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Numerical Solution of Boundary Value Problems for Second Order Fuzzy Linear Differential Equations

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Abstract. In this paper, the numerical solution of the boundary value problem for second order fuzzy linear differential equations is discussed. We consider the fuzzy difference equation to replace the fuzzy differential equation. The numerical solution of the boundary value problem is obtained by calculating the fuzzy difference equation. Finally, an example is given, to verify the effectiveness of this method.

Keywords: Fuzzy number, fuzzy differential equation, fuzzy boundary value problem, fuzzy difference problem

AMS Mathematics Subject Classification (2010): 65L07

1. Introduction

Many engineering system problems are too complex to be directly converted into system equations (groups) to solve, and often involve parameter uncertainties, which often appear as a fuzzy number [1,2]. Therefore, when solving such problems. it is often necessary to convert it into a fuzzy system equation to consider, which makes the solution of fuzzy system equations very important.

In recent years. many scholars have made profound research on second-order fuzzy differential equations given the fuzzy boundary conditions. Wu Q [3] discussed the uncertainty of the two-point boundary value of the second-order differential equation. Using the fuzzy simulation principle and the difference method, the numerical solution of the boundary value problem is obtained. Regan et al. [4] proved a transcendentalresult on the solvability of fuzzy boundary value problems based on the generalized Schauder theorem. WuCX [5] et al. proved that only the extension of the definition of fuzzy numbers, given its new structure and properties, the existence of analytical solutions for fuzzy boundary value problems. Guo et al. (8-9] studied the approximate solutions of second—order linear ‹differential equations with several fuzzy boundary conditions.

This paper mainly studies the boundary value problem of second order fuzzy linear differential equations, under the condition of parameter numbering.

$$\begin{cases} \tilde{y}'' + p(t)\tilde{y}' + q(t)\tilde{y} = g(t), t \in [a,b], \\ \tilde{y}(a) = \tilde{\alpha}, \tilde{y}(b) = \tilde{\beta}, \end{cases}$$

where $\alpha,\beta \in E'$ are fuzzy numbers, $p(t), q(t)$ are coefficient function. Using the difference of each order to replace the corresponding derivative in the equation, thus transforming the boundary value problem of fuzzy differential equation into fuzzy difference problem. And use numerical examples to verify this method.

2. Preliminaries

In this section, we give some definitions and lemmas which will be used later.

2.1. Fuzzy number

Definition 2.1. [1] A fuzzy number is a fuzzy set like $u: R \to I = [0,1]$ which satisfies:

(1) u is upper semi—continuous,

(2) $u(x) = 0$ outside some interval $[c, d]$.

(3) there are real number a,b such that $c \le a \le b \le d$ and

(i) $u(x)$ is monotonic increasing on $[c, a]$,

(ii) $u(x)$ is monotonic decreasing on $[b, d]$

(iii) $u(x) = 1,\ a \le x \le b.$

Let E' be the set of all real fuzzy numbers which are normal, upper semi-continuous, convex and compactly supported fuzzy sets.

Definition 2.2. [2] A fuzzy number u in a parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r), 0 \le r \le 1$ which satisfies the following requirements:

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,

(2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,

(3) $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$

Definition 2.3. [2] Let $x = (\underline{x}(r), \overline{x}(r)), y = \underline{y}(r), \overline{y}(r) \in E', 0 \le r \le 1$ and arbitrary $k \in R$, then

(1) $x = y$ iff $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r)) = \overline{y}(r),$

(2) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r)),$

(3) $x - y = (\underline{x}(r) - \overline{y}(r), \overline{x}(r) - \underline{y}(r)),$

(4) $kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), k \ge 0 \\ (k\overline{x}(r), k\underline{x}(r)), k < 0. \end{cases}$

2.2. Second-order fuzzy boundary value problem

Definition 2.4. [6] Let $x, y \in E'$. If there exists $z \in E'$ such that $x = y + z$, ,then z is called the Hukuhara difference of fuzzy numbers x and y, and it is denoted by $z = x !\ y$.

