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A Deterministic Mathematical Model for the Control of Spread of *Prosopis Juliflora* Plants

***Joel Simon*[1*], *Silas Mirau*[1] and *Livingstone S Luboobi*[2]**

[1]Department of Applied Mathematics and Computational Science
The Nelson Mandela African Institution of Science and Technology
P.O.BOX 447, Arusha, Tanzania. Email silas.mirau@nm-aist.ac
[2]Institute of Mathematical Sciences, Strathmore University
59857-00200, Nairobi, Kenya. Email luboobi@gmail.com
[*]Corresponding author, E-mail: sjoel698@gmail.com

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Abstract. *Prosopis juliflora* plants are the most aggressive invasive species in the world. They spread by animal movement crossing from one place land to another. In this paper a deterministic model to examine the dynamics of *Prosopis julifrola* plants is formulated and presented by adopting a similar approach of a dynamical system as used in epidemiological modeling. The local and global stability analyses of the equilibrium points of the model performed by using next-generation for the basic reproduction number R_0 computation and Lypunov function method. The finding from the study showed that the *Prosopis* free equilibrium of the model is both locally and globally asymptotically stable if and only if the number of secondary infections, is less than unit, that is $R_0 < 1$. Furthermore, the study showed that there exist Prosopis endemic equilibrium for the spread when $R_0 > 1$. The numerical simulation implemented in MATLAB ODE45 algorithm for solving linear ordinary differential equations. The study findings showed that as the number of ingested animals increase, the plant spread increases on land. Based on the findings, the study recommend the application of the model on endemic areas to improve through: Awareness on animal feeding the plant, provision of insight on plant invasion to policy makers and environmental stakeholders to include in environment framework, seminars and environment clubs by visiting community groups an educating them on plant invasion, through this the plant eradication could be achieved.

Keywords: *Prosopis juliflora* plants, mathematical model, Equilibria, Stability, Numerical simulation

AMS Mathematics Subject Classification (2010): 93A30, 00A71

1. Introduction

Prosopis juliflora plants are invasive species in the world originated in Latin America and introduced to Tanzania in 1980's from Taveta Count in Kenya [13]. The plant has invaded Uganda, Kenya and Ethiopia in 1970's. The plant is spread by grazing animals through their movement from one place of land to another as a major agency. However, people, rainfall, wind and floods can spread the plants [1, 2,] . In Tanzania, the areas which have been affected by plants are: Mwanza, Arusha, Kilimanjaro, Dodoma, Tanga and Morogoro.With regards to challenges brought by the plant on environment, it displaces native vegetation and absorption of water sources hard accessibility to economic activities such as: Agricultural, fishing, pastoralism and disaster to biodiversity [3, 13].These challenges brought the need to magnify the efforts to be done upon investigating the magnitude of plant dynamics. However, there are some initiatives that has been taking places to the invaded places on reducing and prevention measure through using chemical and physical means by Center of Agriculture Bioscience International (CABI) to single centers projects in Tanzania, Kenya and Ethiopia. South Africa published National Management Strategies against this invasive plant. Despite of strategies based on statistical and social context, there is no deterministic model that has been employed to this challenge. Most studies [4, 5, 6] have been done on addressing the problem on life science and the dynamics of invasive species basing on statistics and optimization context while others studies [7-9]. [8], developed a mathematical model for estimating the rate of *Prosopis juliflora* plants expansion in the Far region in Ethiopia by using Least square method and secondary data for twenty years. Through this fact, this study decided to use a deterministic model for the control the spread of the plant.

The study aims at presenting the new method of deterministic model by using details of variables and parameters, assumptions and model formulation in section 2. In section three concentrate on calculation of basic of Reproduction number, *Prosopis* free equilibrium, local and global stability analyses and sensitivity. Section four is where results and discussions, numerical analysis of the model is done and finally we conclude with recommendations.

2. Materials and methods

The study involved model formulation in the first phase and model analysis in the second phase. Moreover, the assessment of various parameters on the plant spread was conducted. A mathematical model considered the interactions between land and animal populations in the spread of *Prosopis julifrola* plant. The spread process is denoted by three compartments for the land and two for animal population. In this case, we considered the approach which is similar to dynamical system as applied in epidemiological modeling and ecological modeling. The grazing animals are considered as the main contributor to the spread of the plant to unaffected pieces of land.

2.1. Model formulation

In this study, we consider a method similar to that used in the studies of [10-12]. The significance in formulating a mathematical model of a given real life situation in our daily life is very useful in giving great understanding of the situation. For the dynamics of *Prosopis juliflora* plant. The model is formulated by using the two populations particularly land and animals. The land involves three compartments, which are: susceptible land (S_L), invaded land (I_L) and the reclaimed land (R_L); whereas for the animals were susceptible animals (S_A) and infestation animals (I_A).The susceptible land compartment (S_L) refer to the piece of land which is not yet invaded by the plant. The invaded land, compartment is the number of acres of land which has the seeds of plant that have already germinated and need to be restored; Reclaimed land compartment (R_L) are portions of land in acres that has been invaded by the plant but is successfully being recovered by physical removal, application of chemicals, harvesting, application of leaves sprays to the plant and limitations on the plant growing excessively. As regards to animal population there are susceptible animals (S_A) which are cattle grazing in the invaded land and so are prone in ingesting the plant seeds on the land. The infected animal (I_A), are cattle that have already ingested seeds of the plant and they are the ones that will drop dung containing seeds on land.

1.3. Model assumptions

In our model formulation, we make the following assumptions**:**

- **i.** The plant seeds are majorly spread to susceptible land portions by grazing animals;
- ii. Animal mixing is homogeneous in the considered ecosystem;
- iii. The occurrence of natural calamities can take place at any compartment of land population;
- iv. Recruitment of animals into the system is only through susceptible animal compartment by birth;
- v. The land restoration takes place only on the invaded land;
- vi. The land is claimed from susceptible acres of land.

In Table 1 and Table 2 the variables and parameters used in the model are respectively defined.

Table 1: The five variables for both land and animal populations

Variable	Description
S_L	Number of acres of land which is not yet invaded
I_L	Number of acres of land that has already germinated and need to be restored
R_L	Acres of land that has been invaded by the plant but is successfully recovered by various measures
S_A	Number of animals not yet infested with plant seed
I_A	Number of animals that have ingested plant seeds

1.4. Flow diagram chart

Basing on the dynamics of *Prosopis juliflora* plant in animal population and land according to the assumptions made, the flow diagram for the interactions between the plant, the land and the animals is as in Figure 2.

$\beta_2 I_L S_A$ describes the spread rate of *Prosopis juliflora* plant after interaction between infested land and susceptible animals $\beta_1 I_A S_L$ describes the spread rate of *Prosopis juliflora* plant after interaction between susceptible land and infected animals.

