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Bézier Curves and Surfaces with three Parameters and Extensions in the Triangular Domain

Wang Lu * **and *Zhang Guicang***

School of Mathematics & Statistics, Northwest Normal University
Gansu Lanzhou, 730070, China.
*Corresponding author. Email: 2020211869@nwnu.edu.cn

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Abstract. To define a new basis function to obtain a basis that can inherit the excellent properties of the traditional B-spline method and Bézier method, global and locality of shape adjustment, and can accurately represent the elliptical arc and circle. Firstly, an optimal standard full positive base, the cut angle algorithm, the C^1 and C^2 continuous proof of the base under the quasi-extended Chebyshev space in this paper. Secondly, the base on the rectangular field to the triangular field to obtain the quasi-cubic triangular Bernstein-Bézier base on the triangular field. Thirdly, this base can accurately represent the elliptic arc and circle, and then gives the base cutting algorithm on the triangular domain, and reverse introduce two conditions under which the quasi-cubic triangular Bernstein-Bézier surfaces are G^1 continuous in surface splicing. After a lot of analysis and examples, the new basis function has excellent properties of traditional methods, and can also flexibly adjust the shape parameters to obtain the required curve surface, which meets the actual industrial design requirements.

Keywords: Quasi to extend Chebyshev space, Full positive basis, Shape parameter, Triangular domain

AMS Mathematics Subject Classification (2010): 41A15

1. Introduction

There are two main methods in computer aided geometric design (CAGD), namely Bézier method and B-spline method. They have the advantages of geometric invariance, convex hull, convex preservation, variation and reduction, local support and other excellent properties. However, with the rapid development of modern industry, A lot of industrial geometry modelling requirements already cannot be realized by the simple traditional Bézier method and B-spline method, for this reason, scholars put forward different methods, such as physical and chemical Bézier method and B-spline method, some of these methods have gradual problems, some because of the shape parameters of the setting problem damaged the shape of the curve surface. Among them, the construction of basis functions determines the properties of curves and surfaces.

In order to meet the actual industrial demand, has a large number of literature by adding different parameters for basis function [1-2] or increase the number of basis functions [3-4] and get the Bézier curve in the control under the condition of invariable vertex can also adjust the shape of freedom, such as Yong-Hua Liu [5-6] structure basis function can not only keep curves and the original features, It can also be adjusted by shape parameters [7]. Constructed quadratic triangular Bézier curves with a shape parameter [8-9]. Proposed a class of polynomial curves of trigonometric functions, which can accurately represent some quadratic curves and transcendental curves. The new basis function not only retains the advantages of the traditional basis function, but also represents some transcendental curves. These Spaces mainly include trigonometric function space, hyperbolic function space and exponential function space [10-12]. Although although these methods in some way better than the traditional method, but there is plenty of room to improve, such as whether the new basis function has reduced variation, variation reduction shall protect convexity, is testing whether a curve suitable for one of geometric modelling, design standard, this article through the most specification are all base structure on the basis of the basis function, a new basis function are all positive, Then the curve defined by it has the property of variation reduction.

At the same time, the triangle area curve and surface in the practical application has important value, not only to overcome Bézier curve and surface in the control of the same vertex conditions can not be flexible to adjust the shape of the shortcomings, but also to solve the irregular data points under Bézier curve and surface modeling design problems. Bézier surface of the rectangular domain is in the form of tensor product, and Bézier of the triangular domain is in the form of non-tensor product, there is a lot of literature on the rectangular domain Bézier basis function is defined for the triangular domain [12-23], which Wu Hongyi [18] et al. The triangular domain Bézier with multiple shape parameters, It increases the operation difficulty of curves and surfaces. Yu Liping's [19] curved surface with two shape parameters in triangular domain can be adjusted freely, but it lacks generality.

In this paper, under the framework of quasi-extended Chebyshev space theory, we propose a set of bases whose generating functions are

$$\Phi=\{\sin^2 t,(1-\sin t)^2(1-a\sin t)e^{-\alpha\sin t},(1-\cos t)^2(1-a\cos t)e^{-\beta\cos t}\},$$

and prove that the quasi-bases constitute a set of optimal normal basis. A stable and efficient calculation method is proposed for this basis function. It is proved that the basis function can accurately represent any elliptic arc and parabola. It is proved that this basis function possesses the properties of traditional basis functions, such as integrity, local support, linear independence and full positivity. Then the corresponding curve of the basis function is given, which has the property of variation reduction. In addition, the basis function with four parameters is generated in the triangle domain, and the corresponding surface is given, and the four parameters can adjust it. The G^1 condition of continuous surface splicing in triangular domain is calculated.

2. Quasi-cubic triangular Bézier curves

2.1. Full base

The basis function $(u_0,u_1,\cdots u_n)$ is the full positive basis on the closed interval $[a,b]$. If for any sequence of nodes $a\le t_0<t_1<\cdots<t_n<b$, in the quasi-cubic triangular function space

$$T_{a,\alpha,\beta}=\{1,\sin^2 t,(1-\sin t)^2(1-a\sin t)e^{-\alpha\sin t},(1-\cos t)^2(1-a\cos t)e^{-\beta\cos t}\} \tag{1}$$

is a special form of the space of quasi-cubic triangular functions in [20].

We first prove the differential space

$$DT_{a,\alpha,\beta}=\{2\sin t\cos t, e^{-\alpha\sin t}\cos t(\sin t-1)(2+a+\alpha-(\alpha+3a+a\alpha)\sin t+a\alpha\sin^2 t), \\ -e^{-\beta\cos t}\sin t(\cos t-1)(2+a+\beta-(\beta+3a+a\beta)\cos t+a\beta\cos^2 t)\} \tag{2}$$

is a 3-dimensional Chebyshev space on the closed interval in the space of quasi-cubic triangulated functions $T_{a,\alpha,\beta}$.

Theorem 1. For any $a\in[0,1],\alpha,\beta\in[0,+\infty],t\in[0,\pi/2]$, $DT_{a,\alpha,\beta}$ is a Chebyshev space on the closed interval.

Proof: For any $\xi_i\in R(i=0,1,2)$, Consider the following linear combination

$$\xi_0[2\sin t\cos t]+\xi_1[e^{-\alpha\sin t}\cos t(\sin t-1)(2+a+\alpha-(\alpha+3a+a\alpha)\sin t+a\alpha\sin^2 t)] \\ -\xi_2[e^{-\beta\cos t}\sin t(\cos t-1)(2+a+\beta-(\beta+3a+a\beta)\cos t+a\beta\cos^2 t)]=0. \tag{3}$$

For $t=0$, we know $\xi_1=0$. Similarly, when $t=\pi/2$, we know $\xi_2=0$. Then, $DT_{a,\alpha,\beta}$ is a 3-dimensional Chebyshev space.

The next proof $DT_{a,\alpha,\beta}$ is a 3-dimensional complete extended Chebyshev space on the open interval $(0,\pi/2)$.

For any $t\in[a,b]\subset(0,\pi/2)$, when

$$\begin{aligned} u(t)&=\frac{1}{2}e^{-\alpha\sin t}[(2+a+\alpha)\frac{\cos t}{\sin t}(\alpha+\frac{1}{\sin t})-\cos t(\alpha(3+2\alpha) \\ &\quad +a(3+6\alpha+\alpha^2)-\alpha(5a+\alpha)\sin t+a\alpha^2\sin^2 t)+a\alpha^2\sin 2t]>0 \\ v(t)&=\frac{1}{2}e^{-\beta\cos t}(-\sin t(\beta(3+2\beta)+a(3+6\beta+\beta^2) \\ &\quad -\beta(\beta+a(5+\beta))\cos t+a\beta^2\cos^2 t)-(2+a+\beta)(1+\beta\cos t)\frac{\sin t}{\cos^2 t})>0 \\ u'(t)&=\frac{1}{2}e^{-\alpha\sin t}[(2+a+b)\frac{\cos t}{\sin t}(a+\frac{1}{\sin t})-\cos t(b+a^2(4+b)+a(5+4b) \\ &\quad -3a(a-b)\sin t+b^2\sin^2 t)+a^2b\sin 2t]<0 \\ v'(t)&=\frac{1}{2}e^{-\alpha\sin t}[(2+a+b)\frac{\cos t}{\sin t}(a+\frac{1}{\sin t})-\cos t(b+a^2(4+b)+a(5+4b) \\ &\quad -3a(a-b)\sin t+b^2\sin^2 t)+a^2b\sin 2t]>0 \end{aligned} \tag{4}$$

