Journal of Mathematics and Informatics
Vol. 22, 2022, 49-59
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 30 May 2022
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v22a05208

Journal of **Mathematics and Informatics**

A Further Study on *L*-Fuzzy Covering Rough Sets

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Received 4 April 2022; accepted 28 May 2022

Abstract. For $L=(L,*,\to)$ a complete residuated lattice, a type of L-fuzzy covering rough sets was defined by Li [8] in 2017. In this paper, a further study on rough sets was given. Precisely, a single axiomatic characterization of the L-fuzzy covering rough sets was presented, and the relationships between the L-fuzzy covering rough sets and L-fuzzy relation rough sets were established.

Keywords: L-fuzzy rough set; L-fuzzy covering; Residuated lattice; Axiomatic characterization

AMS Mathematics Subject Classification (2010): 03E72, 94D05

1. Introduction

Rough set theory [15,16], proposed by Pawlak in 1982, is a new tool for dealing with uncertain and incomplete knowledge. The classical Pawlak rough sets are based on equivalent relations, which greatly limits the scope of rough set theory and applications. Hence, many kinds of generalized rough sets were proposed [7, 11, 12, 24-28]. The core concept of (generalized) rough set theory is a pair of approximation operators. There are generally two different approaches to studying those operators: the constructive approach and the axiomatic approach. In the constructive approach, the binary relation, covering, and neighborhood (system) in the domain of discourse are regarded as the original concepts, and the lower (upper) approximation operator is constructed from them. In the axiomatic approach, a pair of abstract approximation operators are put as the initial concepts, then find an axiom set (or even a single axiom) to ensure the existence of binary relation, covering and neighborhood (system) to reproduce the initial approximation operators by the constructive approach.

Fuzzy covering (relation) rough sets are vital generalized rough sets [2-4, 10, 11, 13]. Especially, complete residuated lattice-valued fuzzy covering (relation) rough sets have attracted much attention for their many-valued logic background [1, 8, 9, 14, 17-22] (complete residuated lattice can be regarded as the truth table of many-valued logic [5]). In 2017, consider $L=(L,*,\to)$ a complete residuated lattice, Li [8] introduced a type of L-fuzzy covering rough sets and used axiom sets to characterize them. However, a more interesting single axiomatic characterization has not been given. In addition, it is known from [28] that the covering rough sets and relation rough sets are interrelated closely. Nowadays, the relationships between L-fuzzy covering rough sets and L-fuzzy relation rough sets have not been clarified. In this paper, we shall give a further study on L-fuzzy covering rough set around the above two problems.

The arrangement of this paper is as follows. In Section 2, We recall some the basic concepts and symbols. In Section 3, we give a single axiomatic characterization of the L-fuzzy covering approximation operators. In Section 4, we clarify the relationship between L-fuzzy covering approximation operators and L-fuzzy relation approximation operators. In Section 5, we conclude this paper.

2. Preliminaries

In this section, we shall recall some notions and notation for later use.

2.1. Complete lattice L and L-fuzzy sets

Let X be a nonempty set. And we use $P(X)$ to denote the power set of X, i.e., $P(X)=\{A \mid A\subseteq X\}$.

A complete residuated lattice is an algebra $L=(L,\wedge,\vee,*,0,1)$ fulfills:

(1) $L=(L,\wedge,\vee,0,1)$ is a complete lattice with the least (resp., largest) element 0 (resp., 1),

(2) $(L,*,1)$ is a commutative monoid with 1 as the unit element,

(3) $*$ distributes over arbitrary joins, that is, $\forall a,a_j(j\in J)\in L, a*(\vee_{j\in J} a_j)=\vee_{j\in J}(a*a_j)$.

The binary operation $\to: L\times L\to L$ determined by $a\to b=\vee\{c\in L \mid a*c\le b\}$ is called the residuated implication w.r.t. $*$.

A mapping $A: X\to L$ is called an L-fuzzy set on X [6]. All L-fuzzy sets on X are denoted by L^X. For $a\in L$, we also use a to denote the constant value L-fuzzy set values a. For $A\in P(X)$, we also use A to denote the L-fuzzy set values 1 at $x\in A$ and 0 otherwise.

