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Generalized Penalty Method for a Class of Variational-hemivariational Inequality

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Abstract. In this paper, we consider a class of variational-hemivariational inequality problems with constraints in a reflexive Banach space. This inequality problem involves two nonlinear operators and two nondifferentiable functionals. We introduce the penalty parameter and the penalty operator and change the initial problem into the penalty one, and then use the generalized penalty method to prove the existence result of the solution to the objective inequality.

Keywords: Variational-hemivariational inequality, Nonlinear operators, Generalized penalty method, Existence result.

AMS Mathematics Subject Classification (2010): 49J40, 47J20

1. Introduction

It is very important for us to study contact problems which are always described by variational or hemivariational inequalities that are usually arisen in Mechanics, Physics and Engineering for contact processes are very common phenomenons both in life and industry. The theories of variational inequalities, which involve arguments of monotonicity and convexity that including properties of the subdifferential of a convex function, were firstly studied in the sixties and have been widely developed since then, cf. ([1]-[6]). The conception of hemivariational inequalities have been introduced in the early 1980s by Panagiotopoulos' pioneering works. The studies of this class of inequalities are usually based on properties of the subdifferential in the sense of Clarke and defined for locally Lipschitz functions which may be nonconvex, cf. ([7]-[12]).

Recently, variational-hemivariational inequalities have been studied by more and more researchers. This class of inequalities involves both convex and nonconvex functions, see ([13]-[19]). Our main in this paper is to present a generalized penalty method in the study of a class of variational-hemivariational inequality. The penalty method is effective in the numerical solution of constrained problems. It is a useful tool in proving the existence of the solution to constrained problems, see ([20]-[22]). And the generalized penalty method is the generalization of the penalty method, cf. ([23,24]). The penalty method has also been used to research the history-dependent variational or hemivariational

inequalities, in which convergence of the penalty method is shown as the penalty parameter goes to zero.
The rest of the paper is organized as follows. In Section 2, we recall some basic notions and definitions. In Section 3, we introduce a class of variational-hemivariational inequality, and prove the existence result through the generalized penalty method.

2. Preliminaries
In this section, we recall some definitions and results we need in this paper. More details can be found in the references [25].

For a normed space $\mathbf{V}$, we denote by $\mathbf{V}^*$ its topological dual. We use the notations $\|\cdot\|_{\mathbf{V}}$ and $\|\cdot\|_{\mathbf{V}^*}$ for the norms of the spaces $\mathbf{V}$ and $\mathbf{V}^*$, respectively. $\langle\cdot,\cdot\rangle_{\mathbf{V}^*\times\mathbf{V}}$ represents the dual pairing between $\mathbf{V}^*$ and $\mathbf{V}$. Also, the symbols $\to$ and $\rightharpoonup$ stand for the strong and weak convergence in various spaces, respectively.

We shall consider single-valued operator $A: \mathbf{V} \to \mathbf{V}^*$. The following definitions hold.

Definition 1. *An operator $A: \mathbf{V} \to \mathbf{V}^*$ is called pseudomonotone, if it is bounded and $\mathbf{u}_n \to \mathbf{u}$ in $\mathbf{V}$ together with* $\limsup\langle A\mathbf{u}_n, \mathbf{u}_n - \mathbf{u}\rangle \leq 0$ *imply* $\langle A\mathbf{u}, \mathbf{u} - \mathbf{v}\rangle \leq \liminf_{n\to\infty}\langle A\mathbf{u}_n, \mathbf{u}_n - \mathbf{v}\rangle$ for all $\mathbf{u} \in \mathbf{V}$.