Definition 2.5. [7] Let $f:[a,b]\to E'$ and $t_0\in[a,b]$. We say that f is Hukuhara differential at t_0, if there exists an element $f'(t_0)\in E'$ such that for all $h>0$ sufficiently small, $\exists f(t_0+h)\ominus f(t_0), f(t_0)\ominus f(t_0-h)$ and the limits

$$\lim_{h\to 0}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to 0}\frac{f(t_0)\ominus f(t_0-h)}{h}=f'(t_0).$$

Definition 2.6. [8] The second-order fuzzy differential equation

$$\tilde{y}''=f(t,\tilde{y},\tilde{y}'),t\in[a,b] \tag{2.1}$$

with the fuzzy boundary value conditions

$$\tilde{y}(a)=\tilde{\alpha},\tilde{y}(b)=\tilde{\beta},\tilde{\alpha},\tilde{\beta}\in E' \tag{2.2}$$

Is called the second-order fuzzy boundary value problems, where $\tilde{\alpha}=(\underline{\alpha}(r),\overline{\beta}(r))$.

In this paper, we mainly study the boundary value profiles of second order fuzzy linear differential equations, under the condition of parameter numbering.

$$\begin{cases}\tilde{y}''+p(t)\tilde{y}'+q(t)\tilde{y}=g(t),t\in[a,b]\\ \tilde{y}(a)=\tilde{\alpha},\tilde{y}(b)=\tilde{\beta}\end{cases} \tag{2.3}$$

where $\tilde{\alpha},\tilde{\beta}\in E'$ are fuzzy numbers, $p(t)$, $q(t)$ are coefficient function. And the parameter form is

$$\begin{cases}\underline{y''(t,r)+p(t)y'(t,r)+q(t)y(t,r)=g(t,r)}\\ \underline{y}(a,r)=\underline{\alpha}(r),\underline{y}(b,r)=\underline{\beta}(r)\\ \overline{y''(t,r)+p(t)y'(t,r)+q(t)y(t,r)=g(t,r)}\\ \overline{y}(a,r)=\overline{\alpha}(r),\overline{y}(b,r)=\overline{\beta}(r)\end{cases} \tag{2.4}$$

where $t\in[a,b],0\le r\le 1$.

Lemma 2.1. [9] If the fuzzy boundary value problem (2.3) is satisfied

(1) p(t),q(t),g(t) is continuous on [a,b].

(2) p(t),q(t) is invariant on [a,b].

then equation (2.3) has a unique solution on [a,b].

3. The establishment of the difference method

In this section, we studying the establishment of the difference methods and discussed the solvability of fuzzy problems.

3.1. The fuzzy difference problem

First, we use the following first-order difference quotient to approximate the first derivative, which is

$$\frac{y(t_0+h)-y(t_0)}{h} \approx y'(t),$$

or

$$\frac{y(t_0)-y(t_0-h)}{h} \approx y'(t),$$

or

$$\frac{y(t_0+h)-y(t_0-h)}{2h} \approx y'(t).$$

Then the second derivative can be approximated by the first-order difference quotient of the first-order difference quotient, that is,

$$y''(t) \approx \frac{y(t_0+h)-2y(t_0)+y(t_0-h)}{h^2}.$$

Let the integral interval [a, b] be divided into N equal parts, the step size is $h=\dfrac{b-a}{N}$, Its node is $t_n=t_0+nh, n=0,1,\cdots,N$. Then use the difference quotient instead of the corresponding derivative, and the fuzzy differential equation boundary value problem (2.3) can be discretized into the following fuzzy difference problem.

$$\begin{cases} \dfrac{\tilde{y}_{n+1}-2\tilde{y}_n+\tilde{y}_{n-1}}{h^2}=\tilde{f}(t_n,y_n,\dfrac{y_{n+1}-y_{n-1}}{2h}) \\ y_0=\tilde{\alpha}, y_N=\tilde{\beta}, \tilde{\alpha},\tilde{\beta}\in E' \end{cases} \tag{3.1}$$

According to the different p(t) and q(t) symbols, we discuss the solution of equation (3.1) from the following four cases.