An L-fuzzy set R on $X\times X$ is called an L-fuzzy relation on X.

Proposition 2.1. Let L be a complete residuated lattice.

(1) $1\to a=a, 1*a=a, a\le b\to a, a\le b\Leftrightarrow a\to b=1$.

(2) $(a*b)\to c=a\to(b\to c)=b\to(a\to c)$.

(3) $a\to \wedge_{j\in J} b_j=\wedge_{j\in J}(a\to b_j)$.(4) $\vee_{j\in J} a_j\to b=\wedge_{j\in J}(a_j\to b)$.(5) $a*\wedge_{j\in J} b_j\le \wedge_{j\in J}(a*b_j)$.

(6) $\vee_{j\in J}(a\to b_j)\le a\to \vee_{j\in J} b_j$.(7) $\vee_{j\in J}(a_j\to b)\le \wedge_{j\in J} a_j\to b$.

If we have $(a\to 0)\to 0=a$ for all $a\in L$, then we say the complete residuated lattice L is regular.

Definition 2.1. [8] For $A,B\in L^X$, we define

$$S_{\rightarrow}(A,B)=\underset{x\in X}{\wedge}\left(A(x)\rightarrow B(x)\right), I_{*}(A,B)=\underset{x\in X}{\vee}\left(A(x)*B(x)\right),$$

and call them the $\rightarrow$-subsethood degree and $*$-intersection degree of A,B, respectively.

Lemma 2.1. [8] For $A,B\in L^X, a\in L$.

(O1) $S_{\rightarrow}(A,B)=1\Leftrightarrow A\le B$. (O2) $S_{\rightarrow}(A,a\rightarrow B)=a\rightarrow S_{\rightarrow}(A,B)$.

(O3) $S_{\rightarrow}(A,a*B)\ge a*S_{\rightarrow}(A,B)$.

2.2 L-fuzzy relation rough sets via $*$ and $\rightarrow$

Definition 2.2. [19,20] Let R be an L-fuzzy relation on X. For any $A\in L^X$, the pair of L-fuzzy sets $\left(\underline{R}_{\rightarrow}(A),\overline{R}_{*}(A)\right)$ defined by $\forall x\in X$,

$$\underline{R}_{\rightarrow}(A)(x)=\underset{y\in X}{\wedge}\left(R(x,y)\rightarrow A(y)\right)=S_{\rightarrow}\left(R(x),A\right),$$

$$\overline{R}_{*}(A)(x)=\underset{y\in X}{\vee}\left(R(x,y)*A(y)\right)=I_{*}\left(R(x),A\right)$$

is said to be the L-fuzzy relation rough set of A. The associated mappings $\underline{R}_{\rightarrow}$ and $\overline{R}_{*}$ on L^X are called the lower and upper L-fuzzy relation approximation operators, respectively.

2.2. Covering rough sets and L-fuzzy covering rough sets

Definition 2.3. [10] (1) Let $\boldsymbol{C}\in P(X)$. If $\cup\boldsymbol{C}=X$ and $K\neq\varnothing$ for each $K\in\boldsymbol{C}$, then $\boldsymbol{C}$ is called a covering on X. (2) Let $C\subseteq L^X$. If $\vee C=1$ and $K\neq 0$ for each $K\in C$, then C is called an L-fuzzy covering on X.

Definition 2.4. [28] Let $\boldsymbol{C}$ be a covering on X. For any $x\in X$, the family

$$Md(x)=\{x\in K\in\boldsymbol{C}\mid\forall S\in\boldsymbol{C}, x\in S\wedge S\subseteq K\Rightarrow K=S\}$$

is called the minimal description of x. Furthermore, $\boldsymbol{C}$ is called unary if for any $x\in X$, $|Md(x)|=1$, i.e., there is only one element in $Md(x)$. It is easily observed that $\boldsymbol{C}$ is unary iff $\cap\{K\in\boldsymbol{C}:x\in K\}\in\boldsymbol{C}$ for all $x\in X$.

