All the Solutions of the Diophantine Equation $p^3 + q^y = z^3$ with Distinct Odd Primes $p, q$ when $y > 3$

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Abstract. In this paper, we consider the equation $p^3 + q^y = z^3$ in which $p, q$ assume distinct odd primes and $z$ is a positive integer. Then, for all possible integers $y > 3$, the equation $p^3 + q^y = z^3$ has no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In this paper, we consider the equation $p^3 + q^y = z^3$ in which $p, q$ are distinct odd primes and $z$ is a positive integer. The value $y$ is a positive integer. We now provide a short survey of the equation $p^3 + q^y = z^3$ when $y = 1, 2$ and $3$.

When $y = 1$, we have shown [4] that the equation $p^3 + q = z^3$ has infinitely many solutions. The first four solutions of the equation when $p, q$ are primes and $y = 1$ are:

$$3^3 + 37 = 4^3, \quad 11^3 + 397 = 12^3,$$
$$13^3 + 547 = 14^3, \quad 17^3 + 919 = 18^3.$$ 

When $y = 2$, we have established [5] that the equation $p^3 + q^2 = z^3$ has exactly four solutions. These are:

$$7^3 + 13^2 = (2^3)^3, \quad 7^3 + (7^2)^2 = (2 \cdot 7)^3,$$
$$7^3 + (3 \cdot 7^2)^2 = (2^2 \cdot 7)^3, \quad 7^3 + (3 \cdot 7^2 \cdot 13)^2 = (2 \cdot 7 \cdot 11)^3.$$
Quite surprisingly in all the above solutions of \( p^3 + q^2 = z^3 \), we have \( p = 7 \), where only in the first solution \( q \) is prime.

Let \( y = 3 \). In 1637, Fermat (1601 – 1665) stated that the Diophantine equation \( x^n + y^n = z^n \), with integral \( n \geq 2 \), has no solutions in positive integers \( x, y, z \). This is known as Fermat’s “Last Theorem”. In 1995, 358 years later, the validity of the Theorem was established and published by A. Wiles. Thus, the equation \( p^3 + q^3 = z^3 \) has no solutions in positive integers \( p, q, z \).

Therefore, in this paper we consider possible values \( y \) which satisfy \( y > 3 \). This is done in the following Section 2.

2. All the solutions of \( p^3 + q^3 = z^3 \) when \( p, q \) are distinct odd primes, and \( y > 3 \)

In this section, we show that the equation \( p^3 + q^3 = z^3 \) with distinct odd primes, \( p, q \) and \( y > 3 \) has no solutions.

**Theorem 2.1.** Suppose that \( p, q \) are distinct odd primes. Let \( z \) be a positive integer. Then for all possible values \( y > 3 \) the equation \( p^3 + q^3 = z^3 \) has no solutions.

**Proof:** We shall assume that for some value \( y > 3 \), the equation \( p^3 + q^3 = z^3 \) has a solution and reach a contradiction.

From (1) it follows that
\[
z - p = q^3, \quad z^2 + zp + p^2 = q^8, \quad A < B, \quad A + B = y
\]
where \( A, B \) are non-negative integers, and all conditions in (2) must be satisfied simultaneously.

Let \( A \geq 1 \). Then \( B \geq 3 \) and \( y \geq 4 \). We have from (2) that \( z = p + q^4 \), and hence
\[
z^2 + zp + p^2 = (p + q^4)^2 + (p + q^4)p + p^2 = 3p^2 + 3p \cdot q^4 + (q^4)^2 = q^8.
\]
It then follows from (3) that \( q \mid 3p^2 \). Hence, either \( q = 3 \) or \( q \mid p^2 \) which is impossible. When \( q = 3 \), we have from (3)
\[
3p^2 + 3p \cdot 3^4 + 3^{2A} = 3^a \quad \text{or after simplification}
\]
\[
p^2 + p \cdot 3^4 + 3^{2A-1} = 3^{b-1}.
\]
Since \( p \neq 3 \) (\( p, q \) are distinct), it follows that (4) is impossible. Thus \( A \geq 1 \).

Let \( A = 0 \). Then \( y = B \geq 4 \). From (2) and (3) we obtain
\[
z = p + 1, \quad z^2 + zp + p^2 = 3p^2 + 3p + 1 = q^4, \quad y \geq 4.
\]
Denote \( q^4 - (3p^2 + 3p) = t \). We will now show that \( t \neq 1 \).

In Table 1, the primes \( p, q \) are distinct odd primes and \( y \geq 4 \). For each prime \( p \), the prime \( q \) and the value \( y \) are chosen in such a way that they ensure the smallest possible value \( t \) where \( q^4 - (3p^2 + 3p) = t \).
All the Solutions of the Diophantine Equation \( p^3 + q^y = z^3 \) with Distinct Odd Primes \( p, q \) when \( y > 3 \)

**Table 1.**

<table>
<thead>
<tr>
<th>( p )</th>
<th>( 3p^2 + 3p )</th>
<th>( q )</th>
<th>( y )</th>
<th>( q^y )</th>
<th>( q^y - (3p^2 + 3p) = t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>36</td>
<td>5</td>
<td>4</td>
<td>625</td>
<td>589</td>
</tr>
<tr>
<td>5</td>
<td>90</td>
<td>3</td>
<td>5</td>
<td>243</td>
<td>153</td>
</tr>
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<td>7</td>
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<td>3</td>
<td>5</td>
<td>243</td>
<td>75</td>
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<tr>
<td>11</td>
<td>396</td>
<td>5</td>
<td>4</td>
<td>625</td>
<td>229</td>
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<tr>
<td>13</td>
<td>546</td>
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<td>4</td>
<td>625</td>
<td>79</td>
</tr>
<tr>
<td>17</td>
<td>918</td>
<td>3</td>
<td>7</td>
<td>2187</td>
<td>1269</td>
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<tr>
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<td>1140</td>
<td>3</td>
<td>7</td>
<td>2187</td>
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<td>6561</td>
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<td>5676</td>
<td>3</td>
<td>8</td>
<td>6561</td>
<td>885</td>
</tr>
</tbody>
</table>

As a consequence of the data presented in Table 1, unequivocally, it then follows that the value \( t \) is not equal to 1. In Table 1 we have considered the first thirteen consecutive primes \( p \), and accordingly for each \( p \), the respective prime \( q \) and value \( y \) as mentioned earlier. For all these primes \( p \), the number \( t \) satisfies \( t \geq 75 \). If \( D \) denotes the number of digits of the number \( t \), then for all numbers \( t \) we have that \( D \geq 2 \). Observing that no value \( t \) even has one digit \( (D = 1) \), not to say the least of all values \( D = 1 \), namely \( t = 1 \), it follows that \( t = 1 \) is never attained.

We can therefore state that the equation \( p^3 + q^y = z^3 \) has no solutions.

This concludes the proof of Theorem 2.1. □

**Final remark.** In this paper, we have provided a short concise summary on the equation \( p^3 + q^y = z^3 \) when \( p, q \) are distinct odd primes and \( y = 1, 2, 3 \). Some solutions were exhibited when \( y = 1 \) and \( y = 2 \). We have also established closure to the above equation for all possible values \( y > 3 \) when \( p, q \) are distinct odd primes.

We note that to the best of our knowledge, other authors have not considered equations of the kind \( p^3 + q^y = z^3 \). It is therefore obvious, that no references concerning such equations can be provided.

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