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The Shadowed Set, Intuitionistic Fuzzy Set and their Three-way Decision

***Guangpeng Ta*[1] and *Zengtai Gong*[2*]**

[1,2]College of Mathematics and Statistics, Northwest Normal University
Lanzhou 730070, China
[*]Corresponding author. Email: zt-gong@163.com

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Abstract. The theory of three-way decision provides an effective tool for decision-making under uncertainty and incomplete information when a two-way decision is difficult to make. In this paper, we try to transform an intuitionistic fuzzy set into a pessimistic shadowed set or an optimistic shadowed set by using two intuitionistic fuzzy parameters and the classical shadowed set. Accordingly, the pessimistic three-way decision or the optimistic three-way decision of intuitionistic fuzzy set are investigated by means of the proposed shadowed sets. In addition, a method to calculate the thresholds by the minimum cost principle is proposed and some examples are given for illustrating the validity of the method.

Keywords: Three-way decision, Shadowed set, Intuitionistic fuzzy number

AMS Mathematics Subject Classification (2010): 03E72, 08A72

1. Introduction

The research of the three-way decision theories is closely related to the development of decision theories. As an extension of the traditional two-way decision model, the three-way decision model considers many uncertain factors in the decision process. When the processing information is insufficient to decide whether to accept or reject the decision, the delayed decision is introduced into the decision model as the third decision behavior. Therefore, the three-way decision theory has been widely applied in face recognition [6], medical diagnosis [12], recommendation system [17] and many other fields.

Rough set theory [7], as a mathematical tool to deal with ambiguity and inaccuracy, has developed rapidly both in theories and applications. The most significant difference between rough set method and other theories dealing with uncertain and imprecise problems is that it does not need to provide any prior information other than the data needed to deal with the problem. In 2010, Yao [9, 14] proposed the concept and method of three-way decision based on the concept of the loss function in order to further explain the parameter problem of partition of three regions of probability rough set. In 2017, Yao constructed the three-way approximation of fuzzy set by using the shadowed set proposed by Pedrycz [8], and discussed the relationship between fuzzy set, shadowed set and three-way decision [15]. In 2020, Yao transformed the fuzzy set into the shadowed set and made three-way decision of the fuzzy set [13].

However, it is noted in this paper that the membership grade of fuzzy set [16] cannot represent the neutral state because it represents the two opposites of fuzzy phenomenon at the same time, so the three-way decision constructed with fuzzy set and shadowed set inevitably lose their non-membership grade and other information. As an extension of fuzzy sets, intuitionistic fuzzy sets [1, 5] take into account both membership and non-membership information, which makes intuitionistic fuzzy sets have better expression ability and flexibility than traditional fuzzy sets when dealing with uncertain information, so they complement and develop Zadeh's fuzzy sets. Therefore, by means of the shadowed set and intuitionistic fuzzy parameters, we transform intuitionistic fuzzy set into pessimistic shadowed set and optimistic shadowed set, and proposes pessimistic three-way decision and optimistic three-way decision of intuitionistic fuzzy set. The three-way decision of intuitionistic fuzzy set uses a pair of thresholds to derive three regions. We apply the minimum cost principle to determine a pair of thresholds.

The rest of this paper is organized as follows. Section 2 presents some basic results of the intuitionistic fuzzy sets, intuitionistic fuzzy number and shadowed set. The definition of the pessimistic three-way decision and optimistic three-way decision of intuitionistic fuzzy set are proposed and investigated in the section 3. In section 4, we give a method to calculate the thresholds by the minimum cost principle and illustrate the validity of the method with an example.

2. Preliminaries

In this section, we briefly review the concepts, including the intuitionistic fuzzy set, intuitionistic fuzzy number, $\boldsymbol{\alpha}$ −level set $A_{\boldsymbol{\alpha}}$ of an intuitionistic fuzzy set A on X and shadowed set.

Definition 1. *[1, 5] An intuitionistic fuzzy set is an object having the following form*

$$A=\{\langle\ x,\mu_A(x),\nu_A(x)\rangle|x\in\ X\},$$

which is characterized by a membership function

$$\mu_A: X\to[0,1], x\in X\to\mu_A(x)\in[0,1],$$

and a non-membership function

$$\nu_A: X\to[0,1], x\in X\to\nu_A(x)\in[0,1],$$

with the condition

$$0\le\mu_A(x)+\nu_A\le 1, x\in X,$$

where $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership grade and the non-membership grade of x in A. Moreover, for each intuitionistic fuzzy set A in X, write

$$\pi_A(x)=1-\mu_A(x)-\nu_A, x\in X,$$

then $\pi_A(x)$ is called an indeterminancy grade of x to A.

Definition 2. *[1] Let $A=\{\langle\ x,\mu_A(x),\nu_A(x)\rangle|x\in\ X\}$ be an intuitionistic fuzzy set in the universe X. Give an intuitionistic fuzzy number $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$, the $\boldsymbol{\alpha}-$ level set $A_{\boldsymbol{\alpha}}$ of an intuitionistic fuzzy set A on X is defined by*

$$A_{\boldsymbol{\alpha}}=A_{(\alpha_1,\alpha_2)}=\{x|\mu_A(x)\ge\alpha_1,\nu_A(x)\le\alpha_2, x\in\ X\}.$$

In particular, when $\boldsymbol{\alpha}=(0,1)$ and $(1,0)$, the level set $A_{(0,1)}$ and $A_{(1,0)}$ of an intuitionistic fuzzy set A on X are respectively called the Support set and Core set of intuitionistic fuzzy set A, denoted as $Support(A)$ and $Core(A)$.

Definition 3. *[10] An intuitionistic fuzzy number* $\boldsymbol{\alpha}$ *is wrote as* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, *where* $\alpha_1 \in [0,1], \alpha_2 \in [0,1]$ *and* $0 \le \alpha_1 + \alpha_2 \le 1$.

Definition 4. *[2] Let* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ *be an intuitionistic fuzzy number. The score of* $\boldsymbol{\alpha}$ *can be evaluated by the score function* S *shown as* $S(\boldsymbol{\alpha}) = \alpha_1 - \alpha_2$.