Definition 2.5 [12] Let $\boldsymbol{C}$ be a covering on X. For $A\in P(X)$, the pair of subsets $\left(\underline{\boldsymbol{C}}(A),\overline{\boldsymbol{C}}(A)\right)$ defined by

$$\underline{\boldsymbol{C}}(A)=\cup\{K\mid K\in\boldsymbol{C},K\subseteq A\},\overline{\boldsymbol{C}}(A)=\cap\{K'\mid K\in\boldsymbol{C},A\subseteq K'\},\text{where } K'=X-K$$

is said to be the covering rough set of A. The associated mappings $\underline{\boldsymbol{C}}$ and $\overline{\boldsymbol{C}}$ on $P(X)$ are called the lower and upper covering approximation operators, respectively.

In [8], Li defined a type of L-fuzzy covering rough sets, which can be regarded as the generalization of that in Definition 2.5.

Definition 2.6. [8] Let C be an L-fuzzy covering on X and $A\in L^X$. Then the pair of L-fuzzy sets $\left(\underline{C}_*(A),\overline{C}_\to(A)\right)$ defined by:

$$\underline{C}_*(A)=\bigvee_{K\in C}\left(K*S_\to(K,A)\right),\overline{C}_\to(A)=\bigwedge_{K\in C}\left(K\to I_*(K,A)\right)$$

is said to be the L-fuzzy covering rough sets of A, and $\underline{C}_*$ (resp., $\overline{C}_\to$) is called the L-fuzzy covering lower (resp., upper) approximation operator.

Li also used axiom sets to characterize the lower and upper L-fuzzy covering approximation operators.

Proposition 2.2. [8] For a mapping $p:L^X\to L^X$, there is an L-fuzzy covering C on X s.t. $p=\underline{C}_*$ iff p satisfies

(L1) $p(1)=1$, (L2) $A\le B\Rightarrow p(A)\le p(B)$ for all $A,B\in L^X$, (L3) $p(A)\le A$ for all $A\in L^X$, (L4) $p(A)\le pp(A)$, (L5) $a*p(A)\le p(a*A)$.

Proposition 2.3. [8] Let L be a regular complete residuated lattice and $h:L^X\to L^X$ be a mapping. Then there is an L-fuzzy covering C on X s.t. $h=\overline{C}_\to$ iff h satisfies

(U1) $h(0)=(0)$, (U2) $A\le B\Rightarrow h(A)\le h(B)$ for all $A,B\in L^X$,

(U3) $h(A)\ge A$ for all $A\in L^X$, (U4) $h(A)\ge hh(A)$ for all $A\in L^X$,

(U5) $a\to h(A)\ge h(a\to A)$.

3. The single axiomatic characterizations on *L*-fuzzy covering approximation operators

In this section, we shall use a single axiom to characterize the L-fuzzy covering approximation operators.

3.1. On lower approximation operator $\underline{C}_*$

A mapping $\Phi:L^X\to L^X$ is called order-preserving if for any $A,B\in L^X$, $A\le B$ implies $\Phi(A)\le\Phi(B)$.

Lemma 3.1. Let $\Phi:L^X\to L^X$ be an order-preserving mapping. Then the following two conditions are equivalent.

(1) $a*\Phi(A)\le\Phi(a*A)$ for any $a\in L, A\in L^X$.

(2) $a\to\Phi(A)\ge\Phi(a\to A)$ for any $a\in L, A\in L^X$.

Proof: (1) $\Rightarrow$(2). It follows by $a*\Phi(a\to A)\le\Phi(a*(a\to A))\le\Phi(A)$, which means $\Phi(a\to A)\le a\to\Phi(A)$.

(2) $\Rightarrow$ (1). It follows by $a\to\Phi(a*A)\ge\Phi(a\to(a*A))\ge\Phi(A)$, which means $\Phi(a*A)\le a*\Phi(A)$. □

Remark 3.1. From Lemma 3.1, it is noted that the condition (L5) in Proposition 2.2 can be restated equivalently as: $p(a \to A) \le a \to p(A)$ for all $a \in L, A \in L^X$.

Lemma 3.2. Let $p: L^X \to L^X$ be a mapping. Then (L2)+(L5)⇔ (POD): $\forall A, B \in L^X$, $S_{\to}(A,B) \le S_{\to}(p(A), p(B))$.