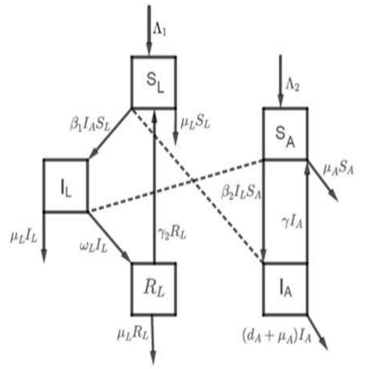


Figure 1: The flow diagram for the spread of *Prosopis juliflora* plant on piece of land by animals that ingest the plant seeds.

Table 2: Parameters and its Description

Parameter	Description
Λ_1	Reclamation rate into the susceptible acres of land portions population
Λ_2	Per capita birth rate of animal population
μ_A	Per capita natural death rate of animal population
d_A	Per capita death rate of animal population induced by ingested seed
μ_L	Land portion used for constructions project and occurrence of natural calamities
ω_L	Rate of progression from invaded land to the reclaimed land
γ	Rate of recovery or restoration of infested animal into susceptible animal
γ_2	Rate of recovery or restoration of invaded land population into susceptible land

Model equations. Through compartmental consideration, as in Figure 1 depicts, we formulated the basic mathematical model which demonstrates the dynamics for *Prosopis juliflora* plant using the following differential equations:

$$\begin{cases} \frac{dS_L}{dt} = \Lambda_1 + \gamma_2 R_L - \beta_1 I_A S_L - \mu_L S_L \\ \frac{dI_L}{dt} = \beta_1 I_A S_L - \mu_L I_L - \omega_L I_L \\ \frac{dR_L}{dt} = \omega_L I_L - \gamma_2 R_L - \mu_L R_L \\ \frac{dS_A}{dt} = \Lambda_2 + \gamma I_A - \beta_2 I_L S_A - \mu_A S_A \\ \frac{dI_A}{dt} = \beta_2 I_L S_A - \gamma I_A - (d_A + \mu_A) I_A \end{cases} \quad (1)$$

with initial conditions:

$$S_L(0) > 0, I_L(0) \geq 0, R_L(0) \geq 0,$$

$$S_A(0) > 0, I_A(0) \geq 0$$

The total size of land is given by: $N_L = S_L + I_L + R_L$ while total number of animal is given by: $N_A = S_A + I_A$

3. Basic properties of the model

Under this section, we checked if our model is mathematically and epidemiologically well posed. Furthermore, we are check the existence of boundedness of the solution.

3.1. Non negativity of solutions

We consider equations for S_L, I_L, R_L for land;

Working with the reclaimed land: R_L

Recalling derivative for reclaimed land from System of differential equation (1) as shown below:

Thus: $\frac{dR_L}{dt} = \omega_L I_L - \gamma_2 R_L - \mu_L R_L$

$$\frac{dR_L}{dt} = \omega_L I_L - (\gamma_2 + \mu_L) R_L$$

From: $I_L(t) \geq 0$

$$\frac{dR_L}{dt} \geq -(\gamma_2 + \mu_L) R_L$$

$$\frac{dR_L}{R_L} \geq -(\gamma_2 + \mu_L) dt$$

$$\int_{R_{L(0)}}^{R_{L(t)}} \frac{dR_L}{R_L} \geq -\int_0^t (\gamma_2 + \mu_L) dt$$

$$\ln\left(\frac{R_L(t)}{R_0(0)}\right) \geq -(\gamma_2 + \mu_L) t$$

$$\left(\frac{R_L(t)}{R_0(0)}\right) \geq e^{-(\gamma_2 + \mu_L)t}$$

$$R_L(t) \geq R_0(0) e^{-(\gamma_2 + \mu_L)t} \geq 0$$

Thus, $R_L \geq 0$

For Susceptible land;

$$\frac{dS_L}{dt} = \Lambda_1 + \gamma_2 R_L - \beta_1 I_L S_L - \mu_L S_L$$

$$\frac{dS_L}{dt} > -\beta_1 I_A S_L - \mu_L S_L$$

$$\frac{dS_L}{dt} > -(\beta_1 I_A + \mu_L) S_L$$

$$\frac{dS_L}{S_L} > -(\beta_1 I_A + \mu_L) dt$$

$$\int_{S_L(0)}^{S_L(t)} \frac{dS_L}{S_L} > -\int (\beta_1 I_A + \mu_L) dt$$

$$\ln\left(\frac{S_L(t)}{S_L(0)}\right) > -(\beta_1 I_A + \mu_L) t$$

$$\frac{S_L(t)}{S_L(0)} > e^{-(\beta_1 I_A + \mu_L)t}$$

$$S_L(t) > S_L(0) e^{-(\beta_1 I_A + \mu_L)t} > 0$$

$$S_L(t) > 0$$

Through applying the same technique, we get:

$S_L(t), I_L(t), R_L(t), S_A(t), I_A(t) \geq 0$

3.2. Invariant region

Invariant region is the one which shows the boundedness of the solution. To determine the region, the animal and land population we considered separately.

For land:

$N_L = S_L + I_L + R_L$

Differentiating size of land compartment, we get:

$$\frac{dN_L}{dt} = \frac{d(S_L + I_L + R_L)}{dt} = \Lambda_1 + \gamma_2 R_L - \beta_1 I_L S_L - \mu_L S_L \qquad (2)$$

$$\frac{dN_L}{dt} = \frac{d(S_L + I_L + R_L)}{dt} = \Lambda_1 + \gamma_2 R_L - \beta_1 I_L S_L - \mu_L S_L$$

$$\frac{dN_L}{dt} = \frac{dS_L}{dt} + \frac{dI_L}{dt} + \frac{dR_L}{dt} = \Lambda_1 - \mu_L S_L - \mu_L I_L - \mu_L R_L$$

$$\frac{d(S_L + I_L + R_L)}{dt} = \Lambda_1 - (S_L + I_L + R_L)\mu_L$$

$$\frac{dN_L(t)}{dt} = \Lambda_1 - N_L(t)\mu_L$$

$$\frac{dN_L(t)}{dt} + \mu_L N_L(t) = \Lambda_1 \quad (3)$$

Equation (3) is the linear ode. Now, to obtain the solution, integrating factor is applied.