Proposition 1. *For a reflexive Banach space $\mathbf{V}$, the following statements hold.*
(a) *If the operator $A: \mathbf{V} \to \mathbf{V}^*$ is bounded, demicontinous and monotone, then A is pseudomonotone.*
(b) *If $A, B: \mathbf{V} \to \mathbf{V}^*$ are pseudomonotone operators, then the sum $A + B: \mathbf{V} \to \mathbf{V}^*$ is pseudomonotone.*

Definition 2. *For a locally Lipschtz function $j: \mathbf{V} \to \mathbb{R}$, we denote by $j^0(\mathbf{u}; \mathbf{v})$ the generalized (Clarke) directioanal derivative of j at the point $\mathbf{u} \in \mathbf{V}$ in the direction $\mathbf{v} \in \mathbf{V}$ defined by*

$$f^0(\mathbf{u}; \mathbf{v}) = \limsup_{\mathbf{y}\to\mathbf{u},\, \lambda\searrow 0} \frac{f(\mathbf{y} + \lambda\mathbf{v}) - f(\mathbf{y})}{\lambda}.$$

The generalized gradient or subdifferential of f at $\mathbf{u}$, denoted by $\partial f(\mathbf{u})$, is a subset of the dual space $\mathbf{V}^*$ given by $\partial f(\mathbf{u}) = \{\xi \in \mathbf{V}^* | f^0(\mathbf{u}; \mathbf{v}) \geq \langle\xi, \mathbf{v}\rangle_{\mathbf{V}^*\times\mathbf{V}}, \forall \mathbf{v} \in \mathbf{V}\}$.

Proposition 2. *Let $\mathbf{V}$ be a real Banach Space, $f: \mathbf{V} \to \mathbb{R}$ be a locally Lipschitz function. For all $(\mathbf{u}, \mathbf{v}) \in \mathbf{V} \times \mathbf{V}$, $(\mathbf{u}_n, \mathbf{v}_n) \in \mathbf{V} \times \mathbf{V}$ such that $(\mathbf{u}_n, \mathbf{v}_n) \to (\mathbf{u}, \mathbf{v})$ in $\mathbf{V} \times \mathbf{V}$, we have* $\limsup f^0(\mathbf{u}_n; \mathbf{v}_n) \leq f^0(\mathbf{u}; \mathbf{v})$.

3. The main result
Let Ω be an open bounded subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ which is Lipschitz continuous. Γ be a measurable subset of $\partial\Omega$. We denote by $\mathbf{x}$ a generic point in Γ and $m(\Gamma)$ the $(d-1)$ dimensional measure of Γ. Given an integer $s > 0$. We use notation $\mathbf{V}$ for a closed subset of $\mathbf{H}^1(\Omega; R^s)$, where $\mathbf{H} = L^2(\Omega; \mathbb{R}^s)$. We denote $\gamma: \mathbf{V} \to L^2(\Gamma; \mathbb{R}^s)$ the trace operator, $\| \gamma \|$ the norm in the space $\left(V, L^2(\Gamma; \mathbb{R}^s)\right)$. $(\mathbf{V}, \mathbf{H}, \mathbf{V}^*)$ forms evolution triple of spaces and the embedding $\mathbf{V} \subset \mathbf{H}$ is compact, and $\mathbf{K}$ is a subset of $\mathbf{V}$. Given operators

$A: \mathbf{V} \to \mathbf{V}^*$, functions $\varphi, j: \Gamma \times \mathbb{R}^s$ and a functional $f: \mathbf{V} \to \mathbb{R}$, we consider the following problem.

Problem(P). Find an element $\mathbf{u} \in \mathbf{K}$ such that

$$\langle A\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle + \int_\Gamma \varphi(\gamma\mathbf{v}) - \varphi(\gamma\mathbf{u}) d\Gamma + \int_\Gamma j^0(\gamma\mathbf{u}; \gamma\mathbf{v} - \gamma\mathbf{u}) d\Gamma \geq \langle f, \mathbf{v} - \mathbf{u}\rangle \quad \forall \mathbf{v} \in \mathbf{K}.$$

We introduce the following hypotheses.

$H(\mathbf{K})$: $\mathbf{K}$ is a nonempty, closed and convex subset of $\mathbf{V}$.