Case 1: p(t)>0 and q(t)>0, the difference form of the fuzzy boundary value problem (2.3) is

$$\begin{cases} \dfrac{\underline{\underline{y_{n+1}}}(r)-2\overline{y_n}(r)+\underline{\underline{y_{n-1}}}(r)}{h^2}+p_n\dfrac{\underline{\underline{y_{n+1}}}(r)-\overline{y_{n-1}}(r)}{2h}+q_n\underline{y_n}(r)=\underline{g_n}(r) \\ \underline{y_0}(r)=\underline{\alpha}(r), \underline{y_N}(r)=\underline{\beta}(r) \\ \dfrac{\overline{y_{n+1}}(r)-2\underline{y_n}(r)+\overline{y_{n-1}}(r)}{h^2}+p_n\dfrac{\overline{y_{n+1}}(r)-\underline{y_{n-1}}(r)}{2h}+q_n\overline{y_n}(r)=\overline{g_n}(r) \\ \overline{y_0}(r)=\overline{\alpha}(r), \overline{y_N}(r)=\overline{\beta}(r) \end{cases}$$

Finishing the following the $2(N-1)\times 2(N-1)$ order matrix equations, in the form of

$$\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}\begin{pmatrix} \underline{Y}_{N-1}(r) \\ \overline{Y}_{N-1}(r) \end{pmatrix}=\begin{pmatrix} \underline{B}_{N-1}(r) \\ \overline{B}_{N-1}(r) \end{pmatrix}$$

where

$$A_1=\begin{pmatrix} q_1h^2 & 1+\frac{h}{2}p_1 & 0 & \cdots & 0 \\ 1 & q_2h^2 & 1+\frac{h}{2}p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q_{N-2}h^2 & 1+\frac{h}{2}p_{N-2} \\ 0 & 0 & \cdots & 1 & q_{N-1}h^2 \end{pmatrix}, A_2=\begin{pmatrix} -2 & 0 & \cdots & 0 \\ -\frac{h}{2} & -2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & -2 \end{pmatrix},$$

$$\underline{Y}_{N-1}(r)=(\underline{y}_1(r),\cdots,\underline{y}_{N-1}(r))^T$$
$$\overline{Y}_{N-1}(r)=(\overline{y}_{N-1}(r),\cdots,\overline{y}_{N-1}(r))^T$$

$$\underline{B}_{N-1}(r)=(\underline{g}_1(r)h^2+\frac{h}{2}p_1\underline{\alpha}(r)-\overline{\alpha}(r),\underline{g}_2(r)h^2,\cdots,\underline{g}_{N-2}(r)h^2,\underline{g}_{N-1}(r)h^2-(1+\frac{h}{2}p_{N-1})\overline{\beta}(r))^T$$
$$\overline{B}_{N-1}(r)=(\overline{g}_1(r)h^2+\frac{h}{2}p_1\overline{\alpha}(r)-\underline{\alpha}(r),\overline{g}_2(r)h^2,\cdots,\overline{g}_{N-2}(r)h^2,\overline{g}_{N-1}(r)h^2-(1+\frac{h}{2}p_{N-1})\underline{\beta}(r))^T$$

Case 2: p(t)<0 and q(t)<0, the difference form of the fuzzy boundary value problem (2.3) is

$$\begin{cases} \dfrac{\underline{\underline{y}}_{n+1}(r)-2\overline{y_n}(r)+\underline{y}_{n-1}(r)}{h^2}-p_n\dfrac{\overline{y_{n+1}}(r)-\underline{\underline{y}}_{n-1}(r)}{2h}-q_n\overline{y_n}(r)=\underline{g}_n(r) \\ \underline{y}_0(r)=\underline{\alpha}(r),\underline{y}_N(r)=\underline{\beta}(r) \\ \dfrac{\overline{y_{n+1}}(r)-2\underline{y}_n(r)+\overline{y_{n-1}}(r)}{h^2}-p_n\dfrac{\underline{\underline{y}}_{n+1}(r)-\overline{y_{n-1}}(r)}{2h}-q_n\underline{y}_n(r)=\overline{g_n}(r) \\ \overline{y_0}(r)=\overline{\alpha}(r),\overline{y_N}(r)=\overline{\beta}(r) \end{cases}$$