Thus, the expression of the Wronskian of $u(t)$ and $v(t)$ gives that

$$w(u,v)=u(t)v'(t)-u'(t)v(t)>0,\forall t\in[0,\pi/2] \tag{5}$$

For $t\in[a,b]$, three weight functions $w_i(t)(i=0,1,2)$ are defined as follows

$$\begin{aligned} w_0(t)&=2\sin t\cos t, \\ w_1(t)&=Au(t)+Bv(t), \\ w_2(t)&=C\frac{W(u,v)}{[Au(t)+Bv(t)]^2}, \end{aligned} \tag{6}$$

where A,B,C are three arbitrary positive real numbers. Both $w_i(t)(i=0,1,2)$ are C^∞ bounded functions on $[a,b]$, and all greater than zero. The ECC-space is defined below

$$\begin{aligned} &u_0(t)=w_0(t),\\ &u_1(t)=w_0(t)\int_a^t w_1(t_1)dt_1,\\ &u_2(t)=w_0(t)\int_a^t w_1(t_1)\int_a^{t_1} w_2(t_2)dt_2dt_1, \end{aligned} \tag{7}$$

It is easy to check that these functions $u_0(t),u_1(t),u_2(t)$ are in fact some linear combinations of

$$\begin{aligned} &\{2\sin t\cos t, e^{-\alpha\sin t}\cos t(\sin t-1)(2+a+\alpha-(\alpha+3a+a\alpha)\sin t+a\alpha\sin^2 t),\\ &-e^{-\beta\cos t}\sin t(\cos t-1)(2+a+\beta-(\beta+3a+a\beta)\cos t+a\beta\cos^2 t)\}, \end{aligned} \tag{8}$$

This implies that space $DT_{a,\alpha,\beta}$ is an ECC-space on $[a,b]$. As $[a,b]$ are arbitrary subintervals of $(0,\pi/2)$, we can draw the conclusion that the space $DT_{a,\alpha,\beta}$ is an ECC-space in $(0,\pi/2)$ [20].

Further, it is necessary to prove a quasi-extended Chebyshev space $DT_{a,\alpha,\beta}$ in the closed intervals $[0,\pi/2]$. First, it is necessary to prove that any non-zero function of any space $DT_{a,\alpha,\beta}$ has at most two zeros on the closed interval $[0,\pi/2]$ (note that the heavy root is calculated up to the double root). Consider any non-zero function in the space $DT_{a,\alpha,\beta}$, For $t\in[0,\pi/2]$,

$$\begin{aligned} &F(t)=C_0[2\sin t\cos t]+C_1[e^{-\alpha\sin t}\cos t(\sin t-1)(2+a+\alpha-(\alpha+3a+a\alpha)\sin t+a\alpha\sin^2 t)]\\ &-C_2[e^{-\beta\cos t}\sin t(\cos t-1)(2+a+\beta-(\beta+3a+a\beta)\cos t+a\beta\cos^2 t)]=0. \end{aligned} \tag{9}$$

Because $DT_{a,\alpha,\beta}$ is a fully extended Chebyshev space on the open interval $(0,\pi/2)$, the function $F(t)$ has at most two zeros in $(0,\pi/2)$. Assuming $t=0$ the zero point of the function $F(t)$, there is $C_1=0$. In this case, if $C_2=0$, then $F(t)$ has a singular zero at 0 and a singular zero at $\pi/2$; If $C_0=0$, The only root of $F(t)$ is $t=0$. If $C_0C_2>0$ ($C_0>0, C_2>0$ perhaps $C_0<0, C_2<0$), The root of $F(t)$ is $t=0$. When $t\in(0,\pi/2)$,

$$F(t)=\sin t[2C_0\cos t-C_2e^{-\beta\cos t}(\cos t-1)(2+a+\beta-(\beta+3a+a\beta)\cos t+a\beta\cos^2 t)]=0, \tag{10}$$
$$e^{-\beta\cos t}(1-\cos t)(2+a+\beta-(\beta+3a+a\beta)\cos t+a\beta\cos^2 t)>0$$

when $t\in(0,\pi/2)$, $F(t)$ is either constant positive or constant negative. When $t\in[0,\pi/2]$, The only root of $F(t)$ is $t\in[0,\pi/2]$. When $C_0C_2<0$ ($C_0>0, C_2<0$ perhaps $C_0<0, C_2>0$),

$$g(t)=2C_0\cos t-C_2e^{-\beta\cos t}(\cos t-1)(2+a+\beta-(\beta+3a+a\beta)\cos t+a\beta\cos^2 t), \tag{11}$$

$$\begin{aligned} g'(t)=\sin t\{&-2C_0+e^{-\beta\cos t}C_2[2+4\beta-\beta^2+4a+2a\beta-(4\beta+2\beta^2+6\alpha+8a\beta+a\beta^2)\cos t\\ &+(\beta^2+6a\beta+2a\beta^2)\cos^2 t-a\beta^2\cos^3 t]\}, \end{aligned} \tag{12}$$

$$h(t)=2+4\beta-\beta^2+4a+2a\beta-(4\beta+2\beta^2+6a+8a\beta+a\beta^2)\cos t+(\beta^2+6a\beta+2a\beta^2)\cos^2 t-a\beta^2\cos^3 t, \tag{13}$$

We can know $h(t)\ge 4$. So when $C_0>0, C_2<0, g'(t)<0$, $g(t)$ decreases monotonically in $t\in(0,\pi/2)$; when $C_0<0, C_2>0, g'(t)>0$, $g(t)$ increases monotonically in $t\in(0,\pi/2)$ .So $g(t)$ has at most one zero on $(0,\pi/2)$. When $t=0$ is the root of $F(t)$, $F(t)$ has at most two zeroes in $[0,\pi/2]$. Similarly, when $t=\pi/2$ is the roof of $F(t)$, $F(t)$ has at most two zeroes on $[0,\pi/2]$. Therefore, the conclusion can be proved.

Because $DT_{a,\alpha,\beta}$ is the 3-dimensional QEC space above, known from the literature [20] in the existence of flowering. Thus, when $a\in[0,1],\alpha,\beta\in(-2,1],t\in[0,\pi/2]$, $T_{a,\alpha,\beta}$ suitable for the design of the curve, and $T_{a,\alpha,\beta}$ is also the optimal standard full positive basis.

3. Construction of B basis

Theorem 2. For any $\alpha,\beta\in[0,+\infty],a\in[0,1],t\in[0,\pi/2]$, The optimal standard full positive basis (B basis) of the function space $T_{a,\alpha,\beta}$ is

$$\begin{cases} T_0=(1-\sin t)^2(1-a\sin t)e^{-\alpha\sin t}, \\ T_1=\cos^2 t-(1-\sin t)^2(1-a\sin t)e^{-\alpha\sin t}, \\ T_2=\sin^2 t-(1-\cos t)^2(1-a\cos t)e^{-\beta\cos t}, \\ T_3=(1-\cos t)^2(1-a\cos t)e^{-\beta\cos t}. \end{cases} \tag{14}$$

It is called the QCT-Bernstein basis function with three parameters.

Proof: For any $\alpha,\beta\in[0,+\infty],a\in[0,1],t\in[0,\pi/2]$, By the parent function

$$\Phi=\{\sin^2 t,(1-\sin t)^2(1-a\sin t)e^{-\alpha\sin t},(1-\cos t)^2(1-a\cos t)e^{-\beta\cos t}\}$$

acquirability

$$\begin{aligned} &\Phi(0)=(0,1,0), && \Phi(1)=(1,0,1),\\ &\Phi'(0)=(0,-(2+a+\alpha),0), && \Phi'(1)=(0,0,2+a+\beta),\\ &\Phi''(0)=(2,\alpha^2+4\alpha+2+4a+2a\alpha,0), && \Phi''(1)=(-2,0,\beta^2+4\beta+2+4a+2a\beta). \end{aligned} \tag{15}$$

So

$$\begin{aligned} &\Pi_0=\Phi(0)=(0,1,0),\Pi_3=\Phi(\pi/2)=(1,0,1),\\ &\{\Pi_1\}=Osc_1\Phi(0)\cap Osc_2\Phi(\pi/2)=(0,0,0),\\ &\{\Pi_2\}=Osc_2\Phi(0)\cap Osc_1\Phi(\pi/2)=(1,0,0). \end{aligned} \tag{16}$$

For $t\in[0,\pi/2]$, $\Phi(t)=\sum_{i=0}^{3}A_i(t)\Pi_i$, so

$$\begin{cases} T_2(t)+T_3(t)=\sin^2 t \\ T_0(t)=(1-a\sin t)(1-\sin t)^2 e^{-\alpha\sin t} \\ T_3(t)=(1-a\cos t)(1-\cos t)^2 e^{-\alpha\cos t} \end{cases} \tag{17}$$

Simultaneous formula (17) and $\sum_{i=0}^{3}T_i=1$, we can get the expression T_0,T_1,T_2,T_3.