Proof. ⇒. Let $A, B \in L^X$ and $a = S_{\to}(A,B)$. Then

$\forall x \in X, a \le A(x) \to B(x) \Rightarrow \forall x \in X, a * A(x) \le B(x) \Rightarrow a * A \le B$, by(L2)

$\Rightarrow h(a*A) \le h(B)$, by(L5) $\Rightarrow a * h(A) \le h(a*A) \le h(B) \Rightarrow a \le S_{\to}\left(h(A), h(B)\right)$.

⇐. Let $A, B \in L^X$ and $a \in L$.

If $A \le B$, then it follows by Lemma 2.1(O1) and (POD) that

$$1 = S_{\to}(A,B) \le S_{\to}\left(h(A), h(B)\right) \Rightarrow h(A) \le h(B),$$

which means (L2) holds. In addition, note that

$$1 \overset{Lemma 2.1(O1)}{=} S_{\to}(a \to A, a \to A) = a \to S_{\to}(a \to A, A)$$
$$\overset{(HOD)}{\le} a \to S_{\to}\left(h(a \to A), h(A)\right) = S_{\to}\left(h(a \to A), a \to h(A)\right),$$

which means $h(a \to A) \le a \to h(A)$, i.e., (L5) holds. □

Theorem 3.1. (The characterization by a single axiom) Let $p: L^X \to L^X$ be a mapping. Then there is an L-fuzzy covering C on X s.t. $p = \underline{C}_*$ iff it satisfies (POM): for any index set T and any $A_t, B_t (t \in T) \in L^X$,

$$p(1) \wedge \bigwedge_{t \in T} S_{\to}(p(A_t), B_t) = \bigwedge_{t \in T} S_{\to}(p(A_t), p(B_t)).$$

Proof: From Proposition 2.2, we need only to verify that (POM)⇔ (L1)-(L5).

⇒. (L1): Take $T = \varnothing$ in (POM), we get $p(1) = 1$ since $\wedge \varnothing = 1$ for $\varnothing \subseteq L$.

For any $A, B \in L^X$, put $T = \{1\}$ and $A_t = A, B_t = B$, then applying (L1) in (POM), we have that $(POM^*): S_{\to}(p(A), B) = S_{\to}(p(A), p(B))$.

(L3): Take $A = B$ in (POM^*), it follows by Lemma 2.1 (O1) that $S_{\to}\left(p(A), A\right) = 1$, i.e., $p(A) \le A$.

(L2)+(L5): Applying (L3) in (POM^*) we get

$S_{\to}(A,B) \le S_{\to}(p(A), B) = S_{\to}(p(A), p(B))$, i.e., (POD) holds. It follows by Lemma3.2 that (L2) and (L5) holds.

(L4): Take $B = p(A)$ in (POM^*), it follows by Lemma2.1(O1) that

$$S_{\to}\left(p(A), pp(A)\right) = 1, \text{ i.e., } p(A) \le pp(A).$$

⇐. At first, we prove that (L2)-(L5) implies (POM^*). Indeed, for any $A, B \in L^X$,

$$S_{\to}\left(p(A),B\right) \overset{(L2)+(L5)=(POD)}{\leq} S_{\to}\left(pp(A),p(B)\right) \overset{(L4)}{\leq} S_{\to}\left(p(A),p(B)\right) \overset{(L3)}{\leq} S_{\to}\left(p(A),B\right).$$

Hence, $S_{\to}(p(A),B) = S_{\to}(p(A),p(B))$, i.e., (POM^{*}) holds. Then together (POM^{*}) and (L1) we obtain (POM). □

3.2. On upper approximation operator $\overline{C}_{\to}$

The following lemma is just a restatement of Lemma 3.2 by replacing p with h.

Lemma 3.3. Let $h: L^X \to L^X$ be a mapping. Then (U2)+(U5)⇔ (HOD): $\forall A,B\in L^X$, $S_{\to}(A,B) \leq S_{\to}\left(h(A),h(B)\right)$.

Remark 3.2. From Lemma3.3, it is noted that the condition (U5) in Proposition 2.3 can be restated equivalently as: $h(a*A) \geq a*h(A)$ for all $a\in L, A\in L^X$.