Finishing the following the $2(N-1)\times 2(N-1)$ order matrix equations, in the form of

$$\begin{pmatrix} C_1 & C_2 \\ C_2 & C_1 \end{pmatrix}\begin{pmatrix} \underline{Y}_{N-1}(r) \\ \overline{Y}_{N-1}(r) \end{pmatrix}=\begin{pmatrix} \underline{D}_{N-1}(r) \\ \overline{D}_{N-1}(r) \end{pmatrix}$$

where

$$C_1=\begin{pmatrix}0 & 1 & \cdots & 0\\ 1-\frac{h}{2}p_1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}, C_2=\begin{pmatrix}-(2+q_1h^2) & -\frac{h}{2}p_1 & \cdots & 0\\ 0 & -(2+q_1h^2) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -(2+q_{N-1}h^2)\end{pmatrix},$$

$$\underline{D}_{N-1}(r)=(\underline{g}_1(r)h^2+\frac{h}{2}p_1\underline{\alpha}(r)-\overline{\alpha}(r),\underline{g}_2(r)h^2,\cdots,\underline{g}_{N-2}(r)h^2,\underline{g}_{N-1}(r)h^2-(1+\frac{h}{2}p_{N-1})\overline{\beta}(r))^T$$

$$\overline{D}_{N-1}(r)=(\overline{g}_1(r)h^2+\frac{h}{2}p_1\overline{\alpha}(r)-\underline{\alpha}(r),\overline{g}_2(r)h^2,\cdots,\overline{g}_{N-2}(r)h^2,\overline{g}_{N-1}(r)h^2-(1+\frac{h}{2}p_{N-1})\underline{\beta}(r))^T$$

Case 3: p(t)<0 and q(t)>0, the difference form of the fuzzy boundary value problem (2.3) is

$$\begin{cases}\dfrac{\underline{y}_{n+1}(r)-2\overline{y}_n(r)+\underline{y}_{n-1}(r)}{h^2}-p_n\dfrac{\overline{y}_{n+1}(r)-\underline{y}_{n-1}(r)}{2h}+q_n\underline{y}_n(r)=\underline{g}_n(r)\\ \underline{y}_0(r)=\underline{\alpha}(r),\underline{y}_N(r)=\underline{\beta}(r)\\ \dfrac{\overline{y}_{n+1}(r)-2\underline{y}_n(r)+\overline{y}_{n-1}(r)}{h^2}-p_n\dfrac{\underline{y}_{n+1}(r)-\overline{y}_{n-1}(r)}{2h}+q_n\overline{y}_n(r)=\overline{g}_n(r)\\ \overline{y}_0(r)=\overline{\alpha}(r),\overline{y}_N(r)=\overline{\beta}(r)\end{cases}$$

Finishing the following the $2(N-1)\times 2(N-1)$ order matrix equations,in the form of

$$\begin{pmatrix}M_1 & M_2\\ M_2 & M_1\end{pmatrix}\begin{pmatrix}\underline{Y}_{N-1}(r)\\ \overline{Y}_{N-1}(r)\end{pmatrix}=\begin{pmatrix}\underline{N}_{N-1}(r)\\ \overline{N}_{N-1}(r)\end{pmatrix}$$

where $M_1=\begin{pmatrix}q_1h^2 & 1 & \cdots & 0\\ 1-\frac{h}{2}p_2 & q_2h^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & q_{N-1}h^2\end{pmatrix} M_2=\begin{pmatrix}-2 & -\frac{h}{2}p_1 & \cdots & 0\\ 0 & -2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -2\end{pmatrix}$,