Thus the integrating factor $I = e^{\int \mu_L dt} = e^{\mu_L t}$

Multiplying equation (3) with integrating factor we get;

$$e^{\mu_L t}\frac{dN_L(t)}{dt} + \mu_L N_L(t)e^{\mu_L t} = \Lambda_1 e^{\mu_L t}$$

$$\frac{d}{dt}\left(\mu_L N_L(t)e^{\mu_L t}\right) = \Lambda_1 e^{\mu_L t}$$

$$d\left(\mu_L N_L(t)e^{\mu_L t}\right) = \Lambda_1 e^{\mu_L t}dt \quad (4)$$

Integrating equation (4) in both side we get:

$$\int d\left(\mu_L N_L(t)e^{\mu_L t}\right) = \int \Lambda_1 e^{\mu_L t}dt$$

$$N_L(t) = \frac{\Lambda_1}{\mu_L} + Ce^{-\mu_l t} \quad (5)$$

By computing C at $t = 0$, we get;

$$C = N_L(0) - \frac{\Lambda_1}{\mu_L}$$

By substituting C equation 5 we get:

$$N_L(t) = \frac{\Lambda_1}{\mu_L} + \left(N_L(0) - \frac{\Lambda_1}{\mu_L}\right)e^{-\mu_l t}$$

Now through considering two cases;

$$N_L(0) > \frac{\Lambda_1}{\mu_L}, \ \ or\ N_L(0) < \frac{\Lambda_2}{\mu_L}.$$

Now, the boundedness condition is $N_L(t) \le \max\{N_L(0), \frac{\Lambda_1}{\mu_L}\}$

Applying the same technique to the remaining equation, we get:

$$N_A(t) \le \frac{\Lambda_2}{\mu_A} + \left(N_A(0) - \frac{\Lambda_2}{\mu_A}\right)e^{-\mu_A t}$$

Now through considering two cases:

Thus:

$$N_A(0) > \frac{\Lambda_2}{\mu_A}, \ \ or\ N_A(0) < \frac{\Lambda_2}{\mu_A}.$$

For animal population, the boundedness condition is:

$$N_A(t) \le \max\left\{N_A(0), \frac{\Lambda_2}{N_A}\right\}$$

Therefore the model system (1) is positive invariant in the region:

$$\Omega = \{S_L, I_L, R_L, S_A, I_A\} \in R^5 : N_L(t) \le \max\left\{N_L(0), \frac{\Lambda_1}{\mu_L}\right\}, \quad N_A(t) \le \max\left\{N_A(0), \frac{\Lambda_2}{N_A}\right\}$$

3.3. Model analysis

In this analysis we to find the existence of Prosopis free equilibrium for the Prosopis dynamics and computation of its Reproduction number.

3.3.1. Existence of prosopis free equilibrium (PFE) point

Now PFE is obtained when we set derivatives equal to zero and $I_L = I_A = R_L = 0$ and hence we solve for S_L *and* S_A as follows:

$$\frac{dS_L}{dt} = \frac{dS_A}{dt} = \frac{dR_L}{dt} = \frac{dI_A}{dt} = \frac{dI_L}{dt} = 0$$

$$\begin{cases} \Lambda_1 - \beta_1 S_L I_A - \mu_L S_L + \gamma_2 R_L = 0 \\ \beta_1 S_L I_A - (\mu_L + \omega_L) I_L = 0 \\ \omega_L I_L - (\gamma_2 + \mu_L) R_L = 0 \\ \Lambda_2 - \beta_2 S_A I_L + \gamma I_L - \mu_A S_A = 0 \\ \beta_2 S_A I_L - \gamma I_A - (d_A + \mu_A) I_A = 0 \end{cases}$$

$\Lambda_1 - \mu_L S_L = 0$, that is: $S_L = \dfrac{\Lambda_1}{\mu_L}$

and

$\Lambda_2 - \mu_A S_A = 0$, that is: $S_A = \dfrac{\Lambda_2}{\mu_A}$

Therefore, there exist a *Prosopis* free equilibrium (PFE) $E^0\left(S_L^0, I_L^0, R_L^0, S_L^0, I_L^0\right)$ which is equal to $\left(\dfrac{\Lambda_1}{\mu_L}, 0, 0, \dfrac{\Lambda_2}{\mu_A}, 0\right)$

3.3.2. The prosopis endemic equilibrium point (PEE)

The *Prosopis* endemic equilibrium PEE of the model system (1) E^* is steady solution in which the spread of the plant persists and in the solution $I_L \ne 0, R_L \ne 0, I_A \ne 0.$

Now from equation system (1) all derivatives are set equal to zero.

Let $E^* = \left(S_L^*, I_L^*, R_L^*, S^*_A, I_A^*\right)$ be the equilibrium and performing computations as follows:

From system (1) for ingested animal, we get:

$$\beta_2 S_A I_L - (\gamma + d_A + \mu_A) I_A = 0$$

Thus :

$$I_A^* = \frac{\beta_2 \Lambda_1 \Lambda_2}{\gamma(\mu_L + \omega_L)(\gamma_2 + \mu_L + \omega_L) + (d_A + \mu_A)(\beta_2 \Lambda_1 + \mu_A \mu_L (\mu_L + \omega_L)(\gamma_2 + \mu_L + \omega_L))}$$

From system (1) for infested land, we get:

$$\beta_1 S_L I_A - (\mu_L + \omega_L) I_L = 0 \geq I_L^* = \frac{\Lambda_1}{\mu_L((\mu_L + \omega_L)(\gamma_2 + \mu_L + \omega_L))}$$

From system (1) for claimed land, we get: $\omega_L I_L - (\gamma_2 + \mu_L) R_L = 0 \geq R_L^* = \dfrac{\omega_L I_L^*}{\gamma_2 + \mu_L}$

From system (1) for susceptible land, we get:

$$\Lambda_1 - \beta_1 S_L^* I_A^* - \mu_L S_L^* + \gamma_2 \left(\frac{\omega_L I_L^*}{\gamma_2 + \mu_L} \right) = 0 \geq S_L^* = \frac{\Lambda_1 (\gamma_2 + \mu_L) + \gamma_2 \omega_L I_L^*}{(\beta_1 I_A^* - \mu_L)(\gamma_2 + \mu_L)}$$

From system (1) for susceptible animal, we get:

$$\Lambda_2 + \gamma I_A^* - \beta_2 S_A^* I_L^* - \mu_A S_A^* = 0$$

Thus:

$$S_A^* = \frac{\Lambda_2 + \gamma I_A^*}{\beta_2 I_L^* + \mu_A}$$

The values of $(S_L^*, I_L^*, R_L^*, S^*_A, I_A^*)$, as shown above, are positive. Thus there is an exist of *Prosopis* endemic equilibrium point.

