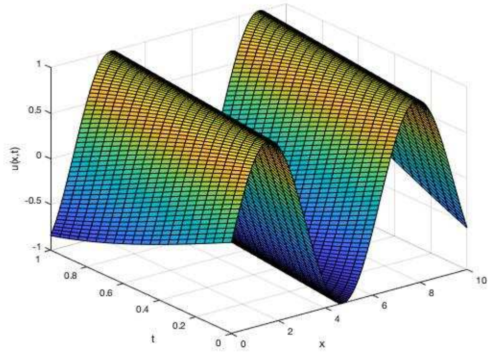
Thus, the series-form approximate analytical solution can be obtained as

$$
\begin{aligned}
u(x,t) &= \lim_{N\to\infty}\sum_{r=0}^{N} u_r(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\ldots \\
&= \sin x\left(1-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-\ldots\right) \\
&\quad -\cos x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}-\ldots\right).
\end{aligned} \tag{28}
$$

The same result was obtained by Pandey and Mishra [16] using HASTM, Shah *et al.* [20] using LADM, Ravi Kanth and Aruna [19] using FDTM and MFDTM.
If we put $\alpha = 1$, in equation (28), we have

$$u(x,t) = \sin(x-t). \tag{29}$$

This result was earlier achieved by Wazwaz [25] using the ADM approach.



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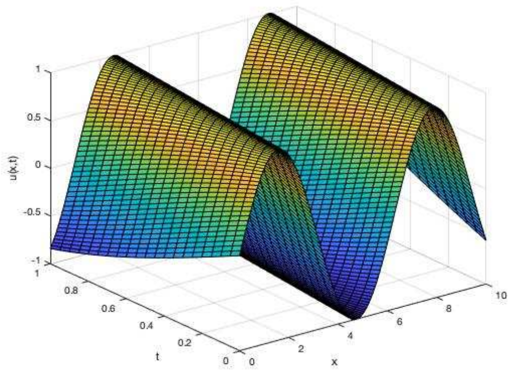


(b)

Figure 1. The surface shows the solution $u(x,t)$ for Example 1: **(a)** The exact solution **(b)** The approximate solution at $\alpha = 1$.

Example 2. Consider the following homogenous third-order time-fractional dispersive partial differential equation in two-dimensional space [20]

$$\frac{\partial^{\alpha}u(x,y,t)}{\partial t^{\alpha}}+\frac{\partial^{3}u(x,y,t)}{\partial x^{3}}+\frac{\partial^{3}u(x,y,t)}{\partial y^{3}}=0,\ \ 0<\alpha\le 1, t>0, \tag{30}$$

with the initial condition

$$u(x,y,0)=\cos(x+y). \tag{31}$$

Taking the Sumudu transform of the above equation (30), we have

$$S\left[\frac{\partial^{\alpha}u(x,y,t)}{\partial t^{\alpha}}\right]=-S\left[\frac{\partial^{3}u(x,y,t)}{\partial x^{3}}+\frac{\partial^{3}u(x,y,t)}{\partial y^{3}}\right]. \tag{32}$$

By using equation (7), we have

$$S\left[u(x,y,t)\right]=u(x,y,0)-\omega^{\alpha}S\left[\frac{\partial^{3}u(x,y,t)}{\partial x^{3}}+\frac{\partial^{3}u(x,y,t)}{\partial y^{3}}\right]. \tag{33}$$

Applying inverse Sumudu transform to the equation (33), we obtain

$$u(x,y,t)=S^{-1}\left[u(x,y,0)\right]-S^{-1}\left[\omega^{\alpha}S\left[\frac{\partial^{3}u(x,y,t)}{\partial x^{3}}+\frac{\partial^{3}u(x,y,t)}{\partial y^{3}}\right]\right]. \tag{34}$$