$$\underline{N}_{N-1}(r)=(\underline{g}_1(r)h^2+\frac{h}{2}p_1\underline{\alpha}(r)-\overline{\alpha}(r),\underline{g}_2(r)h^2,\cdots,\underline{g}_{N-2}(r)h^2,\underline{g}_{N-1}(r)h^2-(1+\frac{h}{2}p_{N-1})\overline{\beta}(r))^T$$

$$\overline{N}_{-1}(r)=(\overline{g}_1(r)h^2+\frac{h}{2}p_1\overline{\alpha}(r)-\underline{\alpha}(r),\overline{g}_2(r)h^2,\cdots,\overline{g}_{N-2}(r)h^2,\overline{g}_{N-1}(r)h^2-(1+\frac{h}{2}p_{N-1})\underline{\beta}(r))^T$$

Case 4: p(t)>0 and q(t)<0, the difference form of the fuzzy boundary value problem (2.3) is

$$\begin{cases} \dfrac{\underline{y}_{n+1}(r)-2\overline{y}_n(r)+\underline{y}_{n-1}(r)}{h^2}+p_n\dfrac{\underline{y}_{n+1}(r)-\overline{y}_{n-1}(r)}{2h}-q_n\overline{y}_n(r)=\underline{g}_n(r) \\ \underline{y}_0(r)=\underline{\alpha}(r),\underline{y}_N(r)=\underline{\beta}(r) \\ \dfrac{\overline{y}_{n+1}(r)-2\underline{y}_n(r)+\overline{y}_{n-1}(r)}{h^2}+p_n\dfrac{\overline{y}_{n+1}(r)-\underline{y}_{n-1}(r)}{2h}-q_n\underline{y}_n(r)=\overline{g}_n(r) \\ \overline{y}_0(r)=\overline{\alpha}(r),\overline{y}_N(r)=\overline{\beta}(r) \end{cases}$$

Finishing the following the $2(N-1)\times 2(N-1)$ order matrix equations, in the form of

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}\begin{pmatrix} \underline{Y}_{N-1}(r) \\ \overline{Y}_{N-1}(r) \end{pmatrix}=\begin{pmatrix} \underline{T}_{N-1}(r) \\ \overline{T}_{N-1}(r) \end{pmatrix}$$

where $S_1=\begin{pmatrix} 0 & 1+\frac{h}{2} & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ $S_2=\begin{pmatrix} -(2+q_1h^2) & 0 & \cdots & 0 \\ -\frac{h}{2}p_2 & -(2+q_2h^2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -(2+q_{N-1}h^2) \end{pmatrix}$,

$$\underline{T}_{N-1}(r)=(\underline{g}_1(r)h^2+\frac{h}{2}p_1\underline{\alpha}(r)-\overline{\alpha}(r),\underline{g}_2(r)h^2,\cdots,\underline{g}_{N-2}(r)h^2,\underline{g}_{N-1}(r)h^2-(1+\frac{h}{2}p_{N-1})\overline{\beta}(r))^T$$

$$\overline{T}_{-1}(r)=(\overline{g}_1(r)h^2+\frac{h}{2}p_1\overline{\alpha}(r)-\underline{\alpha}(r),\overline{g}_2(r)h^2,\cdots,\overline{g}_{N-2}(r)h^2,\overline{g}_{N-1}(r)h^2-(1+\frac{h}{2}p_{N-1})\underline{\beta}(r))^T$$

3.2. Solvability of the fuzzy difference problems

Theorem 3.1. The solution of the boundary value problem (3.1) of fuzzy difference equations is existence and unique.