3.4. Reproduction number of prosopis dynamics

The basic reproduction number is the average of secondary spread of seed plant by single seed plant when introduced in an entirely susceptible land population. We used approach in [14-16] .It determine whether the spread persists or clear out. The spread clears out when $R_0 < 1$ and persist when

$R_0 > 1$. In computing basic reproduction number, we adopt the method similar to that in epidemiological studies. In this case the next generation matrix method in which new spread and transfer functions are considered. If the new spread is mathematically defined by f_i and transfer terms by v_i , then the matrices $\mathbf{F}$ and $\mathbf{V}$ are given by $\mathbf{F} = \dfrac{\partial f_i}{\partial I_i}(X)$ and $\mathbf{V} = \dfrac{\partial v_i}{\partial I_j}$ as defined by [17].

By consideration of infested subpopulation which are:

$$\begin{cases} \dfrac{dI_L}{dt} = \beta_1 S_L I_A - (\mu_L + \omega_L) I_L \\ \\ \dfrac{dI_A}{dt} = \beta_2 S_A I_L - (\gamma + d_A + \mu_A) I_A \end{cases} \qquad (11)$$

Now we find matrix $\mathbf{F}$ and matrix $\mathbf{V}$ where by $\mathbf{F}$ contains force of infestation and $\mathbf{V}$, is the remaining terms.

Let X = (I_L, I_A), $f_1 = \beta_1 S_L I_A$, $f_2 = \beta_2 S_A I_L$, $V_1 = (\mu_L + \omega_L) I_L$, $V_2 = (\gamma + d_A + \mu_A) I_A$

$$f(X) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$f(X) = \begin{pmatrix} \beta_1 S_L I_A \\ \beta_2 S_A I_L \end{pmatrix} \text{ and } \mathbf{V}(X) = \begin{pmatrix} (\mu_L + \omega_L) I_L \\ (\gamma + d_A + \mu_A) I_A \end{pmatrix}$$

By computing partial derivatives on system equation (11) for $\mathbf{F}$ and $\mathbf{V}$ we get:

$$\mathbf{F} = \begin{pmatrix} \dfrac{\partial f_1}{\partial I_L} & \dfrac{\partial f_1}{\partial I_A} \\ \dfrac{\partial f_2}{\partial I_L} & \dfrac{\partial f_2}{\partial I_A} \end{pmatrix} = \begin{pmatrix} 0 & \beta_1 S_L \\ \beta_2 S_A & 0 \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} \dfrac{\partial V_1}{\partial I_L} & \dfrac{\partial V_1}{\partial I_A} \\ \dfrac{\partial V_2}{\partial I_L} & \dfrac{\partial V_2}{\partial I_A} \end{pmatrix} = \begin{pmatrix} \mu_L + \omega_L & 0 \\ 0 & \gamma + d_A + \mu_A \end{pmatrix}$$

Through evaluation of $\mathbf{F}$ and $\mathbf{V}$ at PFE we get:

$$\mathbf{F} = \begin{pmatrix} 0 & \beta_1 S^0{}_L \\ \beta_2 S^0{}_A & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{\beta_1 \Lambda_1}{\mu_L} \\ \dfrac{\beta_2 \Lambda_1}{\mu_A} & 0 \end{pmatrix}$$

$$\mathbf{V}|_{PFE} = \begin{pmatrix} \mu_L + \omega_L & 0 \\ 0 & \gamma + d_A + \mu_A \end{pmatrix}$$

Finding the inverse of $\mathbf{V}$ we get:

$$\mathbf{V}^{-1} = \frac{1}{(\mu_L+\omega_L)(\gamma+d_A+\mu_A)}\begin{pmatrix} \gamma+d_A+\mu_A & 0 \\ 0 & \mu_L+\omega_L \end{pmatrix}$$

Then we compute $\mathbf{FV}^{-1}$ we get:

$$\mathbf{FV}^{-1} = \begin{pmatrix} 0 & \frac{\beta_1\Lambda_1}{\mu_L} \\ \frac{\beta_2\Lambda_1}{\mu_A} & 0 \end{pmatrix}\begin{pmatrix} \frac{\gamma+d_A+\mu_A}{(\mu_L+\omega_L)(\gamma+d_A+\mu_A)} & 0 \\ 0 & \frac{\mu_L+\omega_L}{(\mu_L+\omega_L)(\gamma+d_A+\mu_A)} \end{pmatrix}$$

$$\mathbf{FV}^{-1} = \begin{pmatrix} 0 & \frac{\beta_1\Lambda_1}{\mu_L(\gamma+d_A+\mu_A)} \\ \frac{\beta_2\Lambda_2}{\mu_A(\mu_L+\omega_L)} & 0 \end{pmatrix}$$

We compute the eigenvalues using: $|\mathbf{FV}^{-1}-\lambda|=0$

$$= \begin{pmatrix} -\lambda & \frac{\beta_1\Lambda_1}{\mu_L(\gamma+d_A+\mu_A)} \\ \frac{\beta_2\Lambda_2}{\mu_A(\mu_L+\omega_L)} & -\lambda \end{pmatrix}$$

$$\lambda^2 = \frac{\beta_1\beta_2\Lambda_1\Lambda_2}{\mu_A\mu_L(\mu_L+\omega_L)(\gamma+d_A+\mu_A)}$$

$$\lambda = \pm\sqrt{\frac{\beta_1\beta_2\Lambda_1\Lambda_2}{\mu_A\mu_L(\mu_L+\omega_L)(\gamma+d_A+\mu_A)}}$$

In this case the reproduction number is given by spectral radius $\rho\left(\mathbf{FV}^{-1}\right)$:

Which gives:

$$R_0 = \rho\left(\mathbf{FV}^{-1}\right) = \sqrt{\frac{\beta_1\beta_2\Lambda_1\Lambda_2}{\mu_A\mu_L(\mu_L+\omega_L)(\gamma+d_A+\mu_A)}} \qquad (12)$$

3.5. Local stability of *Prosopis* free equilibrium point (PFE)

In order to get local stability of (PFE), we have to show that the variation matrix $J(E_0)$ of our model system (1) has negative eigenvalues by computing the differentiation of the system The *Prosopis* endemic equilibrium PEE of the model system (1) E^* is steady solution in which the spread of the plant persists and in the solution $I_L \neq 0, R_L \neq 0, I_A \neq 0$.

Now from equation system (1) all derivatives are set equal to zero.