Definition 5. *[4] Let* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ *be an intuitionistic fuzzy number. The accuracy degree of* $\boldsymbol{\alpha}$ *can be evaluated by the accuracy function* H *shown as* $H(\boldsymbol{\alpha}) = \alpha_1 + \alpha_2$.

Definition 6. *[11] Let* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ *and* $\boldsymbol{\beta} = (\beta_1, \beta_2)$ *be two intuitionistic fuzzy numbers,* $S(\boldsymbol{\alpha}) = \alpha_1 - \alpha_2$ *and* $S(\boldsymbol{\beta}) = \beta_1 - \beta_2$ *be two the scores of the intuitionistic fuzzy numbers* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *respectively, and* $H(\boldsymbol{\alpha}) = \alpha_1 + \alpha_2$ *and* $H(\boldsymbol{\beta}) = \beta_1 + \beta_2$ *be two the accuracy degrees of the intuitionistic fuzzy numbers* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *respectively.*

•If $S(\boldsymbol{\alpha}) < S(\boldsymbol{\beta})$, then intuitionistic fuzzy number $\boldsymbol{\alpha}$ is smaller than theintuitionistic fuzzy number $\boldsymbol{\beta}$, denoted by $\boldsymbol{\alpha} < \boldsymbol{\beta}$.

• If $S(\boldsymbol{\alpha}) = S(\boldsymbol{\beta})$, then

(1) If $H(\boldsymbol{\alpha}) = H(\boldsymbol{\beta})$, then intuitionistic fuzzy numbers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are equal, i.e., $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, denoted by $\boldsymbol{\alpha} = \boldsymbol{\beta}$;

(2) If $H(\boldsymbol{\alpha}) < H(\boldsymbol{\beta})$, then intuitionistic fuzzy number $\boldsymbol{\alpha}$ is smaller than the intuitionistic fuzzy number $\boldsymbol{\beta}$, denoted by $\boldsymbol{\alpha} < \boldsymbol{\beta}$;

(3) If $H(\boldsymbol{\alpha}) > H(\boldsymbol{\beta})$, then intuitionistic fuzzy number $\boldsymbol{\alpha}$ is larger than the intuitionistic fuzzy number $\boldsymbol{\beta}$, denoted by $\boldsymbol{\alpha} > \boldsymbol{\beta}$.

Definition 7. *[8] A shadowed set* $\mathbb{S}$ *is defined by a mapping from a nonempty set* X *to the three-valued set* $\{w, g, b\}$, *that is,* $\mathbb{S}: X \to \{w, g, b\}$. *Objects of* X *with membership grade* b *constitute the core of* $\mathbb{S}$ *and objects with membership grade* $\boldsymbol{g}$ *form the shadow of* $\mathbb{S}$.

The core and the shadow of a shadowed set may be empty. When the shadow is an empty set, a shadowed set degenerates to a set. The concept of a shadowed set is used to formulate a three-way approximation of an intuitionistic fuzzy set. This is done by changing fuzzy membership grade and non-membership grade into three values in $\{w, g, b\}$. In other words, we use three-valued set $\{w, g, b\}$ to approximate the intuitionistic fuzzy parameter.

3. Three-way decision and three-way approximation of an intuitionistic fuzzy set

In this section, in order to study the three-way decision of an intuitionistic fuzzy set, we give the transformation from an intuitionistic fuzzy set to a pessimistic shadowed set and optimistic shadowed set [3].

Definition 8. *Let* A *be an intuitionistic fuzzy set over a nonempty set* X.

(1) Give two intuitionistic fuzzy parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) > \boldsymbol{\beta} = (\beta_1, \beta_2)$, we can build a pessimistic shadowed set induced by the intuitionistic fuzzy set A on intuitionistic fuzzy parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as follows.

$$(A \rhd \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = \begin{cases} b, & \mu_A(x) \geq \alpha_1, \nu_A(x) \leq \alpha_2, \\ w, & \mu_A(x) \leq \beta_1, \nu_A(x) \geq \beta_2, \\ g, & otherwise\ . \end{cases}$$

(2) Give two intuitionistic fuzzy parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) > \boldsymbol{\beta} = (\beta_1, \beta_2)$, we can build an optimistic shadowed set induced by the intuitionistic fuzzy set A on intuitionistic fuzzy parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as follows.

$$(A \trianglerighteq \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = \begin{cases} b, & \mu_A(x) \geq \alpha_1, \\ w, & \nu_A(x) \geq \beta_2, \\ g, & otherwise\ . \end{cases}$$

We called that $(A \rhd \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ and $(A \trianglerighteq \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ are respectively pessimistic shadowed set and optimistic shadowed set induced by the intuitionistic fuzzy set, where $\rhd$ and $\trianglerighteq$ denote respectively two transformations from an intuitionistic fuzzy set to a pessimistic shadowed set and optimistic shadowed set.

Without special explanation, pessimistic shadowed set and optimistic shadowed set are generally referred to as the shadowed set induced by intuitionistic fuzzy set A about intuitionistic fuzzy parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, which is expressed as $(A \rhd \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$.

In Definition 8, we use $\{w, g, b\}$ to denote three distinct membership grades of shadowed sets, representing the white, gray, and black objects of a shadowed set. By Definition 8, we map membership grades closer to 1 or non-membership grades closer to 0 to b (black), membership grades closer to 0 or non-membership grades closer to 1 to w (white), and the rest to g (gray).