$H(A)$: $A: \mathbf{V} \to \mathbf{V}^*$ is

(a) pseudomonotone and there exists $\alpha > 0$ such that $\langle A\mathbf{v}, \mathbf{v}\rangle_{\mathbf{V}^* \times \mathbf{V}} \geq \alpha \parallel \mathbf{v} \parallel_{\mathbf{V}^*} \quad \forall \mathbf{v} \in \mathbf{V}$;

(b) strongly monotone, i.e., there exists $m_A > 0$ such that

$$\langle A\mathbf{v}_1 - A\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\rangle_{\mathbf{V}^* \times \mathbf{V}} \geq m_A \parallel \mathbf{v}_1 - \mathbf{v}_2 \parallel_{\mathbf{V}}^* \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}.$$

$H(\varphi)$: $\varphi: \Gamma \times \mathbb{R}^s \to \mathbb{R}$ is such that

(a) $\varphi(\cdot, \xi)$ is measurable on Γ for all $\xi \in \mathbb{R}^s$ and there exists $\overline{e} \in L^2(\Gamma; \mathbb{R}^s)$ such that $\varphi(\cdot, \overline{e}(\cdot)) \in L^2(\Gamma)$;

(b) $\varphi(\mathbf{x}, \cdot)$ is convex for a.e. $\mathbf{x} \in \Gamma$;

(c) there exists $L_\varphi > 0$ such that for all $\xi_1, \xi_2 \in R^s$,

$$|\varphi(\boldsymbol{x}, \xi_1) - \varphi(\boldsymbol{x}, \xi_2)| \leq L_\varphi \left\||\xi_1 - \xi_2|\right\|_{\mathbb{R}^s} \text{ a.e. } \boldsymbol{x} \in \Gamma.$$

$H(j)$: $j: \Gamma \times R$ is such that

(a) $j(\cdot, \xi)$ is measurable on Γ for all $\xi \in \mathbb{R}^s$ and there exists $e \in L^2(\Gamma; \mathbb{R}^s)$ such that $j(\cdot, e(\cdot)) \in L^2(\Gamma)$;

(b) $j(\mathbf{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^s$ for $\mathbf{x} \in \Gamma$;

(c) $\parallel \partial j(\mathbf{x}, \xi) \parallel_{\mathbb{R}^s} \leq c_0 + c_1 \parallel \xi \parallel_{\mathbb{R}^s}$ for a.e. $\mathbf{x} \in \Gamma$, for all $\xi \in \Gamma$ with $c_0, c_1 \geq 0$;

(d) $j^0(\mathbf{x}, \xi_1; \xi_2 - \xi_1) + j^0(\mathbf{x}, \xi_2; \xi_1 - \xi_2) \leq \beta \parallel \xi_1 - \xi_2 \parallel_{\mathbb{R}^s}^2$ for a.e. $\mathbf{x} \in \Gamma$, all $\xi_1, \xi_2 \in \mathbb{R}^s$ with $\beta \geq 0$;

(e) $j^0(\mathbf{x}, \xi; -\xi) \leq d(1 + \parallel \xi \parallel_{\mathbb{R}^s})$ for all $\xi \in \mathbb{R}^s$ a.e. $\mathbf{x} \in \Gamma$ with $d \geq 0$.

$H(f)$: $f \in \mathbf{V}^*$.

$H(s)$: $\beta \parallel \gamma \parallel^2 < m_A$.

Note that Problem(P) is governed by a set of constraints $\mathbf{K}$. Therefore, it is useful to approximate it by a penalty method. Here we introduce the following generalized penalty problem.

Problem(P_n). Find an element $\mathbf{u_n} \in \mathbf{V}$ such that

$$\begin{aligned}\langle A\mathbf{u_n}, \mathbf{v} - \mathbf{u_n}\rangle + \frac{1}{\lambda_n}\langle P_n\mathbf{u}_n, \mathbf{v} - \mathbf{u}_n\rangle \quad &+ \int_\Gamma \varphi(\gamma\mathbf{v}) - \varphi(\gamma\mathbf{u_n}) d\Gamma \\ &+ \int_\Gamma j^0(\gamma\mathbf{u}_n; \gamma\mathbf{v} - \gamma\mathbf{u}_n) d\Gamma \geq \langle f, \mathbf{v} - \mathbf{u}_n\rangle \quad \forall \mathbf{v} \in \mathbf{V}.\end{aligned} \tag{3.1}$$

For the study, we introduce the following assumptions.