Next, for any $\xi_i\in R(i=0,1,2,3)$, it have $\sum_{i=0}^{3}\xi_iT_i(t)=0,\sum_{i=0}^{3}\xi_iT_i'(t)=0$. When $t=0$, it have

$$\begin{cases} \xi_0=0, \\ (\alpha+1)(\xi_0-\xi_1)=0. \end{cases} \tag{18}$$

So $\xi_0=\xi_1=0$, Similarly, $\xi_2=\xi_3=0$. When $t\in[0,\pi/2]$, $T_i(t)(i=0,1,2,3)$ have non-negative. When $t\in(0,\pi/2)$, $T_i(t)(i=0,1,2,3)$ have strict positivity. For $T_i(t)(i=0,1,2,3)$, At the endpoint there are the following properties, $T_0(0)=1$ and $T_0(t)$ have double root at $t=\pi/2$; $T_3(\pi/2)=1$ and $T_3(t)$ have double root at $t=0$; $T_1(t)$ has two roots at $t=0$ and $t=\pi/2$, $T_2(t)$ has two roots at $t=0$ and $t=\pi/2$.

Therefore, we know from literature [20] that the QCT-Bernstein basis $T_i(t), i=0,1,2,3$ with three parameters is the optimal normative full normal basis of function space $T_{a,\alpha,\beta}$. Therefore, the conclusion can be proved. For the sake of discussion, Let's write $T_i(t), i=0,1,2,3$ to $T_i(t;a,\alpha,\beta), i=0,1,2,3$.

3.1. The properties of the basis function

For any $\alpha,\beta \in [0,+\infty], a \in [0,1], t \in [0,\pi/2]$, The following properties of the basis functions are shown

(1) Normative: $\sum_{i=0}^{3} T_i(t;a,\alpha,\beta)=1$

(2) Non-negative: $T_i(t;a,\alpha,\beta) \geq 0, i=0,1,2,3,4$

(3) Symmetry: $T_i(t;a,\alpha,\beta)=T_{3-i}(\pi/2-t;\beta,\alpha), i=0,1$

(4) Endpoint properties: for any $\alpha,\beta \in [2,+\infty)$, have

$$\begin{cases} T_i(0)=T_3(\pi/2)=1,\ \ T_i(0)=T_{3-i}(\pi/2)=0, i=0,1,2,3,4, \\ T_0'(0)=-(2+a+\alpha),\ \ T_1'(0)=2+a+\alpha,\ \ T_i'(0)=0,\ \ i=2,3, \\ T_2'(\pi/2)=-(2+a+\beta),\ \ T_3'(\pi/2)=2+a+\beta, T_i'(\pi/2)=0,\ \ i=0,1. \end{cases} \tag{19}$$

For any $\alpha,\beta \in [k+3,+\infty)$, have

$$\begin{cases} T_0''(0)=\alpha^2+\frac{5}{8}a\alpha^2+2a\alpha+4\alpha+4a+2,\ \ T_1''(0)=-(\alpha^2+2a\alpha+4\alpha+4a+4), i=0,1,2,3,4, \\ T_i''(0)=0, i=2,3, \\ T_2''(\pi/2)=-(\beta^2+2a\beta+4\beta+4a+4),\ \ T_3''(\pi/2)=\beta^2+\frac{5}{8}a\beta^2+2a\beta+4\beta+4a+2, \\ T_i''(0)=0, i=0,1. \end{cases} \tag{20}$$

For any $\alpha,\beta \in [k+3,+\infty)$, have

$$\begin{cases} T_0^{(j)}(0)+T_1^{(j)}(0)=0,\ \ T_i^{(j)}(0)=0,\ \ i=0,1,2,3,4, \\ T_2^{(j)}(\pi/2)+T_3^{(j)}(\pi/2)=0,\ \ T_i^{(j)}(\pi/2)=0,\ \ i=0,1, \end{cases} \tag{21}$$

(5) Linear independence: For any $a,\alpha,\beta \in (2,+\infty)$, $T_i(t;a,\alpha,\beta)$, $i=0,1,2,3,4$ linearly independent.

(6) It has a strict positivity, where $\alpha,\beta \in [0,+\infty], a \in [0,1], t \in (0,\pi/2)$

Proof: Easy to prove prove (3) below, the remaining properties.

When $a \in, \alpha, \beta \in, \xi_i \in R, (i=0,1,2,3)$, it have $\sum_{i=0}^{3} \xi_i T_i(t;a,\alpha,\beta)=0$, the two ends are derived from t, it has $\sum_{i=0}^{4} \xi_i T_i'(t;a,\alpha,\beta)=0, \sum_{i=0}^{4} \xi_i T_i''(t;a,\alpha,\beta)=0$ .when $t=0$,it has $\xi_0=\xi_1=\xi_2=\xi_3=0$. Therefore, the conclusion can be proved.

The different parameter diagram of the basis function is shown below

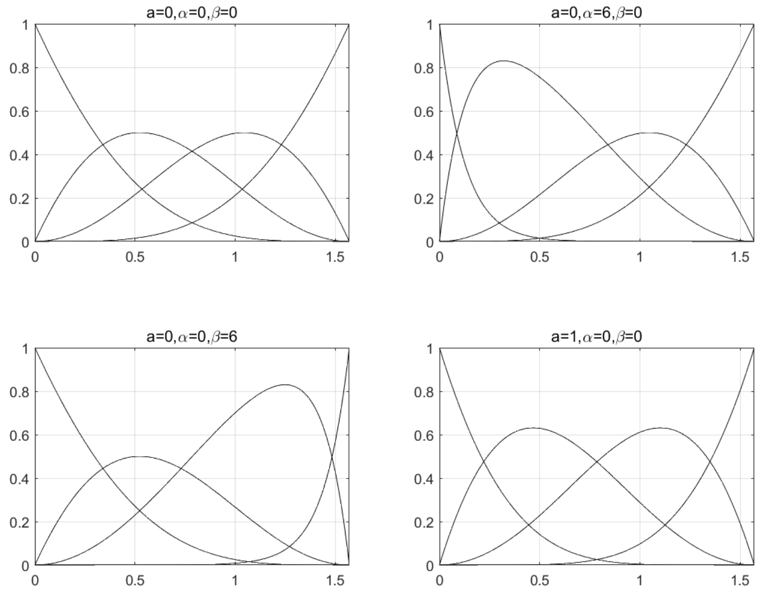

Figure 1: The different parameter diagram of the basis function

4. Curve

4.1. Definition and properties of the Bézier curves of quasi-cubic triangles

Definition 1. For a given control point $P_i \in R^2 / R^3 (i = 0,1,2,3)$, call

$$Q(t;a,\alpha,\beta) = \sum_{i=0}^{3} P_i T_i (t;a,\alpha,\beta) \tag{22}$$

the QCT-Bézier curve with three parameters a,α,β , where $\alpha,\beta \in [0,+\infty), a \in [0,1], t \in [0,\pi/2]$.

4.2. The nature of the curve

(1) Affine invariance, convex inclusion and variation reduction: From its basis function is unit and nonnegative, then the quasi-cubic triangle Bézier curve has affine invariance, convex contracting and variation reduction.

(2) Non-negative: The curve of control vertex P_0, P_1, P_2, P_3 is the same curve as that after the control point is reversed.