Remark 1. *In Definition 8, when $\mu_A(x) + \nu_A(x) = 1, \alpha_1 + \alpha_2 = 1$ and $\beta_1 + \beta_2 = 1$, the intuitionistic fuzzy set A degenerates into a fuzzy set, and shadowed set induced by the intuitionistic fuzzy set A on intuitionistic fuzzy parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ degenerates into the relation between fuzzy set and shadowed set.*

Definition 9. *Let A be an intuitionistic fuzzy set over a nonempty set X.*

(1) Give two intuitionistic fuzzy parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) > \boldsymbol{\beta} = (\beta_1, \beta_2)$, the pessimistic three-way approximation of an intuitionistic fuzzy set A, namely, positive , boundary and negative regions of intuitionistic fuzzy set A are defined by the following regions.

$$POS_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A) = \{x \in X | (A \rhd \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = b\},$$
$$BND_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A) = \{x \in X | (A \rhd \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = g\},$$
$$NEG_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A) = \{x \in X | (A \rhd \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = w\}.$$

(2) Give two intuitionistic fuzzy parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) > \boldsymbol{\beta} = (\beta_1, \beta_2)$, the optimistic three-way approximation of an intuitionistic fuzzy set A, namely, positive , boundary and negative regions of intuitionistic fuzzy set A are defined by regions.

$$POS_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A) = \{x \in X | (A \trianglerighteq \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = b\},$$
$$BND_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A) = \{x \in X | (A \trianglerighteq \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = g\},$$

$$NEG_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A) = \{x \in X | (A \trianglerighteq \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = w\}.$$

Remark 2. *In Definition 9, when* $\boldsymbol{\alpha} = (1,0)$ *and* $\boldsymbol{\beta} = (0,1)$, *the positive , boundary and negative regions of intuitionistic fuzzy set* A *are given respectively by the following regions.*

$$POS_{\{(1,0),(0,1)\}}(A) = Core(A),$$
$$BND_{\{(1,0),(0,1)\}}(A) = Support(A) - Core(A),$$
$$NEG_{\{(1,0),(0,1)\}}(A) = X - Support(A).$$

Remark 3. *In Definitions 8 and 9, when* $\mu_A(x) + \nu_A(x) = 1, \alpha_1 + \alpha_2 = 1$ *and* $\beta_1 + \beta_2 = 1$, *the intuitionistic fuzzy set* A *degenerates into a fuzzy set, and the pessimistic and optimistic three-way approximation of an intuitionistic fuzzy set* A *about intuitionistic fuzzy parameters* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *degenerate into a three-way approximation of fuzzy set.*

(1) Based on the pessimistic three-way approximation of an intuitionistic fuzzy set A, we can make pessimistic three-way decision. That is to say, for $x \in X$,

(P) if $x \in POS_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A)$, i.e., $\mu_{A(x)} \geq \alpha_1$ and $\nu_{A(x)} \leq \alpha_2$, then accept x;

(N) if $x \in NEG_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A)$, i.e., $\mu_{A(x)} \leq \beta_1$ and $\nu_{A(x)} \geq \beta_2$, then reject x;

(B) if $x \in BND_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A)$, then neither accept nor reject x.

(2) Based on the optimistic three-way approximation of an intuitionistic fuzzy set A, we can make optimistic three-way decision. That is, for $x \in X$,

(P) if $x \in POS_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A)$, i.e., $\mu_{A(x)} \geq \alpha_1$, then accept x;

(N) if $x \in NEG_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A)$, i.e., $\nu_{A(x)} \geq \beta_2$, then reject x;

(B) if $x \in BND_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(A)$, then neither accept nor reject x.

By Remark 2, we can construct a qualitative three-way decision of an intuitionistic fuzzy set, which is illustrated with an example.

Example 1. *Let's consider the intuitionistic fuzzy set* $A = \frac{<0,1>}{x_1} + \frac{<0.8,0.1>}{x_2} + \frac{<0.2,0.75>}{x_3} + \frac{<0.5,0.3>}{x_4} + \frac{<1,0>}{x_5}$ *on a nonempty set* $X = \{x_1, x_2, x_3, x_4, x_5\}$. *By Remark 2, we have*

$$POS_{\{(1,0),(0,1)\}}(A) = Core(A) = \{x_5\},$$
$$BND_{\{(1,0),(0,1)\}}(A) = Support(A) - Core(A) = \{x_1, x_2, x_3, x_4\},$$
$$NEG_{\{(1,0),(0,1)\}}(A) = X - Support(A) = \emptyset.$$

Therefore, we can construct the qualitative three-way decision of the intuitionistic fuzzy set A, which are accept x_5 and neither accept nor reject x_1, x_2, x_3 and x_4.

When an object has a high membership grade or a low non-membership grade, we make a positive decision on it. When an object has a low membership grade or a high non-membership grade, we make a negative decision on it. In other cases, we make delayed decision, which needs further study.

4. Determining a pair of thresholds $(\boldsymbol{\alpha}, \boldsymbol{\beta})$

Intuitionistic fuzzy parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ play an important role in the process of constructing a three-way decision of an intuitionistic fuzzy set. In this section, we take the minimum cost principle to determine a pair of thresholds $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

We consider an action based model in order to construct three regions of an intuitionistic fuzzy set. Let $\{a_w, a_g, a_b\}$ represent the set of three actions, where a_b, a_g, and a_w denote the actions that change the membership grade and the non-membership grade to b, g, and w respectively. When the value in $\{w, g, b\}$ is used to approximate the intuitionistic fuzzy number, different intuitionistic fuzzy numbers may correspond to the same value in $\{w, g, b\}$, which leads to the existence of errors. We assume that the cost is related to the error induced by these three actions. Intuitively, there's the greatest cost in changing membership grade 0 and non-membership grade 1 to b and there's the least cost in changing membership grade 1 and non-membership grade 0 to b. On the contrary, there's the least cost in changing membership grade 0 and non-membership grade 1 to w and there's the greatest cost in changing membership grade 1 and non-membership grade 0 to w. For an intuitionistic fuzzy set $A = \{\langle\ x, \mu_A(x), \nu_A(x)\rangle | x \in\ X\}$, as we move intuitionistic fuzzy number $(\mu_{A(x)}, \nu_{A(x)})$ from (0,1) to (1,0), the costs of changing membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$ to b would decrease, and the costs of changing membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$ to w would increase. For g, there's the least cost in changing membership grade 0.5 and non-membership grade 0.5 to g and there's the greatest cost in changing membership grade 1 and non-membership grade 0 to g or changing membership grade 0 and non-membership grade 1 to g. When the membership grade $\mu_A(x)$ from 0.5 increases to 1 or from 0.5 decreases to 0, and the non-membership grade $\nu_A(x)$ from 0.5 increases to 1 or from 0.5 decreases to 0, the costs of changing membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$ to g will increase. In order to formally describe the maximum cost and minimum cost, we assume that the following conditions hold and define some notations as follows.