$H(\lambda_n)$:

(a) $\lambda_n > 0$ for all $n \in \mathbb{N}$;

(b) $\lambda_n \to 0$ as $n \to \infty$.

$H(P_n)$: $P_n: \mathbf{V} \to \mathbf{V}^*$ is bounded, demicontinuous, monotone and coercive for all $n \in \mathbb{N}$.

(H_1): For each $\mathbf{v} \in \mathbf{K}$, there exists a sequence $\{\mathbf{v}_n\} \subset \mathbf{V}$ such that $P_n\mathbf{v}_n = 0_{\mathbf{V}^*}$ for each $n \in \mathbb{N}$ and $\mathbf{v}_n \to \mathbf{v} \in \mathbf{V}$ as $n \to \infty$.

(H_2): There exists an operator $P: \mathbf{V} \to \mathbf{V}^*$ such that
(a) for any sequence $\{\mathbf{u}_n\}$ satisfying $\mathbf{u}_n \rightharpoonup \mathbf{u} \in \mathbf{V}$ and $\limsup\langle P_n \mathbf{u}_n, \mathbf{u}_n - \mathbf{u}\rangle$ we have

$$\liminf_{n\to\infty}\langle P_n \mathbf{u}_n, \mathbf{u}_n - \mathbf{v}\rangle \geq \langle P\mathbf{u}, \mathbf{u} - \mathbf{v}\rangle \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}.$$

(b) $P\mathbf{u} = 0_{\mathbf{V}^*}$ if and only if $\mathbf{u} \in \mathbf{K}$.
(H_3): For each sequences $\{\mathbf{u}_n\}, \{\mathbf{v}_n\}$ satisfying $\mathbf{u}_n \rightharpoonup u \in \mathbf{V}, \mathbf{v}_n \to \mathbf{v} \in \mathbf{V}$, then
(a) $\limsup_{n\to\infty}\big(\varphi(\gamma\mathbf{v}_n) - \varphi(\gamma\mathbf{u}_n)\big) \leq \varphi(\gamma\mathbf{v}) - \varphi(\gamma\mathbf{u})$;
(b) $\limsup_{n\to\infty} j^0(\gamma\mathbf{u}_n; \gamma\mathbf{v}_n - \gamma\mathbf{u}_n) \leq j^0(\gamma\mathbf{u}; \gamma\mathbf{v} - \gamma\mathbf{u})$.

Theorem 1. *Assume $H(A), H(\varphi), H(j), H(f), H(s), H(\lambda_n)(a)$ and $H(P_n)$. Then Problem (P_n) has a unique solution $\mathbf{u} \in \mathbf{V}$.*
Proof:. Let $n \in \mathbb{N}$. Assumptions $H(\lambda_n), H(P_n)$ and Proposition 1 (a) imply that the operator $\frac{1}{\lambda_n} P_n: \mathbf{V} \to \mathbf{V}^*$ is pseudomonotone. Assumption $H(A)$ on the operator A and Proposition 1 (b) shows that the operator $A_n: \mathbf{V} \to \mathbf{V}^*$ defined by $A_n = A + \frac{1}{\lambda_n} P_n$ is pseudomonotone, too. From that P_n is monotone and $\lambda_n > 0$, using assumption $H(A)$ we deduce that A_n is strongly monotone with constant m_A. We can conclude from above that the operator A_n satisfies condition $H(A)$, too. Similar to the, we can prove that Problem (P_n) has a unique solution u_n. □