Theorem 3.2. (The characterization by a single axiom) Let L be a regular complete residuated lattice and $h: L^X \to L^X$ be a mapping. Then there is an L-fuzzy covering C on X s.t. $h = \overline{C}_{\to}$ iff it satisfies (HOM): for any index set T and any $A_t, B_t (t\in T)\in L^X$,

$$h(0) \vee \bigvee_{t\in T} S_{\to}(A_t, h(B_t)) = \bigvee_{t\in T} S_{\to}(h(A_t), h(B_t)).$$

Proof: From Proposition 2.3, we need only to verify that (HOM) ⇔ (U1)-(U5).

⇒. (U1): Take $T=\varnothing$ in (HOM), we get $h(0)=0$ since $\vee\varnothing = 0$ for $\varnothing \subseteq L$.

For any $A,B\in L^X$, put $T=\{1\}$ and $A_t = A, B_t = B$, then applying (U1) in (HOM), we have that $(HOM^{\to}): S_{\to}(A,h(B)) = S_{\to}(h(A),h(B))$.

(U3): Take $A=B$ in $(HOM^{\to})$, it follows by Lemma 2.1 (O1) that $S_{\to}(A,h(A))=1$, i.e., $A\leq h(A)$.

(U2)+(U5): Applying (U3) in $(HOM^{\to})$,we get

$S_{\to}(A,B) \leq S_{\to}(A,h(B)) = S_{\to}(h(A),h(B))$,i.e., (HOD) holds. It follows by Lemma 3.3 that (U2) and (U5) holds.

(U4): Take A = h(B)in $(HOM^{\to})$, it follows by Lemma 2.1 (O1) that

$$S_{\to}\left(hh(B),h(B)\right)=1, \text{ i.e., } hh(B)\leq h(B).$$

⇐. At first, we prove that (U2)-(U5) implies $(HOM^{\to})$. Indeed, for any $A,B\in L^X$,

$$S_{\to}(A,h(B)) \overset{(U2)+(U5)=(HOD)}{\leq} S_{\to}\left(h(A),hh(B)\right) \overset{(U4)}{\leq} S_{\to}\left(h(A),h(B)\right) \overset{(L3)}{\leq} S_{\to}\left(A,h(B)\right).$$

Hence, $S_{\to}(A,h(B)) = S_{\to}\left(h(A),h(B)\right)$,i.e., $(HOM^{\to})$ holds. Then together $(HOM^{\to})$ and (U1) we obtain (HOM). □

4. The relationships between *L*-fuzzy covering approximation operators and *L*-fuzzy relation approximation operators

In this section, we shall prove that some special L-fuzzy covering approximation operators and L-fuzzy relation approximation operators can be mutually induced.

An L-fuzzy relation R on X is called reflexive whenever $\forall x \in X, R(x,x)=1$; and $*$-transitive whenever $\forall x,z \in X, \vee_{y\in X}(R(x,y)*R(y,z)) \le R(x,z)$.

Definition 4.1. Let R be a reflexive L-fuzzy relation on X. Then the family

$$C^R = \{R(x)\in L^X \mid x\in X\}, \text{where } \forall y\in X, R(x)(y)=R(x,y)$$

forms an L-fuzzy covering on X, called the L-fuzzy covering induced by R.

The following theorem shows that R and C^R yield the same L-fuzzy approximation operators if R is reflexive and $*$-transitive.

Theorem 4.1. Let R be a reflexive and $*$-transitive L-fuzzy relation on X. Then $\underline{C}^R_* = \underline{R}_\to$ and $\overline{C}^R_\to = \overline{R}_*$.

Proof: Let $A\in L^X$. We prove below that $\underline{C}^R_*(A) = \underline{R}_\to(A)$ and $\overline{C}^R_\to(A) = \overline{R}_*(A)$.