$\lambda_b > 0$: cost of changing membership grade 0 to b;
$\theta_b > 0$: cost of changing non-membership grade 1 to b;
$\lambda_w > 0$: cost of changing membership grade 1 to w;
$\theta_w > 0$: cost of changing non-membership grade 0 to w;
$\lambda_g > 0$: costs of changing membership grade 1 or 0 to g;
$\theta_g > 0$: costs of changing non-membership grade 1 or 0 to g;
0: costs of changing membership grade 0 and non-membership grade 1 to w,
costs of changing membership grade 0.5 and non-membership grade 0.5 to g,
costs of changing membership grade 1 and non-membership grade 0 to b.

For an intuitionistic fuzzy set $A = \{\langle\ x, \mu_A(x), \nu_A(x)\rangle\ |\ x \in X\}$, in order to calculate the costs of changing membership grade $\mu_A(x)$ to one of $\{w, g, b\}$ and non-membership grade $\nu_A(x)$ to one of $\{w, g, b\}$, respectively, we can use a semantic distance function from [0,1] and $\{w, g, b\}$ to [0,1] in order to scale the maximum costs $\lambda_b, \theta_b, \lambda_w, \theta_w, \lambda_g$ and θ_g.

Let $d: [0,1] \times \{w, g, b\} \to [0,1]$ denote a normalized semantic distance function between values in [0,1] and values in $\{w, g, b\}$ [4].

Definition 10. *Let $A = \{\langle\ x, \mu_A(x), \nu_A(x)\rangle | x \in\ X\}$ be an intuitionistic fuzzy set on nonempty set X. The distances of membership grade $\mu_A(x)$ to one of $\{w, g, b\}$ and non-membership grade $\nu_A(x)$ to one of $\{w, g, b\}$, respectively, are defined as follows.*

$$\begin{aligned}
&d(\mu_A(x), b) = 1 - \mu_A(x),\\
&d(\mu_A(x), g) = abs(2\mu_A(x) - 1),\\
&d(\mu_A(x), w) = \mu_A(x),\\
&d(\nu_A(x), b) = \nu_A(x),\\
&d(\nu_A(x), g) = abs(2\nu_A(x) - 1),\\
&d(\nu_A(x), w) = 1 - \nu_A(x),
\end{aligned} \tag{4.1}$$

where abs denotes the absolute value.

For an intuitionistic fuzzy set $A = \{\langle\ x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, in order to calculate the costs of changing membership grade $\mu_A(x)$ to one of $\{w, g, b\}$ and non-membership grade $\nu_A(x)$ to one of $\{w, g, b\}$, respectively, we can proportionally allocate the maximum costs $\lambda_b, \theta_b, \lambda_w, \theta_w, \lambda_g$ and θ_g using the distance function in Definition 10. Therefore, we can simply multiple the maximum costs by the distance between membership grade $\mu_A(x)$ and one of $\{w, g, b\}$ and the distance between non-membership grade $\nu_A(x)$ and one of $\{w, g, b\}$.

Definition 11. *Let $A = \{\langle\ x, \mu_A(x), \nu_A(x)\rangle | x \in\ X\}$ be an intuitionistic fuzzy set on nonempty set X. For an object $x \in X$ with membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$, the costs of changing membership grade $\mu_A(x)$ to one of $\{w, g, b\}$ and non-membership grade $\nu_A(x)$ to one of $\{w, g, b\}$ are defined respectively as follows.*

$$\begin{aligned}
&C(a_b|\mu_A(x)) = d(\mu_A(x), b)\lambda_b,\\
&C(a_g|\mu_A(x)) = d(\mu_A(x), g)\lambda_g,\\
&C(a_w|\mu_A(x)) = d(\mu_A(x), w)\lambda_w,\\
&C(a_b|\nu_A(x)) = d(\nu_A(x), b)\theta_b,\\
&C(a_g|\nu_A(x)) = d(\nu_A(x), g)\theta_g,\\
&C(a_w|\nu_A(x)) = d(\nu_A(x), w)\theta_w.
\end{aligned}$$

In order to make the calculation meaningful, we need to normalize the distance between membership grade $\mu_A(x)$ and one of $\{w, g, b\}$ and the distance between non-membership grade $\nu_A(x)$ and one of $\{w, g, b\}$ so that the largest distance is 1 and the shortest distance is 0, as given by Eqs. (4.1). It is easy to verify that the distances $d(\mu_A(x), b), d(\mu_A(x), w), d(\nu_A(x), b),$ and $d(\nu_A(x), w)$ are normalized but the distances $d(\mu_A(x), g)$ and $d(\nu_A(x), g)$ need some explanations. If $\mu_A(x) = 1$ or $\mu_A(x) = 0$, then $d(\mu_A(x), g)_{\max} = 1$, and if $\mu_A(x) = 0.5$, then $d(\mu_A(x), g)_{\min} = 0$. The discussion of $d(\nu_A(x), g)$ is the same as the discussion of $d(\mu_A(x), g)$, and we will not repeat it. So the distances $d(\mu_A(x), g)$ and $d(\nu_A(x), g)$ are also normalized.

For an object $x \in X$ with membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$, according to Definitions 10 and 11, the costs of changing membership grade $\mu_A(x)$ to one of $\{w, g, b\}$ and non-membership grade $\nu_A(x)$ to one of $\{w, g, b\}$ are given respectively as follows.

$$\begin{aligned}
&C(a_b|\mu_A(x)) = d(\mu_A(x), b)\lambda_b = (1-\mu_A(x))\lambda_b,\\
&C(a_g|\mu_A(x)) = d(\mu_A(x), g)\lambda_g = abs(2\mu_A(x)-1)\lambda_g,\\
&C(a_w|\mu_A(x)) = d(\mu_A(x), w)\lambda_w = \mu_A(x)\lambda_w,\\
&C(a_b|\nu_A(x)) = d(\nu_A(x), b)\theta_b = \nu_A(x)\theta_b,\\
&C(a_g|\nu_A(x)) = d(\nu_A(x), g)\theta_g = abs(2\nu_A(x)-1)\lambda_g,\\
&C(a_w|\nu_A(x)) = d(\nu_A(x), w)\theta_w = (1-\nu_A(x))\lambda_w.
\end{aligned} \tag{4.2}$$

Let $\psi: X \to \{a_w, a_g, a_b\}$ represent a transformation from an intuitionistic fuzzy set to a shadowed set. For any $x \in X$, we have $\psi(x) \in \{a_w, a_g, a_b\}$, which represents an action we take for x. The set of all transformations are represented by Ψ.