Theorem 2. *Assume $H(A), H(\varphi), H(j), H(f), H(s), H(\lambda_n), H(P_n), (H_1), (H_2)$ and (H_3). Then Problem(P) has a solution.*
Proof. We prove this theorem in the following several steps.
Step1. The sequence $\{\mathbf{u}_n\}$ is bounded. Let $\mathbf{v} \in \mathbf{K}$. Assume that $n \in \mathbb{N}$ is fixed, then assumptions $H(j)(c), H(j)(d), (H_1)$ and Proposition 2 guarantee that

$$\begin{aligned}
&\int_\Gamma j^0(\gamma\mathbf{u}_n; \gamma\mathbf{v}_n - \gamma\mathbf{u}_n) d\Gamma \\
\leq\ &\int_\Gamma \big(j^0(\gamma\mathbf{u}_n; \gamma\mathbf{v}_n - \gamma\mathbf{u}_n) + j^0(\gamma\mathbf{v}_n; \gamma\mathbf{u}_n - \gamma\mathbf{v}_n)\big) d\Gamma - \int_\Gamma j^0(\gamma\mathbf{v}_n; \gamma\mathbf{u}_n - \gamma\mathbf{v}_n) d\Gamma \\
\leq\ &\beta \int_\Gamma \| \gamma\mathbf{u}_n - \gamma\mathbf{v}_n \|_{\mathbb{R}^s}^2 d\Gamma + \int_\Gamma |\max\langle \zeta_n; \gamma\mathbf{u}_n - \gamma\mathbf{v}_n\rangle| d\Gamma,
\end{aligned}$$

where $\zeta_n \in \partial j(\gamma\mathbf{v}_n)$. Therefore,

$$\int_\Gamma j^0(\gamma\mathbf{u}_n; \gamma\mathbf{v}_n - \gamma\mathbf{u}_n) d\Gamma \leq \beta \| \gamma \|^2 \| \mathbf{u}_n - \mathbf{v}_n \|_{\mathbf{V}}^2 + (c_0 m(\Gamma) + c_1 \| \gamma \|) \| \gamma \| \| \mathbf{u}_n - \mathbf{v}_n \|_{\mathbf{V}}. \tag{3.2}$$

On the other hand, we use $H(f)(a)$ and $H(\varphi)(a)$ to see that

$$\int_\Gamma \varphi(\gamma\mathbf{v}_n) - \varphi(\gamma\mathbf{u}_n) d\Gamma \leq L_\varphi \| \gamma \| \| \mathbf{u}_n - \mathbf{v}_n \|_{\mathbf{V}}. \tag{3.3}$$

Next, we test with $\mathbf{v} = \mathbf{u}_n \in \mathbf{V}$ in (3.1.) and take into account the fact that $P_n \mathbf{v}_n = 0_{\mathbf{V}^*}$ and hypotheses $H(A), (H_1)$ and $H(P_n)$ to see that

$$\begin{aligned}
m_A \| \mathbf{u}_n - \mathbf{v}_n \|_{\mathbf{V}}^2 &\leq \langle A\mathbf{v}_n, \mathbf{v}_n - \mathbf{u}_n\rangle + \frac{1}{\lambda_n}\langle P_n\mathbf{u}_n - P_n\mathbf{v}_n, \mathbf{v}_n - \mathbf{u}_n\rangle \\
&\quad + \int_\Gamma \varphi(\gamma\mathbf{v}_n) - \varphi(\gamma\mathbf{u}_n) d\Gamma + \int_\Gamma j^0(\gamma\mathbf{u}_n; \gamma\mathbf{v}_n - \gamma\mathbf{u}_n) d\Gamma + \langle f, \mathbf{u}_n - \mathbf{v}_n\rangle \\
&\leq \langle f - A\mathbf{v}_n, \mathbf{u}_n - \mathbf{v}_n\rangle + \int_\Gamma \varphi(\gamma\mathbf{v}_n) - \varphi(\gamma\mathbf{u}_n) d\Gamma + \int_\Gamma j^0(\gamma\mathbf{u}_n; \gamma\mathbf{v}_n - \gamma\mathbf{u}_n) d\Gamma.
\end{aligned}$$

By virtue of (3.2) and (3.3), we find that

$$m_A \parallel \mathbf{u}_n - \mathbf{v}_n \parallel_{\mathbf{V}}^2 \quad \leq \parallel A\mathbf{v}_n - f \parallel_{\mathbf{V}^*} \parallel \mathbf{u}_n - \mathbf{v}_n \parallel_{\mathbf{V}} + L_\varphi \parallel \gamma \parallel \parallel \mathbf{u}_n - \mathbf{v}_n \parallel_{\mathbf{V}}$$
$$+\beta \parallel \gamma \parallel^2 \parallel \mathbf{u}_n - \mathbf{v}_n \parallel_{\mathbf{V}}^2 + (c_0 m(\Gamma) + c_1 \parallel \gamma \parallel) \parallel \gamma \parallel \parallel \mathbf{u}_n - \mathbf{v}_n \parallel_{\mathbf{V}}.$$