For any $x\in X$, note that

$$\begin{aligned}\underline{C}^R_*(A)(x) &= \vee_{K\in C^R}\left(K(x)*S_\to(K,A)\right) = \vee_{y\in X}\left(R(y)(x)*S_\to(R(y),A)\right)\\ &\overset{y=x}{\ge} R(x,x)*S_\to(R(x),A), by\,reflexivity\\ &= 1*S_\to(R(x),A) = \underline{R}_\to(A)(x).\end{aligned}$$

Conversely,

$$\begin{aligned}\underline{C}^R_*(A)(x) &= \vee_{y\in X}\left(R(y,x)*S_\to(R(y),A)\right) = \vee_{y\in X}\left(R(y,x)*\wedge_{z\in X}\left(R(y,z)\to A(z)\right)\right)\\ &\le \vee_{y\in X}\wedge_{z\in X}\left[R(y,x)*\left(R(y,z)\to A(z)\right)\right], by * -transitivity\\ &\le \vee_{y\in X}\wedge_{z\in X}\left[\left(R(x,z)\to R(y,z)\right)*\left(R(y,z)\to A(z)\right)\right]\\ &\le \wedge_{z\in X}\left(R(x,z)\to A(z)\right) = \underline{R}_\to(A)(x).\end{aligned}$$

Hence $\underline{C}^R_*(A) = \underline{R}_\to(A)$.

For any $x\in X$, note that

$$\begin{aligned}\overline{C}^R_\to(A)(x) &= \wedge_{K\in C^R}\left(K(x)\to I_*(K,A)\right) = \wedge_{y\in X}\left(R(y,x)\to I_*(R(y),A)\right)\\ &\overset{y=x}{\le} R(x,x)\to I_*(R(x),A), by\,reflexivity\\ &= I_*(R(x),A) = \overline{R}_*(A)(x).\end{aligned}$$

Conversely,

$$\begin{aligned}\overline{C}^{R}_{\to}(A)(x) &= \bigwedge_{y\in X}\Big(R(y,x)\to \bigvee_{z\in X}\big(R(y,z)*A(z)\big)\Big)\\ &\geq \bigwedge_{y\in X}\bigvee_{z\in X}\Big(R(y,x)\to\big(R(y,z)*A(z)\big)\Big), by * -transitivity\\ &\geq \bigvee_{z\in X}\Big[\big(R(x,z)\to R(y,z)\big)\to\big(R(y,z)*A(z)\big)\Big]\\ &\geq \bigvee_{z\in X}\big(R(x,z)*A(z)\big)=\overline{R}_{*}(A)(x).\end{aligned}$$

Hence $\overline{C}^{R}_{\to}(A)=\overline{R}_{*}(A)$. □

Let C be an L-fuzzy covering on X. It is well known that the L-fuzzy relation R^{C} on X defined by $\forall x,y\in X$, $R^{C}(x,y)=\bigwedge_{K\in C}\big(K(x)\to K(y)\big)$ is reflexive and $*$-transitive, and called the L-fuzzy relation induced by L-fuzzy covering C.

Definition 4.2. An L-fuzzy covering C on X is called unary if $R^{C}(x)\in C$ for any $x\in X$.

Remark 4.1. When $L=\{0,1\}$, an L-fuzzy covering C degenerates into a crisp covering, and the L-fuzzy set $R^{C}(x)$ degenerates into a crisp set $\wedge\{K\in C: x\in K\}$. Hence from Definition 2.4, we know that the notion of unary L-fuzzy covering C is a generalization of the corresponding crisp notion.

The next theorem shows that C and R^{C} yield the same L-fuzzy approximation operators if C is unary.

Theorem 4.2. Let C be an unary L-fuzzy covering on X. Then $\underline{R}^{C}_{\to}=\underline{C}_{*}$ and $\overline{C}_{\to}=\overline{R}^{C}_{*}$.