Substituting the results from equations (13) to (15) in the equation (34) and making use of the result given by the equation (17), we determine the components of the STIM solution as follows

$$u_0(x,y,t)=S^{-1}\left[u(x,y,0)\right]=\cos(x+y), \tag{35}$$

$$u_1(x,y,t)=-S^{-1}\left[\omega^{\alpha}S\left[\frac{\partial^{3}u_0}{\partial x^{3}}+\frac{\partial^{3}u_0}{\partial y^{3}}\right]\right]=-2\sin(x+y)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{36}$$

$$\begin{aligned}u_2(x,y,t)&=-S^{-1}\left[\omega^{\alpha}S\left[\frac{\partial^{3}(u_0+u_1)}{\partial x^{3}}+\frac{\partial^{3}(u_0+u_1)}{\partial y^{3}}\right]\right]+S^{-1}\left[\omega^{\alpha}S\left[\frac{\partial^{3}u_0}{\partial x^{3}}+\frac{\partial^{3}u_0}{\partial y^{3}}\right]\right]\\&=-4\cos(x+y)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\end{aligned} \tag{37}$$

$$\begin{aligned}u_3(x,y,t)&=-S^{-1}\left[\omega^{\alpha}S\left[\frac{\partial^{3}(u_0+u_1+u_2)}{\partial x^{3}}+\frac{\partial^{3}(u_0+u_1+u_2)}{\partial y^{3}}\right]\right]\\&\quad+S^{-1}\left[\omega^{\alpha}S\left[\frac{\partial^{3}(u_0+u_1)}{\partial x^{3}}+\frac{\partial^{3}(u_0+u_1)}{\partial y^{3}}\right]\right]\\&=8\sin(x+y)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\end{aligned} \tag{38}$$

and so on. The remaining components may be obtained similarly.
Thus, the series-form approximate analytical solution can be obtained as

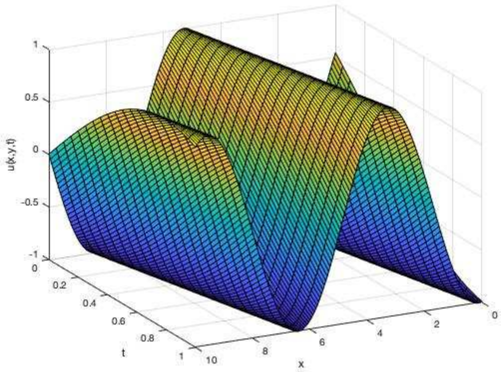
$$\begin{aligned} u(x,y,t) &= \lim_{N\to\infty}\sum_{r=0}^{N} u_r(x,y,t) = u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+u_3(x,y,t)+\ldots \\ &= \cos(x+y)\left(1-\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{16t^{4\alpha}}{\Gamma(4\alpha+1)}-\ldots\right) \\ &\quad -\sin(x+y)\left(\frac{2t^{\alpha}}{\Gamma(\alpha+1)}-\frac{8t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{32t^{5\alpha}}{\Gamma(5\alpha+1)}-\ldots\right). \end{aligned} \tag{39}$$

The same result was obtained by Shah *et al.* [20] using LADM.

If we put $\alpha=1$, in equation (39), we have

$$u(x,y,t)=\cos(x+y+2t). \tag{40}$$

This result was earlier achieved by Shah *et al.* [20] using the LADM approach.



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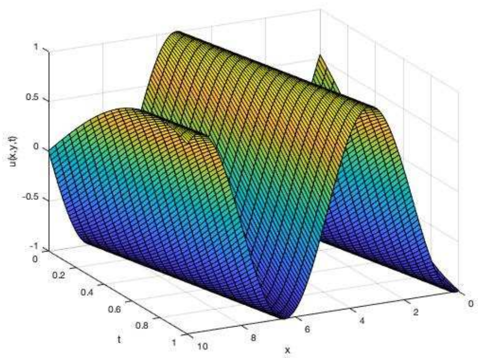


(b)

Figure 2. The surface shows the solution $u(x,y,t)$ for Example 2: **(a)** The exact solution **(b)** The approximate solution at $\alpha=1, y=1$.

Example 3. Consider the following homogenous third-order time-fractional dispersive partial differential equation in two-dimensional space [16,18,19]

$$\frac{\partial^{\alpha}u(x,y,t)}{\partial t^{\alpha}}+2\frac{\partial^{3}u(x,y,t)}{\partial x^{3}}+\frac{\partial^{3}u(x,y,t)}{\partial y^{3}}=0,\ \ 0<\alpha\le 1,\ \ t>0, \tag{41}$$

with the initial condition

$$u(x,y,0)=\cos(x+y). \tag{42}$$

Taking the Sumudu transform of the above equation (41), we have

$$S\left[\frac{\partial^{\alpha}u(x,y,t)}{\partial t^{\alpha}}\right]=-S\left[2\frac{\partial^{3}u(x,y,t)}{\partial x^{3}}+\frac{\partial^{3}u(x,y,t)}{\partial y^{3}}\right]. \tag{43}$$