Let $E^* = \left(S_L^*, I_L^*, R_L^*, S^*_A, I_A^*\right)$ be the equilibrium and performing computations as follows:

From system (1) for ingested animal, we get:

$$\beta_2 S_A I_L - (\gamma + d_A + \mu_A) I_A = 0$$

Thus :

$$I_A{}^* = \frac{\beta_2 \Lambda_1 \Lambda_2}{\gamma(\mu_L + \omega_L)(\gamma_2 + \mu_L + \omega_L) + (d_A + \mu_A)\left(\beta_2 \Lambda_1 + \mu_A \mu_L (\mu_L + \omega_L)(\gamma_2 + \mu_L + \omega_L)\right)}$$

From system (1) for infested land, we get:

$$\beta_1 S_L I_A - (\mu_L + \omega_L) I_L = 0 \geq I_L{}^* = \frac{\Lambda_1}{\mu_L\left((\mu_L + \omega_L)(\gamma_2 + \mu_L + \omega_L)\right)}$$

From system (1) for claimed land, we get: $\omega_L I_L - (\gamma_2 + \mu_L) R_L = 0 \geq R_L{}^* = \dfrac{\omega_L I_L^*}{\gamma_2 + \mu_L}$

From system (1) for susceptible land, we get:

$$\Lambda_1 - \beta_1 S_L{}^* I_A{}^* - \mu_L S_L{}^* + \gamma_2 \left(\frac{\omega_L I_L{}^*}{\gamma_2 + \mu_L}\right) = 0 \geq S_L{}^* = \frac{\Lambda_1(\gamma_2 + \mu_L) + \gamma_2 \omega_L I_L^*}{(\beta_1 I_A^* - \mu_L)(\gamma_2 + \mu_L)}$$

From system (1) for susceptible animal, we get:

$$\Lambda_2 + \gamma I_A{}^* - \beta_2 S_A{}^* I_L{}^* - \mu_A S_A{}^* = 0$$

Thus:

$$S_A{}^* = \frac{\Lambda_2 + \gamma I_A^*}{\beta_2 I_L^* + \mu_A}$$

The values of $\left(S_L{}^*, I_L{}^*, R_L{}^*, S^*{}_A, I_A{}^*\right)$, as shown above, are positive. Thus there is an exist of *Prosopis* endemic equilibrium point with respect to $(S_L, I_L, R_L, S_A, I_A)$ at the *Prosopis* free equilibrium which gives:

$$J(E_0) =$$

$$\begin{pmatrix} -(\beta_1 I_L + \mu_L) & -\beta_1 S_L & \gamma_2 & 0 & 0 \\ \beta_1 S_L & -(\mu_L + \omega_L) & 0 & 0 & 0 \\ 0 & \omega_L & -(\gamma_2 + \mu_L) & 0 & 0 \\ 0 & -(\beta_2 S_A) & 0 & -(\beta_2 I_L + \mu_L) & \gamma \\ 0 & \beta_1 S_A & 0 & \beta_1 I_L & -(\gamma + d_A + \mu_A) \end{pmatrix}$$

Evaluating Jacobian at *Prosopis* free equilibrium point we get:

$$J|_{PFE}(1)=\begin{pmatrix} -\mu_L & -\dfrac{\beta_1\Lambda_1}{\mu_L} & \gamma_2 & 0 & 0 \\ 0 & -(\mu_L+\omega_L) & 0 & 0 & 0 \\ 0 & \omega_L & -(\gamma_2+\mu_L) & 0 & 0 \\ 0 & -\dfrac{\beta_2\Lambda_2}{\mu_A} & 0 & -\mu_A & \gamma \\ 0 & \dfrac{\beta_1\Lambda_2}{\mu_A} & 0 & 0 & -(\gamma+d_A+\mu_A) \end{pmatrix}$$

eigenvalues of Jacobian are $\lambda_1=-\mu_l<0, \lambda_2=-\mu_A<0$;

Also in the second evaluation we get:

$$J|_{PFE}(2)=\begin{pmatrix} (\mu_L+\omega_L) & 0 & 0 \\ \omega_L & -(\gamma_2+\mu_L)\mu_L & 0 \\ \dfrac{\beta_1\Lambda_2}{\mu_A} & 0 & (\gamma_2+d_A+\mu_A) \end{pmatrix}$$

which implies $\lambda_3=-(\mu_L+\omega_L)<0, \lambda_4=-(\gamma_2+\mu_L)<0$ and $\lambda_5=-(\gamma+d_A+\mu_A)<0$.

The *Prosopis* free equilibrium for each population is locally asymptotically stable if and only if the number of secondary infections, is less than unit, that is R_0 < 1 and this is fact because we got negative eigenvalues.

3.6. Global stability of *Prosopis* free equilibrium point (PFE)

The Lyapunov Method and LaSalle's Invariance Principle have been generally used to break down security of self-governing frameworks of differential conditions. Korobeinikov] [10, 18] proposed unequivocal Lyapunov work which was utilized to break down SEIR and SEIS scourge model [19], developed a logarithmic Lyapunov capacity to examine Lotka-Volterra frameworks furthermore, later this capacity was applied by Korobeinikov [20] to dissect endemic equilibrium for SIR, SIRS and SIS pestilence models. Vargas-De-Leon [21] set forward the composite quadratic Lyaponuv capacity to dissect solidness of endemic equilibrium for SIR, SIRS and SIS scourge models and later developed a composite-Volterra capacity to break down endemic equilibrium for the model with backslide. In this work we adopt Lyapunov work. Under this part we study the global behavior of the *Prosopis* endemic equilibrium, E^* for the model system (1).

Theorem 1. The endemic equilibrium point for the *Prosopis* model System (1) is asymptotically Ω if R_0 >1 stable

Proof: We constructed an explicitly Lyapunov function for the model system (1) using [10, 18]. Approaches as it is useful to the most of the Sophisticated Compartmental epidemiological models. In this approach, we construct Lyapunov function of the form:

$$L = \sum a_i \left(X_i - X_i^* ln X_i \right)$$

where a_i is a properly selected positive constant, X_i is the population of the i^{th} is the equilibrium level. We define the Lyapunov function candidate V for the model System (1) as:

$$L = \sum_{i=1}^{5} a_i \left(X_i - X_i^* In\, X_i \right)$$

L= $a_1\left(S_L - S_L^* lnS_L\right) + a_2\left(I_L - I_L^* \ln I_L\right) + a_3\left(R_L - R_L^* \ln R_L\right) + a_4\left(S_A - S_A^* lnS_A\right) +$ $a_5\left(I_A - I_A^* lnS_A\right)$ where $a_1, a_2, \dots, a_6$ are positive constant. The time derivative of the Lyapunov function L is given by:

$$\frac{\partial L}{\partial t} = a_1 \frac{\partial V}{\partial S_L}.\frac{\partial S_L}{\partial t} + a_2 \frac{\partial V}{\partial I_L}.\frac{\partial I_L}{\partial t} + a_3 \frac{\partial V}{\partial R_L}.\frac{\partial R_L}{\partial t} + a_4 \frac{\partial V}{\partial S_A}.\frac{\partial S_A}{\partial t} + a_5 \frac{\partial V}{\partial I_A}.\frac{\partial I_A}{\partial t}$$

$$\frac{\partial L}{\partial t} = a_1\left(1 - \frac{S_L^*}{S_L}\right)\left(\Lambda_1 + \gamma_2 R_L - \beta_1 I_A S_L - \mu_L S_L\right) + a_2\left(1 - \frac{I_L^*}{I_L}\right)\left(\beta_1 I_A S_L - \mu_L I_L - \omega_L I_L\right) +$$

$$a_3\left(1 - \frac{R_L^*}{R_L}\right)\left(\omega_L I_L - \gamma_2 R_L - \mu_L R_L\right)$$

$$+ a_4\left(1 - \frac{S_A^*}{S_A}\right)\left(\Lambda_2 + \gamma I_A - \beta_2 I_L S_A - \mu_A S_A\right) + a_5\left(1 - \frac{I_A^*}{I_A}\right)\left(\beta_2 I_L S_A - \gamma I_A - (d_A + \mu_A) I_A\right)$$