Definition 12. *Each $\psi \in \Psi$ induces a three-way approximation of an intuitionistic fuzzy set. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ be an intuitionistic fuzzy set on nonempty set X. The overall cost of a three-way approximation of the intuitionistic fuzzy set A constructed by ψ is defined as follows.*

$$C_{total}(\psi) = \sum_{x\in X} \left(C(\psi(x)|\mu_A(x)) + C(\psi(x)|\nu_A(x))\right). \tag{4.3}$$

By finding a transformation with minimum overall cost, we can obtain an optimal three-way approximation of intuitionistic fuzzy set. Therefore, we have the following optimization problem

$$\arg\min_{\psi\in\Psi} C_{total}(\psi),$$

where arg selects an argument of C_{total} that produces the minimum overall cost.

By assuming that the cost is positive, according to Eq. (4.3), if action $\psi(x)$ generates the minimum cost for each $x \in X$, then we get the minimum overall cost for the universe X. We should choose an action $\psi(x) \in \{a_w, a_g, a_b\}$ using the following three rules so that the overall cost is minimal. That is to say,

(B) if $C(a_b|\mu_A(x)) \le C(a_w|\mu_A(x)) \wedge C(a_b|\mu_A(x)) \le C(a_g|\mu_A(x))$,
$C(a_b|\nu_A(x)) \le C(a_w|\nu_A(x)) \wedge C(a_b|\nu_A(x)) \le C(a_g|\nu_A(x))$,

then we take action a_b, i.e., changing membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$ to b,

(W) if $C(a_w|\mu_A(x)) \le C(a_g|\mu_A(x)) \wedge C(a_w|\mu_A(x)) \le C(a_b|\mu_A(x))$,
$C(a_w|\nu_A(x)) \le C(a_g|\nu_A(x)) \wedge C(a_w|\nu_A(x)) \le C(a_b|\nu_A(x))$,

then we take action a_w, i.e., changing membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$ to w,

(G) otherwise, we take action a_g, i.e., changing membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$ to g.

According to Eqs. (4.2), for rules (B), (G), and (W), we only need to consider two cases, namely, membership grade $\mu_A(x) \ge 0.5$, non-membership grade $\nu_A(x) \le 0.5$, and membership grade $\mu_A(x) < 0.5$, non-membership grade $\nu_A(x) > 0.5$.

Case 1. Membership grade $\mu_A(x) \ge 0.5$, non-membership grade $\nu_A(x) \le 0.5$.

Rule (B): The first condition of rule (B) is equivalent to $(1-\mu_A(x))\lambda_b \le \mu_A(x)\lambda_w$. Due to $\lambda_b + \lambda_w > 0$, $\mu_A(x) \ge \lambda_b/(\lambda_b + \lambda_w)$ is obtained by calculation. The second condition of rule (B) is equivalent to $(1-\mu_A(x))\lambda_b \leq (2\mu_A(x)-1)\lambda_g$. Because of $\lambda_b + 2\lambda_g > 0$, $\mu_A(x) \geq (\lambda_b+\lambda_g)/(\lambda_b+2\lambda_g)$ is obtained by calculation. Let $\alpha_1 = (\lambda_b+\lambda_g)/(\lambda_b+2\lambda_g)$ and $\gamma_1 = \lambda_b/(\lambda_b+\lambda_w)$, we have $\mu_A(x) \geq \alpha_1$ and $\mu_A(x) \geq \gamma_1$. According to the assumption of positive costs, $\alpha_1 > 0.5$ is obvious. Therefore, $\mu_A(x) \geq \max\{\alpha_1, \gamma_1\}$ is established. The third condition of rule (B) is equivalent to $\nu_A(x)\theta_b \leq (1-\nu_A(x))\theta_w$. Because of $\theta_b+\theta_w > 0$, $\nu_A(x) \leq \theta_w/(\theta_b+\theta_w)$ is obtained by calculation. The fourth condition of rule (B) is equivalent to $\nu_A(x)\theta_b \leq (1-2\nu_A(x))\theta_g$. Because of $\theta_b+2\theta_g > 0$, $\nu_A(x) \leq \theta_g/(\theta_b+2\theta_g)$ is obtained by calculation. Let $\alpha_2 = \theta_g/(\theta_b+2\theta_g)$ and $\gamma_2 = \theta_w/(\theta_b+\theta_w)$, we have $\nu_A(x) \leq \alpha_2$ and $\nu_A(x) \leq \gamma_2$. According to the assumption of positive costs, $\alpha_2 < 0.5$ is obvious. Therefore, $\nu_A(x) \leq \min\{\alpha_2, \gamma_2\}$ is established. Accordingly, rule (B) can be simplified into the following form

(B) If $\mu_A(x) \geq \max\{\alpha_1, \gamma_1\}$ and $\nu_A(x) \leq \min\{\alpha_2, \gamma_2\}$, then we can take action a_b.

Since $\alpha_1 > 0.5$ and $\alpha_2 < 0.5$ imply $\mu_A(x) > 0.5$ and $\nu_A(x) < 0.5$, we do not need to consider $\mu_A(x) > 0.5$ and $\nu_A(x) < 0.5$. The simplified rule (B) uses four thresholds $\alpha_1, \alpha_2, \gamma_1$ and γ_2. In order to further simplify the rule (B) so that only two thresholds are required, we must add some conditions on the costs [15]. By comparing actions a_b and a_g we take for x, we get the thresholds α_1 and α_2. If membership grade is greater than or equal to α_1 and non-membership grade is less than or equal to α_2, then action a_b is better than action a_g. By comparing actions a_b and a_w we take for x, we get the thresholds γ_1 and γ_2, and indicate whether or not action a_b is taken. When $\mu_A(x) \geq \gamma_1$ and $\nu_A(x) \leq \gamma_2$, we take action a_b, and the action we take with respect to x requires further study in other cases. When $\mu_A(x) \geq 0.5$ and $\nu_A(x) \leq 0.5$, intuitively speaking, it is reasonable to expect that one would prefer action a_b to action a_w. Therefore, in order to simplify rule (B), we assume that the threshold γ_1 should be as small as possible and threshold γ_2 should be as large as possible, namely, $\alpha_1 \geq \gamma_1$ and $\alpha_2 \leq \gamma_2$, which are equivalent to the following assumptions on the costs:

$$(s1) \quad \lambda_b\lambda_w + \lambda_g\lambda_w - \lambda_g\lambda_b \geq 0,$$
$$(s2) \quad \theta_b\theta_w + \theta_g\theta_w - \theta_g\theta_b \geq 0.$$