Therefore,

$$(m_A - \beta \parallel \gamma \parallel^2) \parallel \mathbf{u}_n - \mathbf{v}_n \parallel_{\mathbf{V}} \leq \parallel A\mathbf{v}_n - f \parallel_{\mathbf{V}^*} + L_\varphi \parallel \gamma \parallel + (c_0 m(\Gamma) + c_1 \parallel \gamma \parallel) \parallel \gamma \parallel.$$

We use (H_1) to see that $\{\mathbf{v}_n\}$ is bounded. Combine $H(A)(a), H(f)$ and $H(s)$, we know that there exists a positive constant C such that $\parallel \mathbf{u}_n - \mathbf{v}_n \parallel < C$, i.e., $\{\mathbf{u}_n\}$ is bounded in $\mathbf{V}$. Since $\mathbf{V}$ is reflexive Banach space, then there exists an element $\tilde{\mathbf{u}} \in \mathbf{V}$ such that, the subsequence of $\{\mathbf{u}_n\}$, we also presented by $\{\mathbf{u}_n\}$, converges weakly to $\tilde{\mathbf{u}}$. i.e.,

$$\mathbf{u}_n \rightharpoonup \tilde{\mathbf{u}} \in \mathbf{V}. \tag{3.4}$$

Step 2. $\tilde{\mathbf{u}} \in \mathbf{K}$. We use (3.1) to see that

$$\frac{1}{\lambda_n}\langle P_n\mathbf{u}_n, \mathbf{u}_n - \mathbf{v}\rangle \quad \leq \langle A\mathbf{u}_n - A\mathbf{v}, \mathbf{v} - \mathbf{u}_n\rangle + \langle A\mathbf{v} - f, \mathbf{v} - \mathbf{u}_n\rangle$$
$$+ \int_\Gamma \varphi(\gamma\mathbf{v}) - \varphi(\gamma\mathbf{u}_n)d\Gamma + \int_\Gamma j^0(\gamma\mathbf{u}_n; \gamma\mathbf{v} - \gamma\mathbf{u}_n)d\Gamma.$$

From $H(A)$, we know that A is monotone, then

$$\frac{1}{\lambda_n}\langle P_n\mathbf{u}_n, \mathbf{u}_n - \mathbf{v}\rangle \quad \leq \parallel A\mathbf{v} - f \parallel_{\mathbf{V}^*} \parallel \mathbf{u}_n - \mathbf{v} \parallel_{\mathbf{V}} + L_\varphi \parallel \gamma \parallel \parallel \mathbf{u}_n - \mathbf{v} \parallel_{\mathbf{V}}$$
$$+(c_0 m(\Gamma) + c_1 \parallel \gamma \parallel) \parallel \gamma \parallel \parallel \mathbf{u}_n - \mathbf{v} \parallel_{\mathbf{V}}.$$

We now use the boundedness of $\{\mathbf{u}_n\}$ to see that, there exists a positive constant $C_1(\mathbf{v})$ depend on $\mathbf{v}$ such that

$$\frac{1}{\lambda_n}\langle P_n\mathbf{u}_n, \mathbf{u}_n - \mathbf{v}\rangle \leq C_1(\mathbf{v}).$$

We use $H(\lambda_n)$ to see that

$$\limsup_{n\to\infty}\langle P_n\mathbf{u}_n, \mathbf{u}_n - \mathbf{v}\rangle \leq 0. \tag{3.5}$$

Let $\mathbf{v} = \tilde{\mathbf{u}} \in \mathbf{V}$ in (3.5), we get

$$\limsup_{n\to\infty}\langle P_n\mathbf{u}_n, \mathbf{u}_n - \tilde{\mathbf{u}}\rangle \leq 0. \tag{3.6}$$