At endemic equilibrium point:

$$\frac{\partial L}{\partial t} = a_1\left(1 - \frac{S_L^*}{S_L}\right)\left(\beta_1 I_A^* S_L^* + \mu_L S_L^* - \beta_1 I_A S_L + \mu_L S_L\right)$$

$$+ a_2\left(1 - \frac{I_L^*}{I_L}\right)\left(\mu_L I_L^* + \omega_L I_L^* - \mu_L I_L - \omega_L I_L\right) + a_3\left(1 - \frac{R_L^*}{R_L}\right)\left(\mu_L R_L^* - \mu_L R_L\right)$$

$$+ a_4\left(1 - \frac{S_A^*}{S_A}\right)\left(\beta_2 I_L^* S_A^* + \mu_A S_A^* - \beta_2 I_L S_A + \mu_A S_A\right)$$

$$+ a_5\left(1 - \frac{I_A^*}{I_A}\right)\left(\gamma I_A^* + \left(d_A + \mu_A\right) I_A^* - \gamma I_A - \left(d_A + \mu_A\right) I_A\right)$$

Rearranging the terms, we get:

$$\frac{\partial L}{\partial t}=a_1\left(1-\frac{S_L^*}{S_L}\right)\left[-\beta_1 I_A S_L\left(1-\frac{I_A^* S_L^*}{I_A S_L}\right)-\mu_L S_L\left(1-\frac{S_L^*}{S_L}\right)\right]$$
$$+a_2\left(1-\frac{I_L^*}{I_L}\right)\left[-\mu_L I_L\left(1-\frac{I_L^*}{I_L}\right)-\omega_L I_L\left(1-\frac{I_L^*}{I_L}\right)\right]$$
$$+a_3\left(1-\frac{R_L^*}{R_L}\right)\left[-\mu_L R_L\left(1-\frac{R_L^*}{R_L}\right)\right]+a_4\left(1-\frac{S_A^*}{S_A}\right)\left[-\beta_2 I_L S_A\left(1-\frac{I_L^* S_A^*}{I_L S_A}\right)-\mu_A S_A\left(1-\frac{S_A^*}{S_A}\right)\right]$$
$$+a_5\left(1-\frac{I_A^*}{I_A}\right)\left[-\gamma I_A\left(1-\frac{I_A^*}{I_A}\right)-\left(d_A+\mu_A\right)I_A\left(1-\frac{I_A^*}{I_A}\right)\right] \quad (13)$$

Through simplification of (13) we get:

$$\frac{\partial L}{\partial t}=-a_1\mu_l\left(\frac{\left(S_L-S_L^*\right)^2}{S_L}\right)-a_2\left(\mu_L+\omega_L\right)\left(\frac{\left(I_L-I_L^*\right)^2}{I_L}\right)$$
$$-a_3\mu_L\left(\frac{\left(R_L-R_L^*\right)^2}{R_L}\right)-a_4\mu_A\left(\frac{\left(S_A-S_A^*\right)^2}{S_A}\right)-a_5\left(\gamma+d_A+\mu_A\right)\left(\frac{\left(I_A-I_A^*\right)^2}{I_A}\right)+G(\Omega)$$

where $G(\Omega)=\frac{-a_1\beta_1\left(S_L-S_L^*\right)\left(I_A S_L-I_A^* S_L^*\right)}{S_L}-\frac{a_4\beta_2\left(S_A-S_A^*\right)\left(I_L S_A-I_L^* S_A^*\right)}{S_A}$

The function $G(\Omega)$ is a non-positive, thus $G(\Omega)\le 0$ for all Ω .Therefore $\frac{\partial L}{\partial t}\le 0$ in Ω and is zero when $\Omega=\Omega^*$, since $\frac{\partial L}{\partial t}\le 0$ in Ω is zero when $\Omega=\Omega^*$, which implies that the largest compact set in Ω when $\frac{\partial L}{\partial t}=0$ is singleton $\{\Omega^*\}$ which is the endemic equilibrium. By Lassalle's invariant principle [10, 18], it implies that the endemic equilibrium Ω^* is globally asymptotically stable in the interior of Ω when $R_0>1$. R_0 depends on the interactions of infected land and susceptible animals and ingested animal and susceptible land. $R_0>1$ Occurs when susceptible animals interact with infected land and when ingested animals interacts with the susceptible animals. At this point, endemic equilibrium is stable when $R_0>1$.Therefore, these interactions with birth rate of animals makes the endemic equilibrium to be globally asymptotically stable.

3.7. Sensitivity analysis

Though the use of parameter values in Table 2 and the definition of equation (14) its sensitivity index is calculated as shown in Table 3 [24].

Table 3: Parameters and their descriptions

Parameter	Descriptions	**value**	**Source**
Λ_1	Reclamation rate into the susceptible acres of land portions population	0.95	[12]
Λ_2	Per capita birth rate of animal population	0.902	Estimated
μ_A	Per capita natural death rate of animal population	0.0167	[22]
d_A	Per capita death rate of animal population induced by ingested seed	0.005	[11]
μ_L	Land portion used for constructions project and occurrence of natural calamities	0.00548	Estimated
ω_L	Rate of progression from invaded land to the reclaimed land	0.0035	Estimated
γ	Rate of recovery or restoration of infested animal into susceptible animal	0.00005	[12, 14]
γ_2	Rate of recovery or restoration of invaded land population into susceptible land	0.070	[23]

Under this section, we performed the forward sensitivity analysis of Reproduction number R_0 with respect to its parameters to determine which parameter is sensitive to the invasion of the plant on land. In finding a way of reducing the invasion of *Prosopis juliflora* plant, it is better to understand the proportional importance of factors that are reliable for the spread and eradication on the plant invasion. The forward sensitivity index of parameter μ with respect to basic Reproduction number R_0 is denoted by $\tau_{\mu_i}^{R_0}$ .Using the approach in [10, 14, 18].the normalized forward sensitivity index of a parameter μ_i with respect to Reproduction number R_0 is defined by:

$$\alpha_{\mu_i}^{R_0} = \frac{\partial R_0}{\partial \mu_i} x \frac{\mu_i}{R_0} \quad [24, 25] \tag{14}$$

Using the parameters in Table 2 and definition in equation (14), the sensitivity indices of reproduction number with respect to its parameters are given in Table 3.