(3) Endpoint properties:

$$\begin{array}{ll} Q(0;a,\alpha,\beta) = P_0, & Q(\pi/2;a,\alpha,\beta) = P_3 \\ Q'(0;a,\alpha,\beta) = (2+a+\alpha)(P_1 - P_0) & Q'(\pi/2;a,\alpha,\beta) = (2+a+\alpha)(P_3 - P_2) \end{array} \tag{23}$$

In addition, for the arbitrary $a \in [0,1], \alpha,\beta \in [0,+\infty]$ hold

$$\begin{cases} Q(0;a,\alpha,\beta) = P_0, \quad Q(\pi/2;a,\alpha,\beta) = P_3 \\ Q'(0;a,\alpha,\beta) = (\alpha + a + 2)(P_1 - P_0), Q'(\pi/2;a,\alpha,\beta) = (\beta + a + 2)(P_3 - P_2), \\ Q''(0;a,\alpha,\beta) = (\alpha^2 + 2a\alpha + 4\alpha + 4a + 2)(P_0 - P_1) + 2(P_2 - P_1), \\ Q''(\pi/2;a,\alpha,\beta) = (\beta^2 + 2a\beta + 4\beta + 4a + 2)(P_3 - P_2) + 2(P_1 - P_2) \\ Q'''(0;a,\alpha,\beta) = (\alpha^3 + 3a\alpha^2 - 6\alpha^2 - 12a\alpha - 5a - 5\alpha + 2)(P_0 - P_1) \\ Q'''(\pi/2;a,\alpha,\beta) = (\beta^3 + 3a\beta^2 - 6\beta^2 - 12a\beta - 5a - 5\beta + 2)(P_2 - P_3) \end{cases} \tag{24}$$

The endpoint properties show that the proposed cubic triangular Bézier curve interpolates the first two control points and is tangent to the end edge.

Since the curve in this paper has the same properties as the cubic Bézier curve, this curve is called the quasi-cubic triangular Bézier curve.

4.3. Shape control of the quasi-cubic triangular Bézier curve

The quasi-cubic triangular Bézier curve of this paper is rewritten as

$$Q(t;a,\alpha,\beta) = P_1\cos^2 t + P_2\sin^2 t + T_0(t;a,\alpha,\beta)(P_0 - P_1) + T_3(t;a,\alpha,\beta)(P_3 - P_2) \tag{25}$$

Here $T_0(t;a,\alpha,\beta)$ represents the monotonically decrease of the shape parameter α, which means that the quasi-cubic triangular Bézier curve will move along the same direction as the edge vector $P_0 - P_1$ as the α increases. Similarly, when the α decrease occurs, the curve will move in a different direction from the edge vector $P_0 - P_1$. β has a similar effect to α. When α increases or decreases, the curve border vector $P_2 - P_3$ moves in the same direction or in a different direction. When α increases or decreases simultaneously with β, the curve border vector $P_2 - P_1$ moves in the same direction or in a different direction. When a increases, the curve is more inclined to control the polygon, otherwise, away; these indicate that the parameter has tension on the curve. The figure below shows the effect of the different parameters on the curve, where $a \in [0,1], \alpha, \beta \in [0,+\infty],\ \ t \in [0,\pi/2]$.

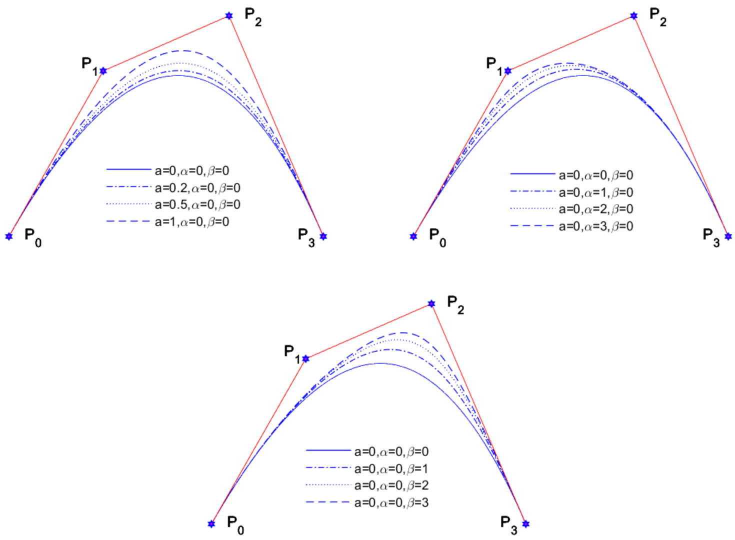

Figure 2: The effect of the different parameters on the curve

4.4. Cut angle algorithm

The advantage of the angle cutting algorithm is stable and efficient. For $t \in [0,\pi/2], a \in [0,1], \alpha, \beta \in [0,+\infty)$, the whole procedure of the angle cutting algorithm is shown below,

$$A(t;\alpha,\beta)=\begin{pmatrix}1-\sin^2 t & 1-\cos^2 t\end{pmatrix}\times\begin{pmatrix}1-\sin t & \sin t & 0\\ 0 & \cos t & 1-\cos t\end{pmatrix}$$

$$\times\begin{pmatrix}\frac{(1-a\sin t)e^{-\alpha\sin t}}{1+\sin t} & \frac{1+\sin t-(1-a\sin t)e^{-\alpha\sin t}}{1+\sin t} & 0 & 0\\ 0 & \frac{(1-\sin^2 t)\sin t}{(1-\sin^2 t)\sin t+(1-\cos^2 t)\cos t} & \frac{(1-\cos^2 t)\cos t}{(1-\sin^2 t)\sin t+(1-\cos^2 t)\cos t} & 0\\ 0 & 0 & \frac{1+\cos t-(1-a\cos t)e^{-\beta\cos t}}{1+\cos t} & \frac{(1-a\cos t)e^{-\beta\cos t}}{1+\cos t}\end{pmatrix}$$

$$\times\begin{pmatrix}Q_0\\ Q_1\\ Q_2\\ Q_3\end{pmatrix}$$

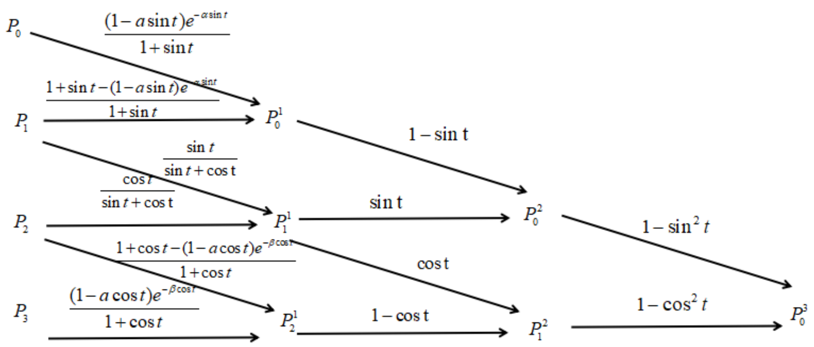

Figure 3: Angle cutting algorithm

4.5. Precise representation of the ellipse versus the parabola

Taking the appropriate shape parameters and control points, for the curves presented here can accurately represent the ellipses and parabola, as well as partially beyond the curve e. g. When $a=1,\alpha=\beta=2$, take the control point is

$$\begin{aligned}&P_0=(x_0+m,y_0), && P_1=(x_0+m,y_0+n/2),\\ &P_0=(x_0+m/2,y_0+n), && P_0=(x_0,y_0+n),\end{aligned}\tag{26}$$

At this point, the parametric equation of the corresponding quasi-cubic triangular Bézier curve $Q(t;1,2,2)$ is

$$\begin{cases}x(t)=x_0+m\cos t,\\ y(t)=y_0+n\sin t,\end{cases} t\in[0,\pi/2]\tag{27}$$

This indicates that the quasi-cubic triangular Bézier curve is a quarter-elliptical arc. When $m=n$, the quasi-triangular Bézier curve $Q(t;1,2,2)$ is a quarter arc. In practical use, when the restriction parameter $t\in[\theta_1,\theta_2]$, any segment of the desired arc is obtained.

When $a=1,\alpha=\beta=2, m-n>0$, take the control point is

$$\begin{aligned}&P_0=(n,e_2n^2+e_1n+e_0), && P_1=(n,e_2n^2+e_1n+e_0),\\ &P_2=((m+n)/2,e_2mn+c_1(m+n)/2+e_0), && P_3=(m,e_2m^2+e_1m+e_0),\end{aligned}\tag{28}$$

At this point, the parametric equation of the corresponding quasi-cubic triangular Bézier curve $Q(t;1,2,2)$ is

$$\begin{cases} x(t) = (n-m)\cos t + m, \\ y(t) = e_2[(n-m)\cos t + m]^2 + e_1[(n-m)\cos t + m]^2 + e_0, \end{cases} t \in [0, \pi/2] \quad (29)$$

This indicates that the quasi-cubic triangular Bézier curve is a parabolic arc.

The figure shows that the Bézier curve can accurately represent the circle and parabolic arc.