By using equation (7), we have

$$S\left[u(x,y,t)\right]=u(x,y,0)-\omega^{\alpha}S\left[2\frac{\partial^{3}u(x,y,t)}{\partial x^{3}}+\frac{\partial^{3}u(x,y,t)}{\partial y^{3}}\right]. \tag{44}$$

Applying inverse Sumudu transform to the equation (44), we obtain

$$u(x,y,t)=S^{-1}\left[u(x,y,0)\right]-S^{-1}\left[\omega^{\alpha}S\left[2\frac{\partial^{3}u(x,y,t)}{\partial x^{3}}+\frac{\partial^{3}u(x,y,t)}{\partial y^{3}}\right]\right]. \tag{45}$$

Substituting the results from equations (13) to (15) in the equation (45) and making use of the results given by the equation (17), we determine the components of the STIM solution as follows

$$u_0(x,y,t)=S^{-1}\left[u(x,y,0)\right]=\cos(x+y), \tag{46}$$

$$u_1(x,y,t)=-S^{-1}\left[\omega^{\alpha}S\left[2\frac{\partial^{3}u_0}{\partial x^{3}}+\frac{\partial^{3}u_0}{\partial y^{3}}\right]\right]=-3\sin(x+y)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{47}$$

$$\begin{aligned}u_2(x,y,t)&=-S^{-1}\left[\omega^{\alpha}S\left[2\frac{\partial^{3}(u_0+u_1)}{\partial x^{3}}+\frac{\partial^{3}(u_0+u_1)}{\partial y^{3}}\right]\right]+S^{-1}\left[\omega^{\alpha}S\left[2\frac{\partial^{3}u_0}{\partial x^{3}}+\frac{\partial^{3}u_0}{\partial y^{3}}\right]\right]\\&=-9\cos(x+y)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\end{aligned} \tag{48}$$

$$\begin{aligned}u_3(x,y,t)&=-S^{-1}\left[\omega^{\alpha}S\left[2\frac{\partial^{3}(u_0+u_1+u_2)}{\partial x^{3}}+\frac{\partial^{3}(u_0+u_1+u_2)}{\partial y^{3}}\right]\right]\\&\quad+S^{-1}\left[\omega^{\alpha}S\left[2\frac{\partial^{3}(u_0+u_1)}{\partial x^{3}}+\frac{\partial^{3}(u_0+u_1)}{\partial y^{3}}\right]\right]\\&=27\sin(x+y)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\end{aligned} \tag{49}$$

and so on. The remaining components may be obtained similarly.

Thus, the series-form approximate analytical solution can be obtained as

$$u(x,y,t)=\lim_{N\to\infty}\sum_{r=0}^{N}u_r(x,y,t)=u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+u_3(x,y,t)+\dots$$

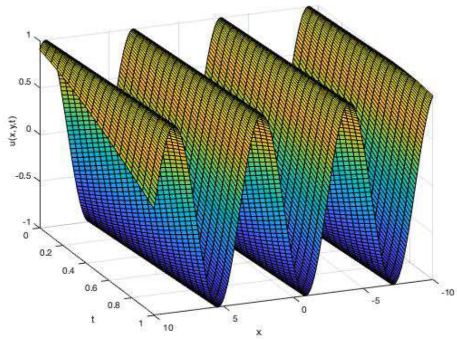
$$\begin{aligned}&=\cos(x+y)\left(1-\frac{9t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{81t^{4\alpha}}{\Gamma(4\alpha+1)}-\dots\right)\\&\qquad-\sin(x+y)\left(\frac{3t^{\alpha}}{\Gamma(\alpha+1)}-\frac{27t^{3\alpha}}{\Gamma(3\alpha+1)}-\dots\right).\end{aligned} \tag{50}$$

The same result was obtained by Pandey and Mishra [16] using HASTM, Prakash and Kumar [18] using FVIM, Ravi Kanth and Aruna [19] using the FDTM and MFDTM.

If we put $\alpha=1$, in equation (50), we have

$$u(x,t)=\cos\,(x+y+3t). \tag{51}$$

This result was earlier achieved by Ravi Kanth and Aruna [19] using the FDTM and MFDTM.