Since the threshold $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is an intuitionistic fuzzy number, we assume that $\alpha_1 + \alpha_2 \leq 1$, which is equivalent to the following assumption on the costs

$$(s3) \quad \lambda_b/\theta_b \leq \lambda_g/\theta_g.$$

Under assumptions (s1), (s2) and (s3), we get the following simplified rule (B).

(B) If $\mu_A(x) \geq \alpha_1$ and $\nu_A(x) \leq \alpha_2$, then we take action a_b, in which two thresholds are used.

Rule (W): When $\mu_A(x) \geq 0.5$ and $\nu_A(x) \leq 0.5$, intuitively speaking, it is reasonable to expect that one would prefer action a_b to action a_w. Therefore, rule (W) is not applicable for $\mu_A(x) \geq 0.5$ and $\nu_A(x) \leq 0.5$ under assumptions (s1), (s2) and (s3).

(G) We take action a_g in other cases.

For Case 1 with $\mu_A(x) \geq 0.5$ and $\nu_A(x) \leq 0.5$, under the assumptions (s1) (s2) and (s3), we can only apply either rule (B) or rule (G) based on two thresholds $\alpha_1 > 0.5$ and $\alpha_2 < 0.5$.

Case 2. Membership grade $\mu_A(x) < 0.5$, non-membership grade $\nu_A(x) > 0.5$. In this case, we first consider rule (W).

Rule (W): The first condition of rule (W) can be expressed as $\mu_A(x)\lambda_w \leq (1 - 2\mu_A(x))\lambda_g$, which is equivalent to $\mu_A(x) \leq \lambda_g/(\lambda_w + 2\lambda_g)$. The second condition of rule (W) can be expressed as $\mu_A(x)\lambda_w < (1 - \mu_A(x))\lambda_b$, which is equivalent to $\mu_A(x) \leq \lambda_b/(\lambda_b + \lambda_w)$, i.e., $\mu_A(x) \leq \gamma_1$. Let $\beta_1 = \lambda_g/(\lambda_w + 2\lambda_g)$, we have $\mu_A(x) \leq \beta_1$. According to the assumption of positive costs, $\beta_1 < 0.5$ is obvious. Therefore, $\mu_A(x) \leq \min\{\beta_1, \gamma_1\}$ is established. The third condition of rule (W) can be expressed as $(1 - \nu_A(x))\theta_w \leq (2\nu_A(x) - 1)\theta_g$, which is equivalent to $\nu_A(x) \geq (\theta_g + \theta_w)/(2\theta_g + \theta_w)$. Let $\beta_2 = (\theta_g + \theta_w)/(2\theta_g + \theta_w)$, we have $\nu_A(x) \geq \beta_2$. According to the assumption of positive costs, $\beta_2 > 0.5$ is obvious. The forth condition of rule (W) can be expressed as $(1 - \nu_A(x))\theta_w < \nu_A(x)\theta_b$, which is equivalent to $\nu_A(x) > \theta_w/(\theta_b + \theta_w)$, i.e., $\nu_A(x) > \gamma_2$. Therefore, $\nu_A(x) \geq \max\{\beta_2, \gamma_2\}$ is established. Accordingly, rule (W) can be simplified into the following rule.

(W) If $\mu_A(x) \leq \min\{\beta_1, \gamma_1\}$ and $\nu_A(x) \geq \max\{\beta_2, \gamma_2\}$, then we can take action a_w.

Since $\beta_1 < 0.5$ and $\beta_2 > 0.5$ imply $\mu_A(x) < 0.5$ and $\nu_A(x) > 0.5$, we do not need to consider $\mu_A(x) < 0.5$ and $\nu_A(x) > 0.5$. The following discussion of rule (W) is similar to that of rule (B) in case 1. In order to further simplify the rule (W) so that only two thresholds are required, we must add some conditions on the costs. By comparing action a_w and action a_g we take for x, we get the thresholds β_1 and β_2. If membership grade is less than or equal to β_1 and non-membership grade is greater than or equal to β_2, then action a_w is better than action a_g. By comparing action a_w and action a_b we take for x, we get the thresholds γ_1 and γ_2, and indicate whether or not action a_w is taken. When $\mu_A(x) < \gamma_1$ and $\nu_A(x) > \gamma_2$, action a_w is taken, and the action we take with respect to x requires further study in other other cases. When $\mu_A(x) < 0.5$ and $\nu_A(x) > 0.5$, intuitively speaking, it is reasonable to expect that one would prefer action a_w to action a_b. Therefore, in order to simplify rule (W), we assume that the threshold γ_1 should be as large as possible and threshold γ_2 should be as small as possible, namely, $\beta_1 \leq \gamma_1$ and $\beta_2 \geq \gamma_2$, which are equivalent to the following assumptions on the costs:

$$(s4) \quad \lambda_b\lambda_w + \lambda_g\lambda_b - \lambda_g\lambda_w \geq 0,$$
$$(s5) \quad \theta_b\theta_w + \theta_g\theta_b - \theta_g\theta_w \geq 0.$$

Since the threshold $\boldsymbol{\beta} = (\beta_1, \beta_2)$ is an intuitionistic fuzzy number, we assume that $\beta_1 + \beta_2 \leq 1$, which is equivalent to the following assumption on the costs

$$(s6) \quad \lambda_g/\theta_g \leq \lambda_w/\theta_w.$$

Under assumptions (s4), (s5) and (6), we get the following simplified rule (W).

(W) If $\mu_A(x) \leq \beta_1$ and $\nu_A(x) \geq \beta_2$, then we take action a_w, in which two thresholds are used.