Proof: Let $A\in L^{X}$. We prove that $\underline{R}^{C}_{\to}(A)=\underline{C}_{*}(A)$ and $\overline{C}_{\to}(A)=\overline{R}^{C}_{*}(A)$.
Note that for any $x\in X$,

$$\begin{aligned}\underline{R}^{C}_{\to}(A)(x) &= \bigwedge_{y\in X}\big(R^{C}(x,y)\to A(y)\big)=\bigwedge_{y\in X}\Big(\bigwedge_{K\in C}\big(K(x)\to K(y)\big)\to A(y)\Big)\\ &\geq \bigwedge_{y\in X}\bigvee_{K\in C}\Big(\big(K(x)\to K(y)\big)\to A(y)\Big)\geq \bigwedge_{y\in X}\bigvee_{K\in C}\Big(K(x)*\big(K(y)\to A(y)\big)\Big)\\ &\geq \bigvee_{K\in C}\Big(K(x)*\bigwedge_{y\in X}\big(K(y)\to A(y)\big)\Big)=\underline{C}_{*}(A)(x),\end{aligned}$$

which means $\underline{R}^{C}_{\to}(A)\geq\underline{C}_{*}(A)$. On the other hand,

$$\underline{C}_*(A)(x)=\bigvee_{K\in C}\left(K(x)*\bigwedge_{y\in X}(K(y)\to A(y))\right), by\ unary\ condition$$
$$\overset{K=R^C(x)}{\geq} R^C(x,x)*\bigwedge_{y\in X}\left(R^C(x,y)\to A(y)\right)$$
$$=1*\bigwedge_{y\in X}\left(R^C(x,y)\to A(y)\right)=\underline{R}^C_{\to}(A)(x),$$

which means $\underline{C}_*(A)\geq\underline{R}^C_{\to}(A)$. Hence $\underline{C}_*=\underline{R}^C_{\to}$.

Note that for any $x\in X$,

$$\overline{R}^C_*(A)(x)=\bigvee_{y\in X}\left(R^C(x,y)*A(y)\right)=\bigvee_{y\in X}\left(\bigwedge_{K\in C}(K(x)\to K(y))*A(y)\right)$$
$$\leq\bigvee_{y\in X}\bigwedge_{K\in C}\left((K(x)\to K(y))*A(y)\right)\leq\bigvee_{y\in X}\bigwedge_{K\in C}\left(K(x)\to(K(y)*A(y))\right)$$
$$\leq\bigwedge_{K\in C}\left(K(x)\to\bigvee_{y\in X}(K(y)*A(y))\right)=\overline{C}_{\to}(A)(x),$$

which means $\overline{C}_{\to}(A)\geq\overline{R}^C_*(A)$. On the other hand,

$$\overline{C}_{\to}(A)(x)=\bigwedge_{K\in C}\left(K(x)\to\bigvee_{y\in X}(K(y)*A(y))\right), by\ unary\ condition$$
$$\overset{K=R^C(x)}{\leq} R^C(x,x)\to\bigvee_{y\in X}\left(R^C(x,y)*A(y)\right)=1\to\bigvee_{y\in X}\left(R^C(x,y)*A(y)\right)$$
$$=\overline{R}^C_*(A)(x),$$

which means $\overline{C}_{\to}(A)\leq\overline{R}^C_*(A)$ .Hence $\overline{C}_{\to}=\overline{R}^C_*$. □

5. Concluding remarks

In this paper, a further study on L-fuzzy covering rough sets was given. The single axiom characterization on L-fuzzy covering approximation operators was presented, and the connections between L-fuzzy covering approximation operators and the L-fuzzy relation approximation operators were constructed.

Acknowledgments. This work is supported by the National Natural Science Foundation of China (No. 12171220) and the Natural Science Foundation of Shandong Province (No. ZR2020MA042).

The authors are also thankful to the reviewers for their critical comments on the improvement of the paper.

Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. All the authors contributed equally to this work.

REFERENCES

1. Y.L. Bao, H.L. Yang and Y.H. She, Using one axiom to characterize L-fuzzy rough approximation operators based on residuated lattices, *Fuzzy Sets and Systems,* 336 (2018) 87–115.