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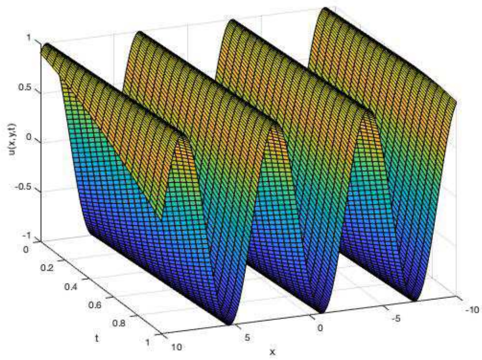


(b)

Figure 3. The surface shows the solution $u(x,y,t)$ for Example 3 : **(a)** The exact solution **(b)** The approximate solution at $\alpha=1, y=1$.

Example 4. Consider the following non-homogenous third-order time-fractional dispersive partial differential equation in one-dimensional space [19,20]

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\frac{\partial^{3}u(x,t)}{\partial x^{3}}=-\sin\pi x\sin t-\pi^{3}\cos\pi x\cos t,\ 0<\alpha\le 1, t>0, \tag{52}$$

with the initial condition

$$u(x,0)=\sin\pi x. \tag{53}$$

Taking the Sumudu transform of the above equation (52), we have

$$S\left[\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}\right]=-S\left[\sin\pi x\sin t+\pi^{3}\cos\pi x\cos t\right]-S\left[\frac{\partial^{3}u(x,t)}{\partial x^{3}}\right]. \tag{54}$$

By using equation (7), we have

$$S\left[u(x,t)\right]=u(x,0)-\omega^{\alpha}S\left[\sin\pi x\sin t+\pi^{3}\cos\pi x\cos t\right]-\omega^{\alpha}S\left[\frac{\partial^{3}u(x,t)}{\partial x^{3}}\right]. \tag{55}$$

Applying inverse Sumudu transform to the equation (55), we obtain

$$\begin{aligned}u(x,t)=S^{-1}\left[u(x,0)\right]-S^{-1}\left[\omega^{\alpha}S\left[\sin\pi x\sin t+\pi^{3}\cos\pi x\cos t\right]\right]\\ -S^{-1}\left[\omega^{\alpha}S\left[\frac{\partial^{3}u(x,t)}{\partial x^{3}}\right]\right].\end{aligned} \tag{56}$$

Substituting the results from equations (13) to (15) in the equation (56) and making use of the result given by the equation (17), we determine the components of the STIM solution as follows

$$\begin{aligned}u_0(x,t)&=S^{-1}\left[u(x,0)\right]-S^{-1}\left[\omega^{\alpha}S\left[\sin\pi x\sin t+\pi^{3}\cos\pi x\cos t\right]\right]\\ &=\sin\pi x-\sin\pi x\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{t^{\alpha+3}}{\Gamma(\alpha+4)}+\frac{t^{\alpha+5}}{\Gamma(\alpha+6)}-\ldots\right)\\ &\quad-\pi^{3}\cos\pi x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{t^{\alpha+4}}{\Gamma(\alpha+5)}-\ldots\right),\end{aligned} \tag{57}$$

$$\begin{aligned}u_1(x,t)&=-S^{-1}\left[\omega^{\alpha}S\left[\frac{\partial^{3}v_0}{\partial x^{3}}\right]\right]=\pi^{3}\cos\pi x\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\ &\quad-\pi^{3}\cos\pi x\left(\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)}+\frac{t^{2\alpha+5}}{\Gamma(2\alpha+6)}-\ldots\right)\\ &\quad+\pi^{6}\sin\pi x\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)}+\frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)}-\ldots\right),\end{aligned} \tag{58}$$

$$u_2(x,t) = -S^{-1}\left[\omega^{\alpha} S\left[\frac{\partial^3(u_0+u_1)}{\partial x^3}\right]\right] + S^{-1}\left[\omega^{\alpha} S\left[\frac{\partial^3 u_0}{\partial x^3}\right]\right] = -\pi^6 \sin\pi x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+\pi^6 \sin\pi x\left(\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \frac{t^{3\alpha+3}}{\Gamma(3\alpha+4)} + \frac{t^{3\alpha+5}}{\Gamma(3\alpha+6)} - \ldots\right)$$
$$+\pi^9 \cos\pi x\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)} - \ldots\right), \tag{59}$$

and so on. The remaining components may be obtained similarly.