Rule (B): When $\mu_A(x) < 0.5$ and $\nu_A(x) > 0.5$, intuitively speaking, it is reasonable to expect that one would prefer action a_w to action a_b. Therefore, rule (B) is not applicable for $\mu_A(x) < 0.5$ and $\nu_A(x) > 0.5$ under assumptions (s4), (s5) and (s6).

(G) We take action a_g in other cases.

For Case 2 with $\mu_A(x) < 0.5$ and $\nu_A(x) > 0.5$, under the assumptions (s4) (s5) and (s6), we can only apply either rule (W) or rule (G) based on two thresholds $\beta_1 < 0.5$ and $\beta_2 > 0.5$.

By combining the assumptions (s3) and (s6) in case 1 and 2, we have $\lambda_b/\theta_b \leq \lambda_g/\theta_g \leq \lambda_w/\theta_w$.

By combining rules of the case 1 and 2, under the assumptions of positive costs, (s1), (s2), (s4), (s5) and $\lambda_b/\theta_b \leq \lambda_g/\theta_g \leq \lambda_w/\theta_w$, we can derive the following three rules.

(B) If $\mu_A(x) \geq \alpha_1$ and $\nu_A(x) \leq \alpha_2$, then we take action a_b, i.e., changing membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$ to b,

(W) If $\mu_A(x) \leq \beta_1$ and $\nu_A(x) \geq \beta_2$, then we take action a_w, i.e., changing membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$ to w,

(G) We take action a_g in other cases, i.e., changing membership grade $\mu_A(x)$ and non-membership grade $\nu_A(x)$ to g.

According to the three simplified rules, the optimal transformation is given by

$$\psi_{optimal}(x) = \begin{cases} a_b, & \mu_A(x) \geq \alpha_1, \nu_A(x) \leq \alpha_2, \\ a_w, & \mu_A(x) \leq \beta_1, \nu_A(x) \geq \beta_2, \\ a_g, & otherwise \end{cases},$$

where

$$\alpha_1 = \frac{\lambda_b+\lambda_g}{\lambda_b+2\lambda_g}, \alpha_2 = \frac{\theta_g}{\theta_b+2\theta_g}, \beta_1 = \frac{\lambda_g}{\lambda_w+2\lambda_g}, \beta_2 = \frac{\theta_g+\theta_w}{2\theta_g+\theta_w}. \tag{4.4}$$

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$, it can be easily verified that $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) > \boldsymbol{\beta} = (\beta_1, \beta_2)$ by Definition 6. Therefore, we can conclude that a pessimistic three-way approximation of an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ induced by the optimal transformation. In fact, the pessimistic shadowed set is given as follows.

$$(A \rhd \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = \begin{cases} b, & \mu_A(x) \geq \alpha_1, \nu_A(x) \leq \alpha_2, \\ w, & \mu_A(x) \leq \beta_1, \nu_A(x) \geq \beta_2, \\ g, & otherwise \end{cases}.$$

An optimistic three-way approximation of an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ induced by the optimal transformation is defined. In fact, the optimistic shadowed set is given as follows.

$$(A \unrhd \mathbb{S})_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = \begin{cases} b, & \mu_A(x) \geq \alpha_1, \\ w, & \nu_A(x) \geq \beta_2, \\ g, & otherwise \end{cases}.$$

Our derivation not only gives a pessimistic shadowed set and optimistic shadowed set, but also determines a pair of thresholds $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ according to the minimum cost principle.

In the special case $\lambda_b = \lambda_g = \lambda_w > 0$ and $\theta_b = \theta_g = \theta_w > 0$, the assumptions (s1), (s2), (s4), (s5) and $\lambda_b/\theta_b \leq \lambda_g/\theta_g \leq \lambda_w/\theta_w$ hold. According to Eqs. (4.4), we get $\alpha_1 = 2/3, \alpha_2 = 1/3, \beta_1 = 1/3$ and $\beta_2 = 2/3$. Therefore, $\boldsymbol{\alpha} = (2/3, 1/3)$ and $\boldsymbol{\beta} = (1/3, 2/3)$, it can be easily verified that $\boldsymbol{\alpha} = (2/3, 1/3) > \boldsymbol{\beta} = (1/3, 2/3)$ by Definition 6.

In the special case $\lambda_b = \lambda_w = 2\lambda_g > 0$ and $\theta_b = \theta_w = 2\theta_g > 0$, the assumptions (s1), (s2), (s4), (s5) and $\lambda_b/\theta_b \leq \lambda_g/\theta_g \leq \lambda_w/\theta_w$ hold. According to Eqs. (4.4), we get

$\alpha_1 = 3/4, \alpha_2 = 1/4, \beta_1 = 1/4$ and $\beta_2 = 3/4$. Therefore, $\boldsymbol{\alpha} = (3/4,1/4)$ and $\boldsymbol{\beta} = (1/4,3/4)$, it can be easily verified that $\boldsymbol{\alpha} = (3/4,1/4) > \boldsymbol{\beta} = (1/4,3/4)$ by Definition 6.

Example 2. *Let's consider the intuitionistic fuzzy set*

$$A = \frac{<0,1>}{x_1} + \frac{<0.8,0.1>}{x_2} + \frac{<0.2,0.75>}{x_3} + \frac{<0.5,0.3>}{x_4} + \frac{<1,0>}{x_5} + \frac{<0.75,0.22>}{x_6} + \frac{<0.21,0.77>}{x_7}$$

on a nonempty set $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. *Based on the minimum cost principle, the required thresholds* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *are determined by constructing the optimal three-way approximation of an intuitionistic fuzzy set. We assume that* $\lambda_b = 0.4$ $\lambda_w = 0.6, \lambda_g = 0.2, \theta_b = 0.6, \theta_w = 0.4$ *and* $\theta_g = 0.2$. *It follows that* $\lambda_b\lambda_w + \lambda_g\lambda_w - \lambda_g\lambda_b = 0.28 > 0, \theta_b\theta_w + \theta_g\theta_w - \theta_g\theta_b = 0.2 > 0, \lambda_b\lambda_w + \lambda_g\lambda_b - \lambda_g\lambda_w = 0.2 > 0, \theta_b\theta_w + \theta_g\theta_b - \theta_g\theta_w = 0.28 > 0$ *and* $\lambda_b/\theta_b = 2/3 < \lambda_g/\theta_g = 1 < \lambda_w/\theta_w = 3/2$. *Thus, they satisfy assumptions (s1), (s2), (s4), (s5) and* $\lambda_b/\theta_b \le \lambda_g/\theta_g \le \lambda_w/\theta_w$. *According to Eqs. (4.4), we get the parameters as follows.*