2. T. Deng, Y. Chen and W. Xu, et al. A Novel Approach to Fuzzy Rough Sets Based on a Fuzzy Covering, *Information Sciences*, 177 (2007) 2308–2326.
3. L. D′eer and C. Cornelis, A comprehensive study of fuzzy covering-based rough set models: Definitions, properties and interrelationships, *Fuzzy Sets and Systems*, 336 (2018) 1–26.
4. D. Dubois and H. Prade, Rough fuzzy sets and fuzzy rough sets, *International Journal of General Systems*, 17 (1990) 191–208.
5. S. Gottwald, *A Treatise on Many-Valued Logics*, Research Studies Press, Baldock, 2001.
6. J.A. Goguen, L-Fuzzy Sets, *Journal of Mathematical Analysis and Applications*, 18 (1967) 145–174.
7. M. Kondo, On the structure of generalized rough sets, *Information Sciences*, 176 (2005) 589-600.
8. L.Q. Li, Q. Jin, Kai. Hu, et al. The axiomatic characterizations on L-fuzzy covering-based approximation operators, *International Journal of General Systems*, 46(4)(2017) 332–353.
9. L.Q. Li, B.X. Yao, J.M. Zhan and Q. Jin, L-fuzzifying approximation operators derived from general L-fuzzifying neighborhood systems, *International Journal of Machine Learning And Cybernetics*, 12(5)(2021) 1343–1367 .
10. T.J. Li, Y.Leung and W.X. Zhang, Generalized Fuzzy Rough Approximation Operators Based on Fuzzy Coverings, *International Journal of Approximate Reasoning*, 48 (2009) 836–856.
11. G.L. Liu, Using one axiom to characterize rough set and fuzzy rough set approximations, *Information Sciences*, 223 (2013) 285–296.
12. G.L. Liu, The Relationship Among Different Covering Approximations, *Information Sciences*, 250 (2013) 178–183.
13. N.N. Morsi and M.M. Yakout, Axiomatics for fuzzy rough sets, *Fuzzy Sets and Systems*, 100 (1998) 327–342.
14. B. Pang and J.S. Mi, Using single axioms to characterize L-rough approximate operators with respect to various types of L-relations, *International Journal of Machine Learning and Cybernetics*, 11 (2020) 1061–1082.
15. Z. Pawlak, Rough Set, *Int. J. Comput. Inf. Sci*, 11 (1982) 341–356.
16. Z. Pawlak, *Rough Set: Theoretical Aspects of Reasoning About Data*, Kluwer Academic Publishers, Boston, 1991.
17. J.S. Qiao and B.Q. Hu, On $(\odot, *)$-fuzzy rough sets based on residuated and co-residuated lattices, *Fuzzy Sets and Systems*, 336 (2018) 54–86.
18. J.S. Qiao and B.Q. Hu, Granular variable recision L-fuzzy rough sets based on residuated lattices, *Fuzzy Sets Systems*, 336 (2018) 148–166.
19. A.M. Radzikowska and E.E. Kerre, Fuzzy Rough Sets Based on Residuated Lattices, *Transactions on Rough Sets II. LNCS*, 3135 (2004) 278–296.
20. Y.H. She and G.J. Wang, An axiomatic approach of fuzzy rough sets based on residuated lattices, *Computers and Mathematics with Applications*, 58 (2009) 189–201.
21. C.Y. Wang, X.G. Zhang and Y.H. Wu, New results on single axiom for L-fuzzy rough approximation operators, *Fuzzy Sets and Systems*, 380 (2020) 131–149.

22. W. Wei, B. Pang and J.S. Mi, Axiomatic characterizations of L-valued rough sets using a single axiom, *Information Sciences*, 580 (2021) 283–310.
23. H.S. Wu and G.L. Liu, The relationships between topologies and generalized rough sets, *International Journal of Approximate Reasoning*, 119 (2020) 313–324.
24. Y.Y. Yao, Constructive and algebraic methods of the theory of rough sets, *Information Sciences*, 109(1998) 21–47.
25. Y.Y. Yao and B.X. Yao, Covering based rough set approximations, *Information Sciences*, 200 (2012) 91–107.
26. F.F Zhao and L.Q Li, Reduction of Rough Set Based on Generalized Neighborhood System Operator, *Journal of Mathematics and Informatics*, 6(2016) 67–72.
27. F.F. Zhao and L.Q. Li, Axiomatization on generalized neighborhood system-based rough sets, *Soft Computing*, 22(18) (2018) 6099–6110.
28. W.Zhu, Relationship between generalized rough sets based on binary relation and covering, *Information Sciences*, 179 (2009) 210–225.