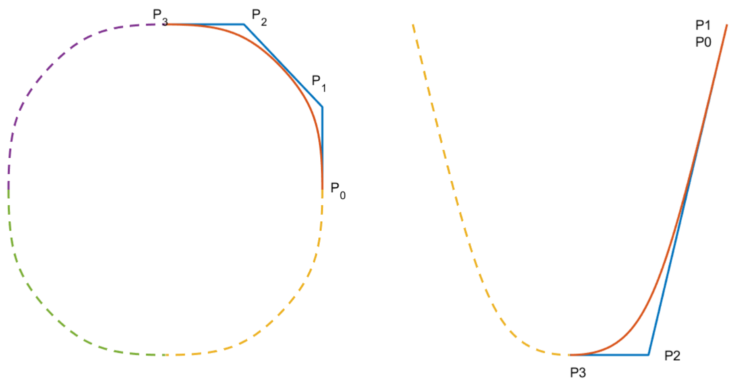

Figure 4: The circle and parabolic arc

4.6. The splicing of quasi cubic triangular Bézier curves

If the two trigonometric Bézier curves with three parameters

$$Q_1(t; a, \alpha, \beta) = \sum_{i=0}^{3} P_i T_i(t; a_1, \alpha_1, \beta_1) \quad (30)$$

and

$$Q_2(t; a_2, \alpha_2, \beta_2) = \sum_{i=0}^{3} q_i T_i(t; a_2, \alpha_2, \beta_2) \quad (31)$$

Respectively, for the two curve segments to reach C^0 continuous, the control point is required to meet $P_3 = q_0$, then the two curves are first at node $u_1 < u_2 < u_3$ and expressed as

$$G(u) = \begin{cases} Q_1(\frac{\pi}{2} \times \frac{u-u_1}{h_1}; \alpha_1, \beta_1), u \in [u_1, u_2], \\ Q_2(\frac{\pi}{2} \times \frac{u-u_2}{h_2}; \alpha_2, \beta_2), u \in [u_2, u_3], \end{cases} \quad (32)$$

where $h_i = u_{i+1} - u_i, i = 1,2$.

Theorem 3. For arbitrary $a_i \in [0,1], \alpha_i, \beta_i \in [0, +\infty](i = 1,2)$. The curve $G(u)$ is continuous as C^1 at node u_2, if condition

$$P_3 = q_0 = \frac{(a_2 + \alpha_2 + 2)h_1 q_1 + (a_1 + \beta_1 + 2)h_2 P_2}{(a_2 + \alpha_2 + 2)h_1 + (a_1 + \beta_1 + 2)h_2} \quad (33)$$

holds. Further, for arbitrary $a_i \in [0,1], \alpha_i, \beta_i \in [0, +\infty](i = 1,2)$. The curve $G(u)$ is continuous as C^2 at node u_2, if the conditional above and

$$q_2 = \frac{1}{2\alpha_2 h_1^2}\{2(\beta_1 + a_1)h_1 h_2 + h_2[\alpha_2 h_2(2a_1\beta_1 + 4\beta_1 + \beta_1{}^2 + 4a_1 + 2) + \beta_1 h_1(2a_2\alpha_2 + 4\alpha_2 + \alpha_2{}^2 + 4a_2 + 2)]\}$$
$$(P_3 - P_2) + \frac{h_2^2}{h_1^2}(P_1 - P_2) + P_3 \tag{34}$$

hold simultaneously.

Proof: For arbitrary $a_i \in [0,1], \alpha_i, \beta_i \in [0,+\infty](i=1,2)$, Easy to verify

$$\begin{aligned}
&Q(u_2^-) = P_3,\\
&Q(u_2^+) = q_0,\\
&Q'(u_2^-) = \frac{\pi}{2}\frac{(a_1 + \beta_1 + 2)}{h_1}(P_3 - P_2),\\
&Q'(u_2^+) = \frac{\pi}{2}\frac{(a_2 + \beta_2 + 2)}{h_2}(q_1 - q_0),\\
&Q''(u_2^-) = (\frac{\pi}{2h_1})^2[(2a_1\beta_1 + 4\beta_1 + \beta_1{}^2 + 4a_1 + 2)(P_3 - P_2) + 2(P_1 - P_2)],\\
&Q''(u_2^+) = (\frac{\pi}{2h_2})^2[(2a_2\alpha_2 + 4\alpha_2 + \alpha_2{}^2 + 4a_2 + 2)(q_0 - q_1) + 2(q_2 - q_1)].
\end{aligned} \tag{35}$$

From this can get $Q(u_2^-) = Q(u_2^+), Q'(u_2^-) = Q'(u_2^+), Q''(u_2^-) = Q''(u_2^+)$. Therefore, the conclusion can be proved.

Shown is the splicing of the curve, and for the C^1 splicing condition, the shape parameter is $a_1 = 0, \alpha_1 = 1, \beta_1 = 0$ and $a_2 = 0, \alpha_2 = 1, \beta_2 = 1$.

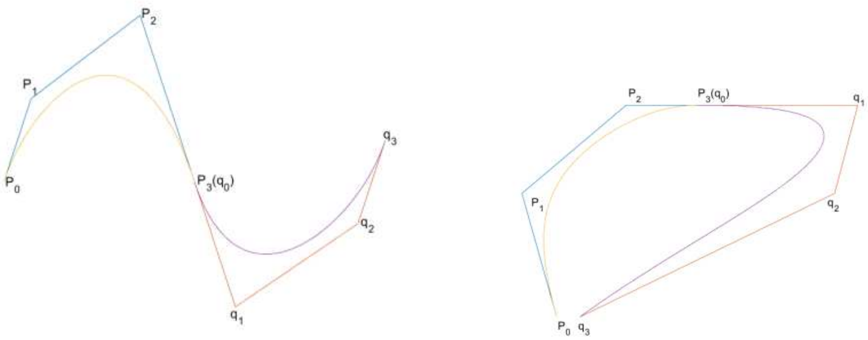

Figure 5: The splicing of the curve $a_1 = 0, \alpha_1 = 1, \beta_1 = 0$ and $a_2 = 0, \alpha_2 = 1, \beta_2 = 1$

5. Construction of the basis of the triangular domain

Definition 2. For any $a \in [0,0.1], \alpha, \beta, \gamma \in [0,+\infty], D = \{(u,v,w) | u + v + w = \pi/2, u \geq 0, v \geq 0, w \geq 0\}$.

The following ten trigonal polynomials are called as basis functions with four exponential parameters a, α, β, γ over the triangular domain D .

$$\begin{cases} T^3_{3,0,0}(u,v,w;a,\alpha,\beta,\gamma)=(1-a\cos u)(1-\cos u)^2 e^{-\alpha\cos u}, \\ T^3_{0,3,0}(u,v,w;a,\alpha,\beta,\gamma)=(1-a\cos v)(1-\cos v)^2 e^{-\beta\cos v}, \\ T^3_{0,0,3}(u,v,w;a,\alpha,\beta,\gamma)=(1-a\cos w)(1-\cos w)^2 e^{-\gamma\cos w}, \\ T^3_{2,1,0}(u,v,w;a,\alpha,\beta,\gamma)=\cos w\sin v(1-\cos u)\dfrac{1+\cos u-(1-\cos u)e^{-\alpha\cos u}}{\cos u}(1+a), \\ T^3_{2,0,1}(u,v,w;a,\alpha,\beta,\gamma)=\cos v\sin w(1-\cos u)\dfrac{1+\cos u-(1-\cos u)e^{-\alpha\cos u}}{\cos u}(1+a), \\ T^3_{1,2,0}(u,v,w;a,\alpha,\beta,\gamma)=\cos w\sin u(1-\cos v)\dfrac{1+\cos v-(1-\cos v)e^{-\beta\cos v}}{\cos v}(1+a), \\ T^3_{0,2,1}(u,v,w;a,\alpha,\beta,\gamma)=\cos u\sin w(1-\cos v)\dfrac{1+\cos v-(1-\cos v)e^{-\beta\cos v}}{\cos v}(1+a), \\ T^3_{1,0,2}(u,v,w;a,\alpha,\beta,\gamma)=\cos v\sin u(1-\cos w)\dfrac{1+\cos w-(1-\cos w)e^{-\gamma\cos w}}{\cos w}(1+a), \\ T^3_{0,1,2}(u,v,w;\alpha,\beta,\gamma)=\cos u\sin v(1-\cos w)\dfrac{1+\cos w-(1-\cos w)e^{-\gamma\cos w}}{\cos w}(1+a), \\ T^3_{1,1,1}(u,v,w;\alpha,\beta,\gamma)=1-\sum\limits_{\substack{i+j+k=3,\\ i\cdot j\cdot k\neq 1}} T^3_{i,j,k}(u,v,w;\alpha,\beta,\gamma). \end{cases} \tag{36}$$

When $a\in[0,0.1],\alpha,\beta,\gamma\in[0,+\infty]$, we have

$$\begin{aligned} &\lim_{u\to\pi/2}\frac{1+\cos u-(1-\cos u)e^{-\alpha\cos u}}{\cos u}(1+a)=(\alpha+2)(a+1), \\ &\lim_{u\to\pi/2}\frac{1+\cos v-(1-\cos v)e^{-\beta\cos v}}{\cos v}(1+a)=(\beta+2)(a+1), \\ &\lim_{u\to\pi/2}\frac{1+\cos w-(1-\cos w)e^{-\lambda\cos v}}{\cos w}(1+a)=(\gamma+2)(a+1). \end{aligned} \tag{37}$$

This means that basis functions make sense on triangular domains.