$$\alpha_1 = \frac{\lambda_b+\lambda_g}{\lambda_b+2\lambda_g} = \frac{0.2+0.4}{0.4+2\times0.2} = \frac{0.6}{0.8} = 0.75,$$
$$\alpha_2 = \frac{\theta_g}{\theta_b+2\theta_g} = \frac{0.2}{0.6+2\times0.2} = \frac{0.2}{1} = 0.2,$$
$$\beta_1 = \frac{\lambda_g}{\lambda_w+2\lambda_g} = \frac{0.2}{0.6+2\times0.2} = \frac{0.2}{1} = 0.2,$$
$$\beta_2 = \frac{\theta_g+\theta_w}{2\theta_g+\theta_w} = \frac{0.2+0.4}{2\times0.2+0.4} = \frac{0.6}{0.8} = 0.75.$$

Therefore, $\boldsymbol{\alpha} = (0.75,0.2)$ and $\boldsymbol{\beta} = (0.2,0.75)$, and it can be easily verified that $\boldsymbol{\alpha} = (0.75,0.2) > \boldsymbol{\beta} = (0.2,0.75)$ by Definition 6.

The pessimistic shadowed set $(A \rhd \mathbb{S})_{\{(0.75,0.2),(0.2,0.75)\}}$ as a pessimistic three-way approximation of an intuitionistic fuzzy set is given by

$$(A \rhd \mathbb{S})_{\{(0.75,0.2),(0.2,0.75)\}}(x) = \begin{cases} b, & \mu_A(x) \ge 0.75, \nu_A(x) \le 0.2, \\ w, & \mu_A(x) \le 0.2, \nu_A(x) \ge 0.75, \\ g, & otherwise \ . \end{cases}$$

According to Definition 9, we have

$POS_{\{(0.75,0.2),(0.2,0.75)\}}(A) = \{x \in X | (A \rhd \mathbb{S})_{\{(0.75,0.2),(0.2,0.75)\}}(x) = b\} = \{x_2, x_5\}$,
$BND_{\{(0.75,0.2),(0.2,0.75)\}}(A) = \{x \in X | (A \rhd \mathbb{S})_{\{(0.75,0.2),(0.2,0.75)\}}(x) = g\} = \{x_4, x_6, x_7\}$,
$NEG_{\{(0.75,0.2),(0.2,0.75)\}}(A) = \{x \in X | (A \rhd \mathbb{S})_{\{(0.75,0.2),(0.2,0.75)\}}(x) = w\} = \{x_1, x_3\}$.

Therefore, we can construct the pessimistic three-way decision, which are accept x_2 and x_5, reject x_1 and x_3 and neither accept nor reject x_4, x_6 and x_7.

The optimistic shadowed set $(A \trianglerighteq \mathbb{S})_{\{(0.75,0.2),(0.2,0.75)\}}$ as an optimistic three-way approximation of an intuitionistic fuzzy set is given by

$$(A \trianglerighteq \mathbb{S})_{\{(0.75,0.2),(0.2,0.75)\}}(x) = \begin{cases} b, & \mu_A(x) \ge 0.75, \\ w, & \nu_A(x) \ge 0.75, \\ g, & otherwise \ . \end{cases}$$

According to Definition 9, we have

$POS_{\{(0.75,0.2),(0.2,0.75)\}}(A) = \{x \in X | (A \trianglerighteq \mathbb{S})_{\{(0.75,0.2),(0.2,0.75)\}}(x) = b\} = \{x_2, x_5, x_6\}$,
$BND_{\{(0.75,0.2),(0.2,0.75)\}}(A) = \{x \in X | (A \trianglerighteq \mathbb{S})_{\{(0.75,0.2),(0.2,0.75)\}}(x) = g\} = \{x_4\}$,
$NEG_{\{(0.75,0.2),(0.2,0.75)\}}(A) = \{x \in X | (A \trianglerighteq \mathbb{S})_{\{(0.75,0.2),(0.2,0.75)\}}(x) = w\} = \{x_1, x_3, x_7\}$.

Therefore, we can construct the optimistic three-way decision, which are accept x_2, x_5 and x_6, reject x_1, x_3 and x_7 and neither accept nor reject x_4.

Eqs. (4.4) can be modified as follows.

$$\alpha_1 = \frac{1+\lambda_g/\lambda_b}{1+2\lambda_g/\lambda_b}, \alpha_2 = \frac{\theta_g/\theta_b}{1+2\theta_g/\theta_b}, \beta_1 = \frac{\lambda_g/\lambda_w}{1+2\lambda_g/\lambda_w}, \beta_2 = \frac{\theta_g/\theta_w+1}{2\theta_g/\theta_w+1}. \tag{4.5}$$

We can see that the ratios λ_g/λ_b and θ_g/θ_b determine $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and ratios λ_g/λ_w and θ_g/θ_w determine $\boldsymbol{\beta} = (\beta_1, \beta_2)$. For different maximum costs $\lambda_b, \theta_b, \lambda_w, \theta_w, \lambda_g$ and θ_g, if the ratios λ_g/λ_b, θ_g/θ_b, λ_g/λ_w and θ_g/θ_w remain unchanged, then $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ remain unchanged by Eqs. (4.5). Therefore, it is possible for different maximum costs to determine a pair of equal thresholds $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

5. Conclusions and remarks

In this paper, we investigate the method that an intuitionistic fuzzy set transforms into a pessimistic shadowed set or an optimistic shadowed set. In addition, the pessimistic three-way decision and optimistic three-way decision of intuitionistic fuzzy set are proposed by the defined shadowed sets. The results show that an intuitionistic fuzzy set could be divided into three parts and it has obvious advantages in decision making.

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