Then, using (3.4), (3.6) and $(H_2)(a)$, we have

$$\liminf_{n\to\infty}\langle P_n\mathbf{u}_n, \mathbf{u}_n - \mathbf{v}\rangle \geq \langle P\tilde{\mathbf{u}}, \tilde{\mathbf{u}} - \mathbf{v}\rangle \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}. \tag{3.7}$$

The inequalities (3.5) and (3.7) together imply that $\langle P\tilde{\mathbf{u}}, \tilde{\mathbf{u}} - \mathbf{v}\rangle \leq 0 \quad \forall \mathbf{v} \in \mathbf{V}$. Let $\mathbf{v} = \tilde{\mathbf{u}} - \mathbf{w}, \forall \mathbf{w} \in \mathbf{V}$, we have $\langle P\tilde{\mathbf{u}}, \mathbf{w}\rangle = 0, \mathbf{w} \in \mathbf{V}$. Thus, $P\tilde{\mathbf{u}} = 0$, and by $(H_2)(b), \tilde{\mathbf{u}} \in \mathbf{K}$.

Step3. $\tilde{\mathbf{u}} \in \mathbf{K}$ is a solution to Problem(P). Let $\mathbf{v}$ be a given element in $\mathbf{V}$. From (H_1), we have that for each $\mathbf{v} \in \mathbf{K}$, there exists a sequence $\{\mathbf{v}_n\} \subset \mathbf{V}$ such that $P_n\mathbf{v}_n = 0_{\mathbf{V}^*}$ for each $n \in \mathbb{N}$. Then, let $\mathbf{v} = \mathbf{v}_n$ in (3.1), we have

$$\langle A\mathbf{u}_n, \mathbf{u}_n - \mathbf{v}_n\rangle \quad \leq \frac{1}{\lambda_n}\langle P_n\mathbf{u}_n - P_n\mathbf{v}_n, \mathbf{v}_n - \mathbf{u}_n\rangle + \int_\Gamma \varphi(\gamma\mathbf{v}_n) - \varphi(\gamma\mathbf{u}_n)d\Gamma$$
$$+ \int_\Gamma j^0(\gamma\mathbf{u}_n; \gamma\mathbf{v}_n - \gamma\mathbf{u}_n)d\Gamma + \langle f, \mathbf{u}_n - \mathbf{v}_n\rangle.$$

Using $H(P_n)$, we infer that

$$\langle A\mathbf{u}_n,\mathbf{u}_n-\mathbf{v}_n\rangle \quad \le \langle f,\mathbf{u}_n-\mathbf{v}_n\rangle+\int_\Gamma \varphi\,(\gamma\mathbf{v}_n)-\varphi(\gamma\mathbf{u}_n)d\Gamma$$
$$+\int_\Gamma j^0\,(\gamma\mathbf{u}_n;\gamma\mathbf{v}_n-\gamma\mathbf{u}_n)d\Gamma. \tag{3.8}$$

Then, using (34) and assumptions $(H_2)(a),(H_3)$ we find that

$$\limsup_{n\to\infty}\int_\Gamma \varphi\,(\gamma\mathbf{v}_n)-\varphi(\gamma\mathbf{u}_n)d\Gamma \quad \le \int_\Gamma \limsup_{n\to\infty}\big(\varphi(\gamma\mathbf{v}_n)-\varphi(\gamma\mathbf{u}_n)\big)d\Gamma$$
$$\le \int_\Gamma \varphi\,(\gamma\mathbf{v})-\varphi(\gamma\tilde{\mathbf{u}})d\Gamma. \tag{3.9}$$

$$\limsup_{n\to\infty}\int_\Gamma j^0\,(\gamma\mathbf{u}_n;\gamma\mathbf{v}_n-\gamma\mathbf{u}_n)d\Gamma \quad \le \int_\Gamma \limsup_{n\to\infty} j^0(\gamma\mathbf{u}_n;\gamma\mathbf{v}_n-\gamma\mathbf{u}_n)d\Gamma$$
$$\le \int_\Gamma j^0\,(\gamma\tilde{\mathbf{u}};\gamma\mathbf{v}-\gamma\tilde{\mathbf{u}})d\Gamma. \tag{3.10}$$