Thus, the series-form approximate analytical solution can be obtained as

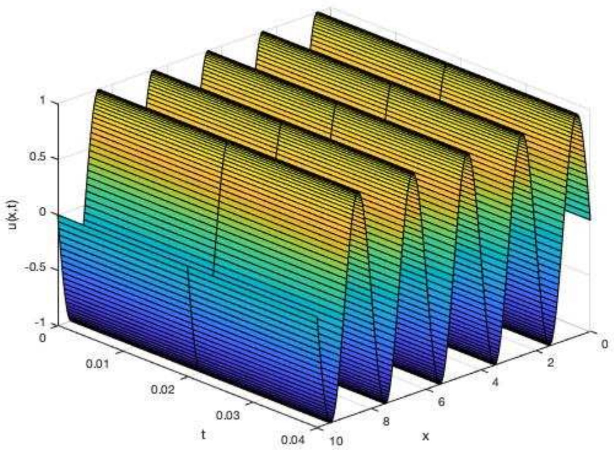
$$u(x,t) = \lim_{N\to\infty}\sum_{r=0}^{N} u_r(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$
$$= \sin\pi x - \sin\pi x\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{t^{\alpha+5}}{\Gamma(\alpha+6)} - \ldots\right)$$
$$-\pi^3 \cos\pi x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} - \ldots\right)$$
$$+\pi^3 \cos\pi x \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \pi^3 \cos\pi x\left(\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} + \frac{t^{2\alpha+5}}{\Gamma(2\alpha+6)} - \ldots\right)$$
$$+\pi^6 \sin\pi x\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)} - \ldots\right)$$
$$-\pi^6 \sin\pi x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \pi^6 \sin\pi x\left(\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \frac{t^{3\alpha+3}}{\Gamma(3\alpha+4)} + \frac{t^{3\alpha+5}}{\Gamma(3\alpha+6)} - \ldots\right)$$
$$+\pi^9 \cos\pi x\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)} - \ldots\right). \tag{60}$$

The same result was obtained by Shah *et al.* [20] using LADM, Ravi Kanth and Aruna [19] using FDTM and MFDTM.

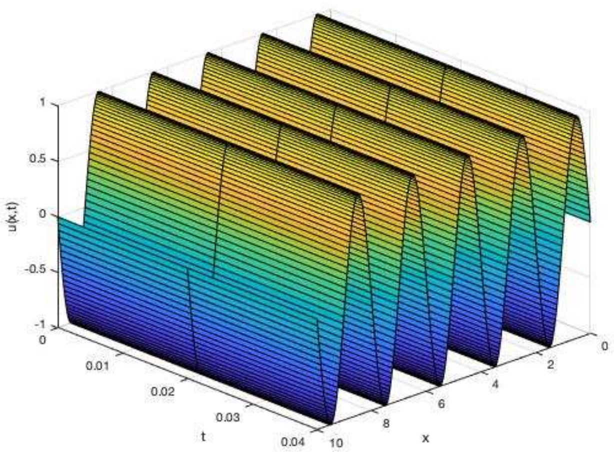
If we put $\alpha = 1$, in equation (60), we have

$$u(x,t) = \sin\pi x \cos t. \tag{61}$$

This result was earlier achieved by Wazwaz [25] using the ADM approach.



(a)



(b)

Figure 4. The surface shows the solution $u(x,t)$ for Example 4 : **(a)** The exact solution **(b)** The approximate solution at $\alpha = 1$.

5. Concluding remarks

In this paper, we have used the iterative Sumudu transform method to find series form approximate analytical solutions to third-order time-fractional dispersive partial differential equations in one and higher dimensions. The time fractional derivatives are considered in the Caputo sense. Moreover, the results of this study are presented graphically, and the solution graphs demonstrate that the approximate solutions are closely connected to the exact solutions.

Acknowledgments. The authors are very thankful to the reviewer for meticulous efforts to improve the manuscript. Also, Ms. Priyanka, one of the authors, would like to thank the University Grants Commission (UGC), New Delhi, for providing her a Junior Research Fellowship (JRF).

Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. All the authors contributed equally to this work.

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