Table 2: Sensitivity indices for R_0

Parameter	**Value**	**Sensitivity Index**
Λ_1	0.95	+ 0.5000
Λ_2	0.0902	+ 0.5000
β_1	0.000062	+ 0.5000

β_2	0.000059	+ 0.5000
μ_A	0.00237	-0.6597
μ_L	0.00548	$-$ 0.8051
ω_L	0.0035	$-$ 0.1949
d_{A}	0.00005	$-$ 0.00034
Parameter	Value	Sensitivity Index
γ	0.005	-0.3369

The positive indices show that the basic reproduction number is directly proportional to the values of its parameter. On the other hand, the negatives indices indicate that the basic Reproduction number is inversely proportional to the parameters.

In Table 3: Λ_1, Λ_2, β_1, β_2 have positive indices which means its production number increases as it increases and vice-versa. Under this situation, implies that the reproduction number is directly proportional to birth rate of animals, interaction between ingested animals and Susceptible land, in this case, it shows that as the interaction between ingested animal and susceptible land increases it leads to more spread of the *Prosopis juliflora* plants. However, the negative indexed parameters are: μ_A, μ_L, ω_L, d_{A} and γ. Due to this result, the natural death rate of animals, land other uses rate most sensitive negative parameters of the model, this shows that the secondary infection of land by the plants will decrease as ingested animal decreases.

4. Results and discussion

This section explains more on the findings from the study after being implemented in the MATLAB ODE45 through Numerical simulation as it explained below.

4.1. Numerical simulation

In order to determine which parameter is sensitive to the spread of *Prosopis juliflora* plants done by sensitivity analysis, we simulate the model using the parameters found in Table 3

i. The effects of *Prosopis juliflora* plants dynamics on land

Figure 2 shows the simulation of *Prosopis juliflora* dynamics model which implanted in the MATLAB as the figure itself shows.

This is the dynamics of *Prosopis juliflora* plants which shows that the ingested animals increase from 150 to 240 for 8 months which leads to decreases of susceptible animals particularly from 150 to 135 animals for the mentioned period. Moreover, for the same period an infected land increased from 150 acres of land to 250 acres which resulted into decrease of susceptible land from 150 acres to 90 acres. This indicates that, there is high spread of Prosopis juliflora plants and therefore, control measures and strategies are needed to minimize or prevent the occurring situation.

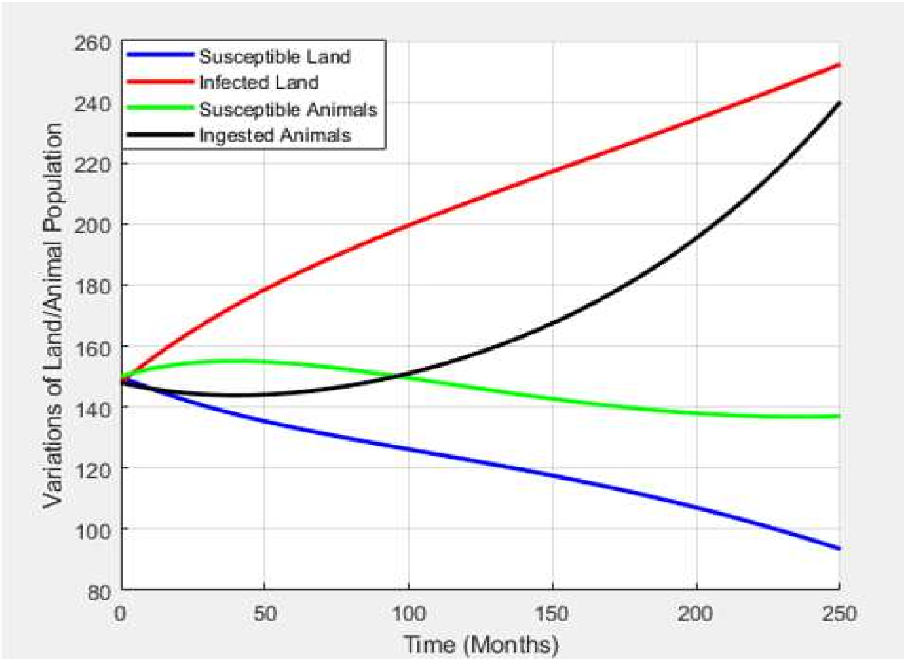
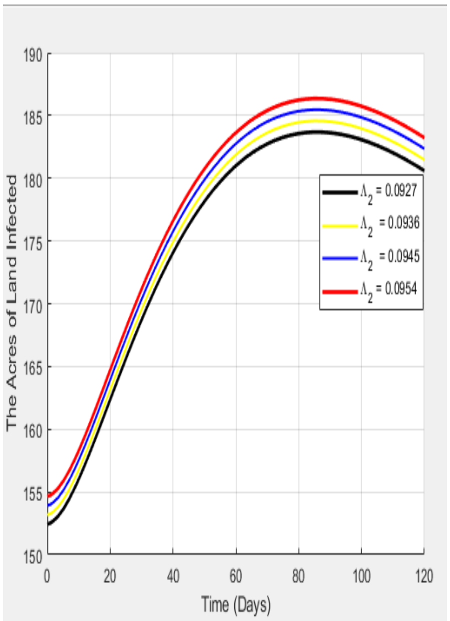


Figure 2: Dynamics of Prosopis juliflora plants on land and animal population

ii. The effects of birth rate of animals in spreading the plant on Land



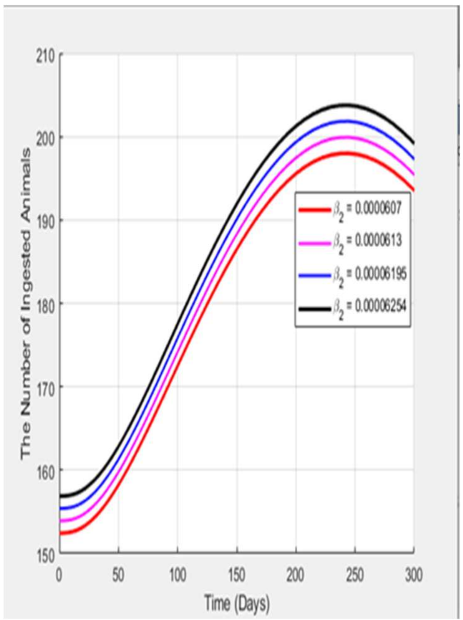


Figure 3: (a) Effects of variation of Λ_2 on land infected land (b) Effects of variation β_2 with