Moreover,

$$\limsup_{n\to\infty}\langle f,\mathbf{u}_n-\mathbf{v}_n\rangle=\langle f,\tilde{\mathbf{u}}-\mathbf{v}\rangle. \tag{3.11}$$

From (3.8)-(3.11), we have that

$$\limsup_{n\to\infty}\langle A\mathbf{u}_n,\mathbf{u}_n-\mathbf{v}_n\rangle \quad \le \int_\Gamma \varphi\,(\gamma\mathbf{v})-\varphi(\gamma\tilde{\mathbf{u}})d\Gamma+\int_\Gamma j^0\,(\gamma\tilde{\mathbf{u}};\gamma\mathbf{v}-\gamma\tilde{\mathbf{u}})d\Gamma+\langle f,\tilde{\mathbf{u}}-\mathbf{v}\rangle. \tag{3.12}$$

Next, $H(A)$, (H_1),(3.4)and imply that $\langle A\mathbf{u}_n,\mathbf{v}-\mathbf{u}_n\rangle\to 0$, as $n\to\infty$. Hence, writing

$$\langle A\mathbf{u}_n,\mathbf{u}_n-\mathbf{v}_n\rangle=\langle A\mathbf{u}_n,\mathbf{u}_n-\mathbf{v}\rangle+\langle A\mathbf{u}_n,\mathbf{v}-\mathbf{u}_n\rangle.$$

We deduce that

$$\limsup_{n\to\infty}\langle A\mathbf{u}_n,\mathbf{u}_n-\mathbf{v}_n\rangle=\limsup_{n\to\infty}\langle A\mathbf{u}_n,\mathbf{u}_n-\mathbf{v}\rangle.$$

This inequality combine with inequality (3.12) yields

$$\limsup_{n\to\infty}\langle A\mathbf{u}_n,\mathbf{u}_n-\mathbf{v}\rangle \quad \le \int_\Gamma \varphi\,(\gamma\mathbf{v})-\varphi(\gamma\tilde{\mathbf{u}})d\Gamma+\int_\Gamma j^0\,(\gamma\tilde{\mathbf{u}};\gamma\mathbf{v}-\gamma\tilde{\mathbf{u}})d\Gamma+\langle f,\tilde{\mathbf{u}}-\mathbf{v}\rangle. \tag{3.13}$$

for all $\mathbf{v}\in\mathbf{K}$. Now, choosing $\mathbf{v}=\tilde{\mathbf{u}}\in\mathbf{K}$ in (3.13) and using Proposition 2 we obtain that

$$\limsup_{n\to\infty}\langle A\mathbf{u}_n,\mathbf{u}_n-\tilde{\mathbf{u}}\rangle\le 0.$$

This inequality together with (3.4) and Definition 1 lead

$$\liminf_{n\to\infty}\langle A\mathbf{u}_n,\mathbf{u}_n-\mathbf{v}\rangle\ge\langle A\tilde{\mathbf{u}},\tilde{\mathbf{u}}-\mathbf{v}\rangle. \tag{3.14}$$

for all $\mathbf{v}\in\mathbf{V}$. We use (3.13) and (3.14) to see that

$$\langle A\tilde{\mathbf{u}},\tilde{\mathbf{u}}-\mathbf{v}\rangle \quad \le \int_\Gamma \varphi\,(\gamma\mathbf{v})-\varphi(\gamma\tilde{\mathbf{u}})d\Gamma+\int_\Gamma j^0\,(\gamma\tilde{\mathbf{u}};\gamma\mathbf{v}-\gamma\tilde{\mathbf{u}})d\Gamma+\langle f,\tilde{\mathbf{u}}-\mathbf{v}\rangle.$$

for all $\mathbf{v}\in\mathbf{K}$. This means $\tilde{\mathbf{u}}$ is a solution to Problem(P), i.e., $\tilde{\mathbf{u}}=\mathbf{u}$. □

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