Proof: First, we can eliminate the first-order difference in the fuzzy difference equation by appropriate transformation of the independent variables.

$$\begin{cases} \dfrac{\underline{y}_{n+1}(r)-2\overline{y}_n(r)+\underline{y}_{n-1}(r)}{h^2}+q_n\underline{y}_n(r)=\underline{g}_n(r) \\ \underline{y}_0(r)=\underline{\alpha}(r),\underline{y}_N(r)=\underline{\beta}(r) \\ \dfrac{\overline{y}_{n+1}(r)-2\underline{y}_n(r)+\overline{y}_{n-1}(r)}{h^2}+q_n\overline{y}_n(r)=\overline{g}_n(r) \\ \overline{y}_0(r)=\overline{\alpha}(r),\overline{y}_N(r)=\overline{\beta}(r) \end{cases}$$

Just prove that the corresponding homogeneous linear equations

$$\begin{cases} \dfrac{\underline{y}_{n+1}(r)-2\overline{y}_n(r)+\underline{y}_{n-1}(r)}{h^2}+q_n\underline{y}_n(r)=0 \\ \underline{y}_0(r)=0, \underline{y}_N(r)=0 \\ \dfrac{\overline{y}_{n+1}(r)-2\underline{y}_n(r)+\overline{y}_{n-1}(r)}{h^2}+q_n\overline{y}_n(r)=0 \\ \overline{y}_0(r)=0, \overline{y}_N(r)=0 \end{cases}$$

have only zero solutions. Obviously, The positive and negative minimum values of $\tilde{y}_n$ can only be $\tilde{y}_0$ or $\tilde{y}_N$ .Also known by the boundary condition $\tilde{y}_0=\tilde{y}_N=[0,0]$, all $\tilde{y}_n=[0,0]$.

4. Numerical example

Example 4.1. Consider the boundary value problem of fuzzy differential equations as follows

$$\begin{cases} \tilde{y}''-\tilde{y}=t, t\in[0,1] \\ \tilde{y}(0)=(0.1-0.1r,-0.1+0.1r) \\ \tilde{y}(1)=(-0.1r,1+0.1r) \end{cases}$$

The exact solution is follows

$$\begin{cases} \underline{Y}(t,r)=(0.1-0.1r)cost+(-0.1r)sint-t \\ \overline{Y}(t,r)=(-0.1+0.1r)cost+(1+0.1r)sint-t \end{cases}$$

Converting the boundary value problem of fuzzy differential equations into boundary value problems of fuzzy difference equations, take the step size of 0.2, the node $t_n=\frac{n}{5},(n=0,1,2,3,4,5)$, then its matrix form is

$$\begin{pmatrix} 0 & 1.1 & 0 & 0 & -1.96 & 0 & 0 & 0 \\ 1 & 0 & 1.1 & 0 & 0 & -1.96 & 0 & 0 \\ 0 & 1 & 0 & 1.1 & 0 & 0 & -1.96 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1.96 \\ -1.96 & 0 & 0 & 0 & 0 & 1.1 & 0 & 0 \\ 0 & -1.96 & 0 & 0 & 1 & 0 & 1.1 & 0 \\ 0 & 0 & -1.96 & 0 & 0 & 1 & 0 & 1.1 \\ 0 & 0 & 0 & -1.96 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \\ \overline{y}_1 \\ \overline{y}_2 \\ \overline{y}_3 \\ \overline{y}_4 \end{pmatrix} = \begin{pmatrix} 0.092-0.1r \\ 0.016 \\ 0.024 \\ 0.032 \\ -0.092-0.1r \\ 0.016 \\ 0.024 \\ 0.032 \end{pmatrix}$$

Then, the solution of the boundary value problem of fuzzy differential equations is

$$\begin{cases} \underline{Y}(t,r) = (0.045+0.100r)+(-0.157+0.083r)t+(-0.032+0.060r)t^2+(-0.090+0.030r)t^3 \\ \overline{Y}(t,r) = (-0.135+0.100r)+(-0.003+0.083r)t+(-0.142+0.060r)t^2+(-0.033+0.030r)t^3 \end{cases}$$

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Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. All the authors contributed equally to this work.

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