Lemma 1. For $u+v+w=\pi/2$, it have $1-(\sin^2 u+\sin^2 v+\sin^2 w)=2\sin u\sin v\sin w$.

5.1. Properties of the basis of the triangular domain

Theorem 5. When the triangular field $a\in[0,0.1]$ is taken for a fixed value, the base has the following properties,

(1) $\sum_{i+j+k=3} T^3_{i,j,k}(u,v,w;\alpha,\beta,\gamma)=1$

(2) Non-negative: when $i,j,k\in N, i+j+k=3$, it have $T^3_{i,j,k}(u,v,w;\alpha,\beta,\gamma)\geq 0$

(3) Symmetry:

$$\begin{aligned} &T^3_{i,j,k}(u,v,w;\alpha,\beta,\gamma)=T^3_{j,i,k}(u,v,w;\beta,\alpha,\gamma)= \\ &T^3_{j,k,i}(u,v,w;\beta,\gamma,\alpha)=T^3_{i,k,j}(u,v,w;\alpha,\gamma,\beta)= \\ &T^3_{k,i,j}(u,v,w;\gamma,\alpha,\beta)=T^3_{i,j,k}(u,v,w;\gamma,\beta,\alpha) \end{aligned} \tag{38}$$

(4) Boundary property:

When the parameter α,β,γ, one of which is 0, the base function (36) will degenerate into the corresponding base function (14) with three parameters.

(5) Linear independence: $\{T^3_{i,j,k}(u,v,w;\alpha,\beta,\gamma), i+j+k=3\}$ linearly independent.

Proof: The following properties (3) and (22). The remaining properties are easy to prove.

(2) For any $a \in [0,0.1], \alpha, \beta, \gamma \geq 0, i, j, k \in N, i+j+k=3$ and $i \cdot j \cdot k \neq 1$, it have $T^3_{i,j,k}(u,v,w;\alpha,\beta,\gamma) \geq 0$, and it is given by the lemma 1.

(5) For any $a \in [0,0.1], \alpha, \beta, \gamma \geq 0, \xi_{i,j,k} \in R, (i,j,k \in N, i+j+k=3)$, Consider linear combinations

$$\sum_{i+j+k=3} \lambda_{i,j,k} T^3_{i,j,k}(u,v,w;\alpha,\beta,\gamma) = 0 \tag{39}$$

When $w=0$, we have $\sum_{i=0}^{3} \xi_{i,(3-i),0} T_i(u;\alpha,\beta) = 0$. Thus the basis function is linearly independent, then $\xi_{i,(3-i),0} = 0 (i=0,1,2,3)$. Similarly $\xi_{i,0,(3-i)} = \xi_{0,i,(3-i)} = 0 (i=0,1,2,3)$, then $\xi_{1,1,1} = 0$. Therefore, the conclusion can be proved.

The graph gives the graph of the base with the exponential parameter values of $a=0, \alpha=\beta=\gamma=2$.

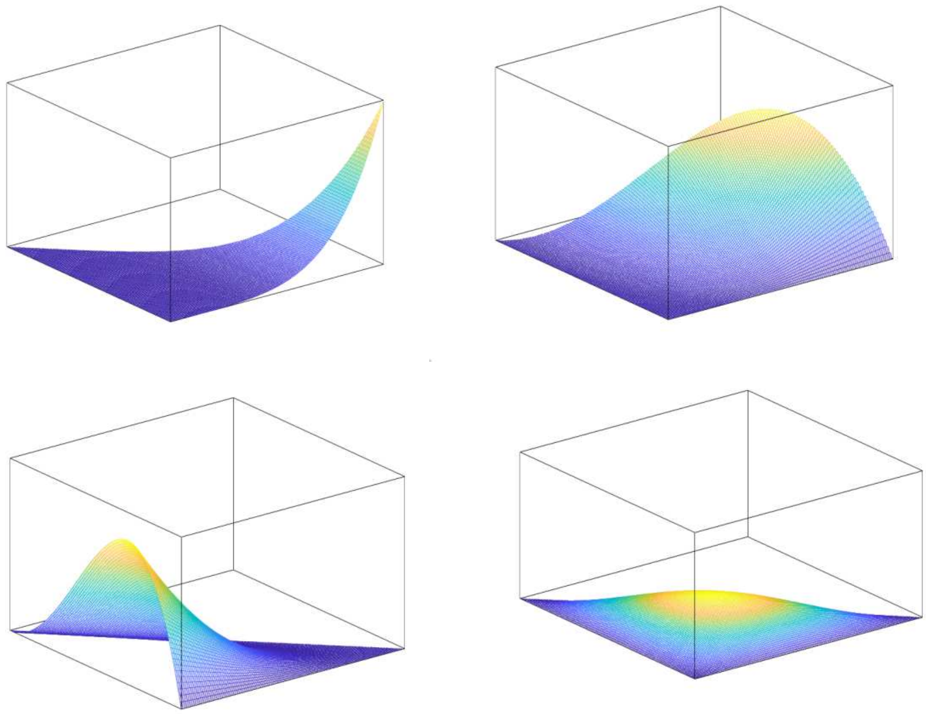

Figure 6: The splicing of the graph of the base with the exponential parameter values of $a=0, \alpha=\beta=\gamma=2$. Followed by $T_{3,0,0}, T_{0,2,1}, T_{0,1,2}, T_{1,1,1}$

5.2. A curved piece on a triangle

Definition 3. For any real number $\alpha,\beta,\gamma \in [0,+\infty], D = \{(u,v,w) | u+v+w = \pi/2, u \geq 0, v \geq 0, w \geq 0\}$, for a fixed value $a \in [0,0.1]$, given a control vertex $P_{i,j,k} \in R^3 (i,j,k \in N, i+j+k=3)$ is called a surface on a triangular domain $R(u,v,w) = \sum_{i+j+k=3} T^3_{i,j,k}(u,v,w;\alpha,\beta,\gamma) P_{i,j,k}(u,v,w) \in D$.

From the properties of the properties of the corresponding surfaces:

(1) Affine invariance and convex inclusion:
It can be derived from the basis function as being unitary and non-negative.
(2) Interpolation of angular point properties:
By calculating the available $R(\pi/2,0,0)=P_{3,0,0}, R(0,\pi/2,0)=P_{0,3,0}, R(0,0,\pi/2)=P_{0,0,3}$
(3) Angle-point tangent-plane properties:
When $w=\pi/2-u-v$, we have

$$\begin{aligned}
&\frac{\delta R(u,v,w)}{\delta u}\Big|_{(\pi/2,0,0)}=(2+\alpha+a)(P_{3,0,0}-P_{2,0,1})\\
&\frac{\delta R(u,v,w)}{\delta v}\Big|_{(\pi/2,0,0)}=(2+\alpha+a)(P_{2,1,0}-P_{2,0,1})\\
&\frac{\delta R(u,v,w)}{\delta u}\Big|_{(0,\pi/2,0)}=(2+\beta+a)(P_{1,2,0}-P_{0,2,1})\\
&\frac{\delta R(u,v,w)}{\delta v}\Big|_{(0,\pi/2,0)}=(2+\beta+a)(P_{0,3,0}-P_{0,2,1})\\
&\frac{\delta R(u,v,w)}{\delta u}\Big|_{(0,0,\pi/2)}=(2+\gamma+a)(P_{1,0,2}-P_{0,0,3})\\
&\frac{\delta R(u,v,w)}{\delta v}\Big|_{(0,0,\pi/2)}=(2+\gamma+a)(P_{0,1,2}-P_{0,0,3})
\end{aligned} \tag{40}$$