The Figure 3 (a) shows that as per capita birth rate of animals increases it results into increase of infected land as from 152 acres of land to 187 acres within four months. Now this is very serious issue of an invasion of species and therefore there is need of extra efforts towards resolving the problem. On the other hand Figure 3 (b) shows that the interaction force between the susceptible animals and the infected land as it increases it results to an increase of ingested animals.

iii. The effects of ingested animals on susceptible animals and infected land on susceptible land

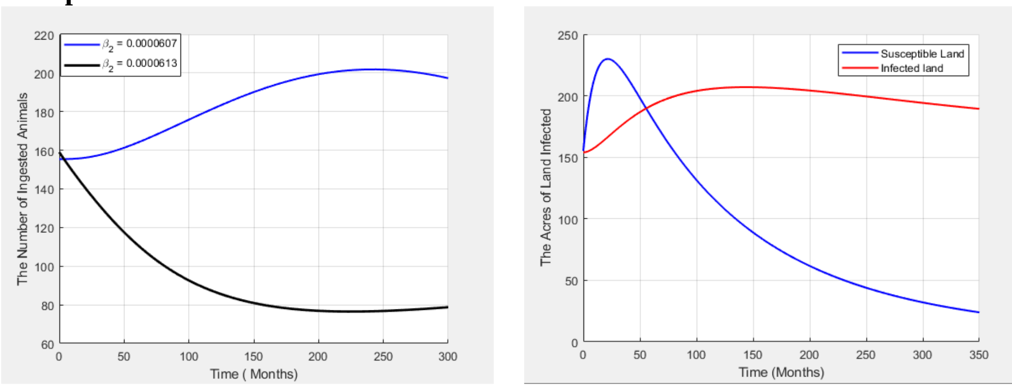


Figure 4: *(a) Effects of Ingested Animals and Susceptible Animals (b) Effects of Infested Land and* Susceptible

Figure 4(a) show that as animals ingested increases it leads to decreases of susceptible animals while Figure 4 (b) shows that as infected land increases it results to deceases of susceptible. In facts both Figures shows that its relationship are inversely propotional.

iv. The effect of ingested Animals on Acres of land

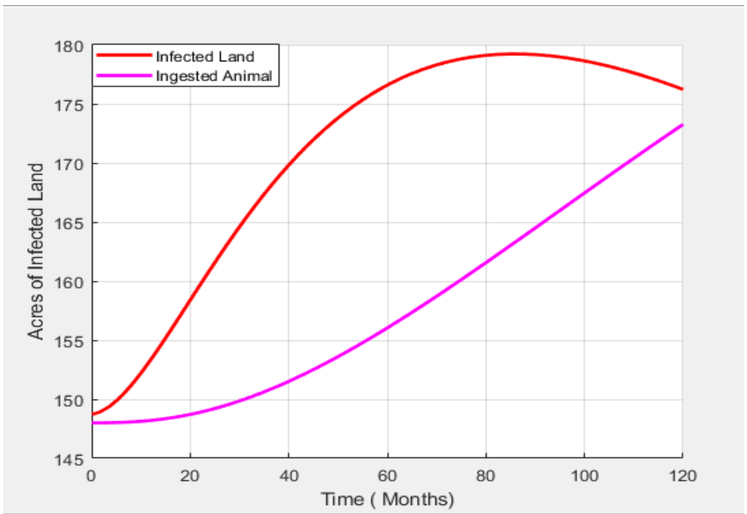


Figure 5: Variation of Ingested Animals on Acres of Land

Figure 5 show that as number of ingested animals increases it results to an increase of acres of infected land particularly from 148 acres to 178 acres which is equal to 20% rate of infection per month. In this case there is a need of extra efforts to rectify the situation on reducing or prevention on plants spread.

v. **Determination impact of each of parameters on the control of the spread of *Prosopis juliflora* plant**

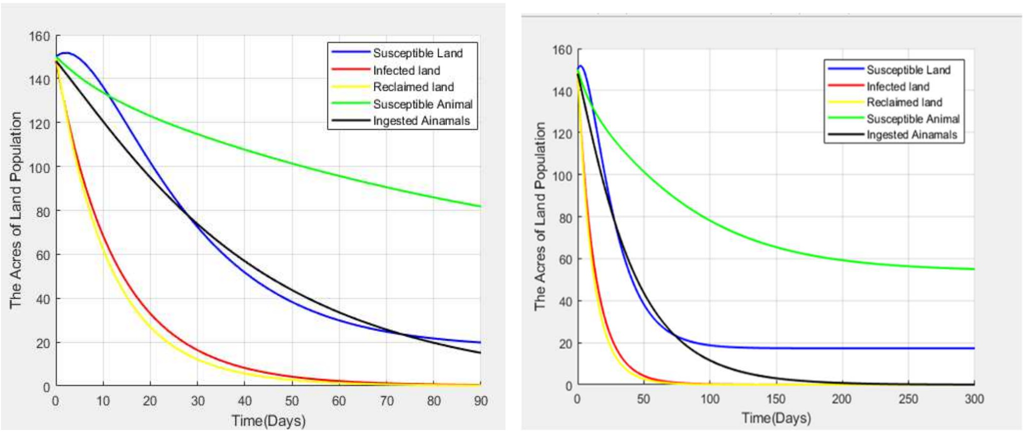


Figure 6: (a) Control of Prosopis juliflora within 1.5 months, (b) Control of Prosopis juliflora within 10 months

Figure 2 show that the spread was increasing before measures taken while Figure 6(a) showed that after applying measure of reducing and eradicating the plant within short time particular one month and half. Through applying the method for long time it managed to prevent a plant from spreading as shown in Figure 6(b) above.Morever this attained by varying the progression rate of invaded land to reclaimed land by 61.43%, ω_L from 0.0035 to 0.035 through certain time as in this case ten months.

5. Conclusion

The study developed a deterministic model for the *Prosopis* juliflora dynamics which gives a better understanding on how plants invasion dynamics affect land and other native species. Moreover, basic reproduction number is computed and sensitivity analysis with their parameters performed as well. The study shows that when $R_0 < 1$ *Prosopis* free equilibrium is locally asymptotically stable. Furthermore the findings showed that the rate of plant spread due to interactions force on infested land and susceptible animals (β_2) and to interactions force on infected animals and susceptible land (β_2) with birth rate of animals on the reclaimed land are the most sensitive parameters on plants spread dynamics. As the interaction either between infested land and susceptible animals or susceptible land and infected animals are proportional to an increase rate of the plant spread. This is well supported by numerical simulation. The study recommends that strategies towards the control and prevention on plant invasion have to be implemented.

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