This indicates that the tangent plane of the surface at the three angular points $(\pi/2,0,0),(0,\pi/2,0),(0,0,\pi/2)$ is generated by the control point
$P_{3,0,0},P_{2,1,0},P_{1,2,0};P_{0,3,0},P_{1,2,0},P_{0,2,1};P_{0,0,3},P_{1,0,2},P_{0,1,2}$.
(4) Boundary property:
When $w=0$ and a is a fixed value, $R(u,v,w)$ degenerates to the curve

$$R(u,v,0)=\sum_{i=0}^{3}P_{i,i-3,0}A_i(u;\alpha,\beta) \tag{41}$$

with parameters α,β as defined in the following equation. Similarly, $R(u,0,w)$ is the curve of α,γ, $R(0,v,w)$ is the curve of β,γ. Therefore, the boundary curve of QCT-Bernstein-Bézier surface with parameters can accurately represent a parabolic arc and elliptical arc.
(5) Shape adjustment properties:
Since the fixed parameter $a(a\in[0,1])$ of QCT-Bernstein-Bézier surface contains three exponential parameter α,β,γ, the shape of the surface can be adjusted by changing the value of α,β,γ, when the control grid is fixed. When the value of α,β,γ is increased, the surface will approach the control grid, so α,β,γ has a tension effect. In addition, it is easy to know from the boundary properties of the surface that each boundary curve $R(0,v,w),R(u,0,w)$ and $R(u,v,0)$ are only related to two of the parameters, independent of the other parameter. This indicates that a change of one parameter can only affect the shape of two of the boundary curves. Figure 7 shows the effect of the different exponential parameters on the QCT-Bernstein-Bézier surfaces.

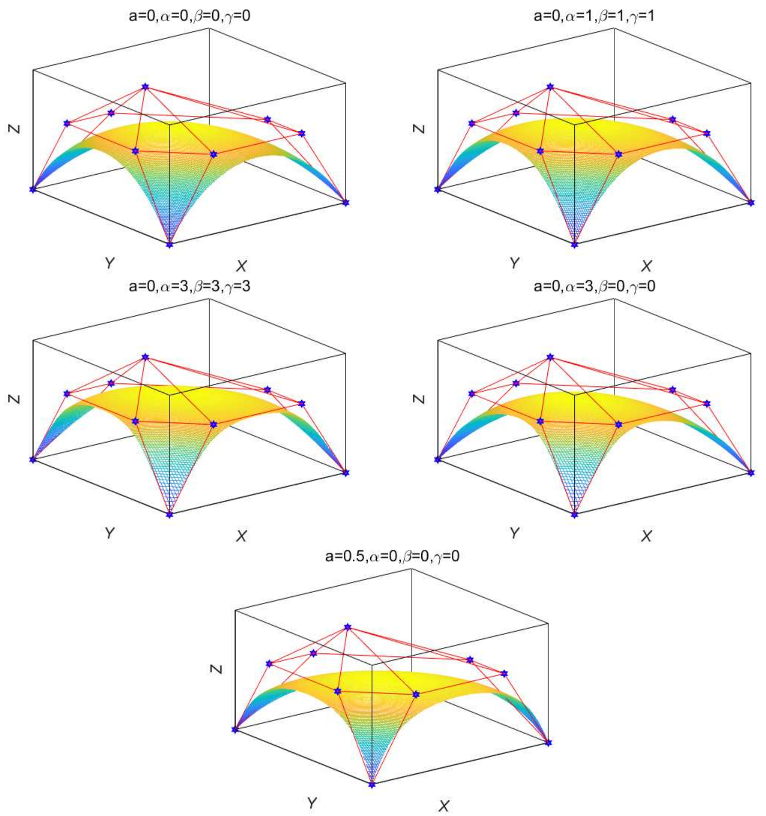

Figure 7: The effect of the different exponential parameters on the QCT-Bernstein-Bézier surfaces

5.3. The De Casteljau-type algorithm on the triangular domain

Below we present a De Casteljau-type algorithm for efficient and stable generating surfaces. For any $(u,v,w)\in D$,

When

$$\begin{aligned}
f_1(u,v,w) &:= \frac{\sin u\cos w(\sin^2 u+\sin^2 v+\sin^2 w)}{\cos w(\sin u+\sin v)(\sin^2 u+\sin^2 v+\sin^2 w)+\sin w(\sin^2 u+\sin^2 v)},\\
f_2(u,v,w) &:= \frac{\sin v\cos w(\sin^2 u+\sin^2 v+\sin^2 w)}{\cos w(\sin u+\sin v)(\sin^2 u+\sin^2 v+\sin^2 w)+\sin w(\sin^2 u+\sin^2 v)},\\
f_3(u,v,w) &:= \frac{\sin w(\sin^2 u+\sin^2 v)}{\cos w(\sin u+\sin v)(\sin^2 u+\sin^2 v+\sin^2 w)+\sin w(\sin^2 u+\sin^2 v)},\\
g_1(u,v,w) &:= (1-\cos u)(\sin^2 u+\sin^2 v+\sin^2 w),\\
g_2(u,v,w) &:= \sin v\cos w(\sin^2 u+\sin^2 v+\sin^2 w)+\sin u\sin v\sin w,\\
g_3(u,v,w) &:= \cos v\sin w(\sin^2 u+\sin^2 v+\sin^2 w)+\sin u\sin v\sin w,
\end{aligned} \tag{42}$$

and

$$P_{2,0,0}^{1} := \frac{(1-a\cos u)e^{-\alpha\cos u}}{1+\cos u}P_{3,0,0} + \frac{[1+\cos u-(1-a\cos u)e^{-\alpha\cos u}]\sin v\cos w}{(1+\cos u)\cos u}P_{2,1,0}$$
$$+\frac{[1+\cos u-(1-a\cos u)e^{-\alpha\cos u}]\cos v\sin w}{(1+\cos u)\cos u}P_{2,0,1},$$
$$P_{0,2,0}^{1} := \frac{[1+\cos v-(1-a\cos v)e^{-\beta\cos v}]\sin u\cos w}{(1+\cos v)\cos v}P_{1,2,0} + \frac{(1-a\cos v)e^{-\beta\cos v}}{1+\cos v}P_{0,3,0} \tag{43}$$
$$+\frac{[1+\cos v-(1-a\cos v)e^{-\beta\cos v}]\cos u\sin w}{(1+\cos v)\cos v}P_{0,2,1},$$
$$P_{0,0,2}^{1} := \frac{[1+\cos w-(1-a\cos w)e^{-\gamma\cos w}]\sin u\cos v}{(1+\cos w)\cos w}P_{1,0,2} + \frac{[1+\cos w-(1-a\cos w)e^{-\gamma\cos w}]\cos u\sin v}{(1+\cos w)\cos w}P_{0,1,2}$$
$$+\frac{(1-a\cos w)e^{-\gamma\cos w}}{1+\cos w}P_{0,0,3},$$

The expression for the rewritable surface is

$$\begin{aligned}R(u,v,w) = {} & \frac{1-\cos^2 u}{\sin^2 u+\sin^2 v+\sin^2 w}[g_1(u,v,w)P_{2,0,0}^{1} + g_2(u,v,w)P_{1,1,0}^{1} + g_3(u,v,w)P_{1,0,1}^{1}] + \\ & \frac{1-\cos^2 v}{\sin^2 u+\sin^2 v+\sin^2 w}[g_1(v,u,w)P_{0,2,0}^{1} + g_2(v,u,w)P_{1,1,0}^{1} + g_3(v,u,w)P_{0,1,1}^{1}] + \\ & \frac{1-\cos^2 w}{\sin^2 u+\sin^2 v+\sin^2 w}[g_1(w,v,u)P_{0,0,2}^{1} + g_2(w,v,u)P_{0,1,1}^{1} + g_3(w,v,u)P_{1,0,1}^{1}].\end{aligned} \tag{44}$$

These can be written as

$$\begin{aligned}P_{1,0,0}^{2} &:= g_1(u,v,w)P_{2,0,0}^{1} + g_2(u,v,w)P_{1,1,0}^{1} + g_3(u,v,w)P_{1,0,1}^{1}\\ P_{0,1,0}^{2} &:= g_2(v,u,w)P_{1,1,0}^{1} + g_1(v,u,w)P_{0,2,0}^{1} + g_3(v,u,w)P_{0,1,1}^{1}\\ P_{0,0,1}^{2} &:= g_3(w,v,u)P_{1,0,1}^{1} + g_2(w,v,u)P_{0,1,1}^{1} + g_1(w,v,u)P_{0,0,2}^{1}\end{aligned} \tag{45}$$

Then, the expression of QCT-Bernstein-Bézier surface with parameters can be further rewritten as

$$R(u,v,w) = \frac{1-\cos^2 u}{\sin^2 u+\sin^2 v+\sin^2 w}P_{1,0,0}^{2} + \frac{1-\cos^2 v}{\sin^2 u+\sin^2 v+\sin^2 w}P_{0,1,0}^{2} + \frac{1-\cos^2 w}{\sin^2 u+\sin^2 v+\sin^2 w}P_{0,0,1}^{2} := P_{0,0,0}^{3}. \tag{46}$$

For arbitrary $u+v+w=\pi/2$, it is easily verifies $f_1(u,v,w)+f_2(u,v,w)+f_3(u,v,w)=1$ and $g_1(u,v,w)+g_2(u,v,w)+g_3(u,v,w)=1$

5.4. Blending surfaces

Where $a\in[0,0.1]$ there is a fixed value , we can define two QCT-Bernstein-Bézier surfaces as

$$R_1(u,v,w) = \sum_{i+j+k=3} T_{i,j,k}^{3}(u,v,w;\alpha_1,\beta,\gamma)P_{i,j,k},(u,v,w)\in D \tag{47}$$

and

$$R_2(u,v,w) = \sum_{i+j+k=3} T_{i,j,k}^{3}(u,v,w;\alpha_2,\beta,\gamma)q_{i,j,k},(u,v,w)\in D. \tag{48}$$

When the control point satisfies

$$P_{0,j,k} = q_{0,j,k}\ ,\ j,k\in N, j+k=3\,, \tag{49}$$

the two surfaces have common boundary, that is $R_1(u,v,w)=R_2(u,v,w), v+w=\pi/2$. At this point, the surface is formed by the splicing of two surfaces and is C^0 continuous.

When we take the derivative of $R_1(0,v,\pi/2-v)$ common boundary curve, with respect to v, we get

$$\begin{aligned}\frac{dR_1(0,v,\pi/2-v)}{dv} &= [-e^{-\beta\cos v}(\cos v-1)\sin v(2+a+\beta-(3a+(a+1)\beta)\cos v+a\beta\cos^2 v)](P_{0,3,0}-P_{0,2,1}) \\ &+2\sin v\cos v(P_{0,2,1}-P_{0,1,2})+[-e^{-\gamma\sin v}(\sin v-1)\cos v(2+a+\gamma-(3a+(a+1)\gamma)\sin v \\ &+a\gamma\sin^2 v)](P_{0,1,2}-P_{0,0,3}).\end{aligned} \tag{50}$$

When we take the derivative of both surface pieces $R_1(u,,v,\pi/2-u-v)$ and $R_2(u,,v,\pi/2-u-v)$ with respect to u, we can obtain

$$\begin{aligned}\left.\frac{\delta R_1(u,v,\pi/2-v-u)}{\delta u}\right|_{u=0} &= [-e^{-\beta\cos v}(\cos v-1)\sin v(2+a+\beta-(3a+(a+1)\beta)\cos v+a\beta\cos^2 v)](P_{0,3,0}-P_{0,2,1}) \\ &+2\sin v\cos v(P_{0,2,1}-P_{0,1,2})+[-e^{-\gamma\sin v}(\sin v-1)\cos v(2+a+\gamma-(3a+(a+1)\gamma)\sin v \\ &+a\gamma\sin^2 v)](P_{0,1,2}-P_{0,0,3}).\end{aligned} \tag{51}$$

$$\begin{aligned}\left.\frac{\delta R_2(u,v,\pi/2-v-u)}{\delta u}\right|_{u=0} &= [-e^{-\beta\cos v}(\cos v-1)\sin v(2+a+\beta-(3a+(a+1)\beta)\cos v+a\beta\cos^2 v)](P_{0,3,0}-P_{0,2,1}) \\ &+2\sin v\cos v(P_{0,2,1}-P_{0,1,2})+[-e^{-\gamma\sin v}(\sin v-1)\cos v(2+a+\gamma-(3a+(a+1)\gamma)\sin v \\ &+a\gamma\sin^2 v)](P_{0,1,2}-P_{0,0,3}).\end{aligned} \tag{52}$$

If two surfaces, pieces $R_1(u,v,w)$ and $R_2(u,v,w)$ are continuous to G^1, the two vectors given in the preceding two equations are collinear to any variable v, and can be expressed as

$$\left.\frac{\delta R_2(u,v,\pi/2-u-v)}{\delta u}\right|_{u=0}=\lambda\frac{dR_1(0,v,\pi/2-v)}{dv}+\mu\left.\frac{\delta R_1(u,v,\pi/2-u-v)}{\delta u}\right|_{u=0} \tag{53}$$

Where λ,μ is an arbitrary constant. The following conditions are obtained

$$\begin{aligned} q_{1,2,0}-q_{0,2,1} &= \lambda(P_{0,3,0}-P_{0,2,1})+\mu(P_{1,2,0}-P_{0,2,1}), \\ q_{1,1,1}-q_{0,2,1} &= \lambda(P_{0,2,1}-P_{0,1,2})+\mu(P_{1,1,1}-P_{0,1,2}), \\ q_{1,0,2}-q_{0,0,3} &= \lambda(P_{0,1,2}-P_{0,0,3})+\mu(P_{1,0,2}-P_{0,0,3}).\end{aligned} \tag{54}$$

To sum up, the following theorem can be obtained.

Theorem 6. For any $\alpha,\beta,\gamma\in[0,+\infty],i=1,2$, surfaces $R_1(u,v,w)$ and $R_2(u,v,w)$ are G^1 continuous if (49) and (54) are satisfied for the control vertices.

Figure 8 shows the splicing of two surfaces with different shape parameters.

6. Conclusion

In this paper, we prove that a new QCT basis with three shape parameters exists in the framework of quasi-extended Chebyshev space. This basis function not only has the excellent properties of Bézier basis, but also can freely adjust the parameters to achieve the desired shape of curves and surfaces as follows: Ellipse, parabola, circle and other beyond the curve, but also with full positive and variation reduction, such good properties. Secondly, the Angle cutting algorithm is applied to make the curve

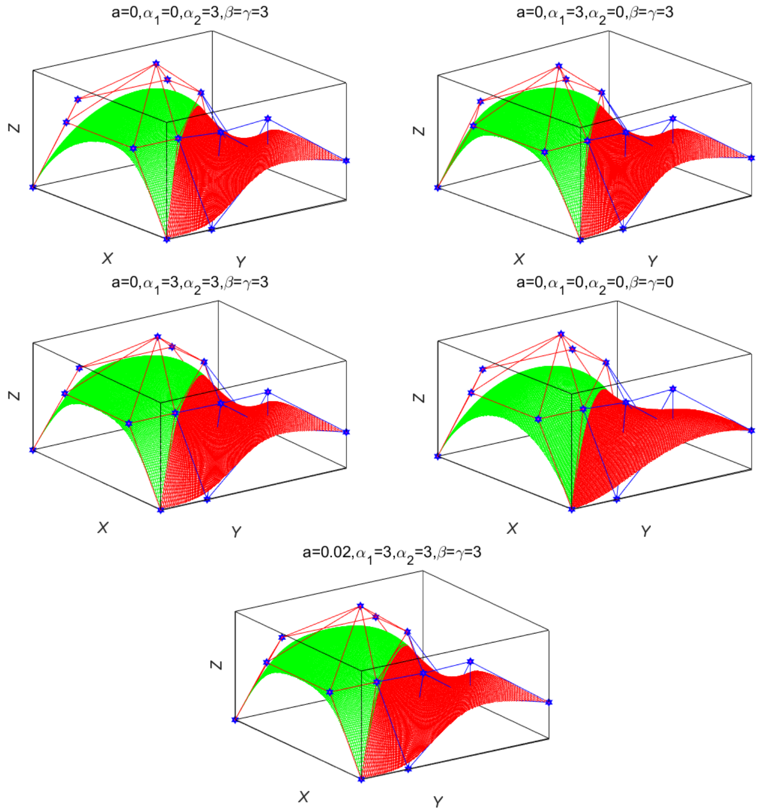

Figure 8: The splicing of two surfaces with different shape parameters

construction more stable and efficient. At the same time, the curve formed by this basis has the condition that C^0 continuous and C^1 continuous. After that, the QCT basis applies to the triangular domain, and a new triangular basis is obtained, which has many excellent properties, which are listed in the paper. Then the Angle cutting algorithm on the triangular domain is obtained, which makes the algorithm more excellent. The surface formed by the base of the triangle has the advantage of G^1 continuity in splicing. In addition, this base also has sharp points, inflection points, etc., which can be further studied. This article is not repeated, and the results will be described in another article.

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