Journal of Mathematics and Informatics
Vol. 17, 2019, 131-144
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 19 November 2019
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.151av17a12

Journal of **Mathematics and Informatics**

(λ,μ)-Fuzzy Ideal of (λ,μ)-Fuzzy Subring

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Received 1 November 2019; accepted 5 November 2019

Abstract: (λ,μ)-fuzzy ideal of a ring is a very general fuzzy algebraic structure since it includes many well-known fuzzy ideals as its special cases. In this paper, by replacing a ring with a (λ,μ)-fuzzy subring, we introduce the notion of (λ,μ)-fuzzy ideal of a (λ,μ)-fuzzy subring. We prove that the proposed notion has nice characterizations and properties. Particular, it is verified that our (λ,μ)-fuzzy ideal are preserved under homomorphic image and preimage, and are closed under many familiar algebraic operations such as intersection, sum and product.

Keywords: fuzzy algebra, (λ,μ)-fuzzy subring, (λ,μ)-fuzzy ideal, algebraic operation, homomorphism

AMS Mathematics Subject Classification (2010): 08A72

1. Introduction

The theory of fuzzy sets, proposed by Zadeh [1], has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill defined to admit precise mathematical analysis by classical methods and tools. There are many extensions of fuzzy sets. For example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets, bipolar fuzzy sets, cubic sets etc. The fuzzification of algebraic structure was introduced by Rosenfeld [2] and the notion of fuzzy group was proposed in 1971. The concept of ideal is one of the most important concepts in rings or semigroups. Liu [3] defined and studied fuzzy subrings as well as fuzzy ideals in rings. In 1983, Liu gave some operations of intersection, sum, product and quotient of fuzzy ideals, and then introduced a quasi-ideal between general fuzzy ideal and distinct ideal, and proved a series of properties of the operations of fuzzy ideals [4]. Kuroki [5,6,7] studied various fuzzy ideals in semigroups. Swamy and Swamy [8] studied fuzzy prime ideals in rings in 1988. Kumbhojkar established the correspondence theorems for fuzzy ideals [9]. In general, the study of fuzzy algebra always takes the classical algebraic system as the domain to study its fuzzy subset. This greatly limits the development of fuzzy algebra. In order to change this unfavorable situation, a fundamental method is to define fuzzy groups and fuzzy rings by defining fuzzy operations. Demirci gave the concept of smooth group [10]. Yuan [11] gave a new binary fuzzy operation and introduced the definition of fuzzy group based on the fuzzy binary operation, and obtained some characteristic properties.

With the generalization of concepts of fuzzy subgroup and fuzzy subring, Bhakart redefined fuzzy subgroup and fuzzy ideal with “belong to” and “quasi-coincident with” relation between a fuzzy point and a fuzzy subset, and introduced the concepts of $(\in,\in\vee q)$-fuzzy subgroup [12], $(\in,\in\vee q)$-fuzzy normal subgroup [13] and $(\in,\in\vee q)$-fuzzy subring [14]. Davvaz introduced the concepts of $(\in,\in\vee q)$-fuzzy subnearrings and ideals [15]. Jun characterized semigroups by $(\in,\in\vee q)$-fuzzy bi-ideals [16]. In this sense, Rosenfeld fuzzy subgroup is special case of $(\in,\in\vee q)$-fuzzy subgroup, which can be called (0,0.5)-fuzzy subgroup or (0,0.5)-fuzzy subring.

The more general concept than$(\in,\in\vee q)$-fuzzy idea is $(\in,\in\vee qk)$-fuzzy ideals. Shabir characterized semigroups by $(\in,\in\vee qk)$-fuzzy ideals [17] and Khan characterized ordered semigroups by $(\in,\in\vee qk)$-fuzzy generalized bi-ideals [18]. To generalize the concepts of fuzzy subgroup and fuzzy ideal further, Yuan introduced the concept of fuzzy subgroup with threshold [19]. Meanwhile, we introduced the concepts of (λ,μ)-fuzzy normal subgroup [20] and (λ,μ)-fuzzy ideal [21], and studied the characterization of (λ,μ)-fuzzy prime ideal [22]. Then we characterized regular semigroups by using (λ,μ)-fuzzy ideal [23], and Li studied some properties of (λ,μ)-fuzzy subgroups systematically [24].

In this paper, we shall focus our aim on (λ,μ)-fuzzy ideal. As we observed in [21], that a (0,1)-fuzzy subring (resp., (0,1)-fuzzy ideal) is precisely a fuzzy subring (resp., fuzzy ideal) in Liu [3], and a (0,0.5)-fuzzy subring (resp., (0,0.5)-fuzzy ideal) is precisely a $(\in,\in\vee q)$-fuzzy subring (resp., $(\in,\in\vee q)$-fuzzy ideal) in Bhakat [14]. So, (λ,μ)-fuzzy ideal presents a fairly extensive framework for fuzzy ideal. In addition, as mentioned earlier, that replaceing the classical domain of algebra system with fuzzy domain is a very popular method in the study of fuzzy algebra. Note that the notion of (λ,μ)-fuzzy ideal was defined on a classical ring R. Inspired by these two observations, in this paper, we will develop a theory of (λ,μ)-fuzzy ideal of a (λ,μ)-fuzzy subring.

The organization of this paper is as follows. In section 2, we briefly recall some notions and conclusions about (λ,μ)-fuzzy subrings and (λ,μ)-fuzzy ideals of a classical ring. In section 3, we will introduce the notion of (λ,μ)-fuzzy ideal of a (λ,μ)-fuzzy subring A of a ring R. We will verify that when $A=\bar{1}$ (where, $\bar{1}$ denotes the constant fuzzy subset of R values 1), the (λ,μ)-fuzzy ideal of A reduces to that of R. This shows that our notion is a reasonable generalization of the corresponding notion of a classical ring. We shall also prove that (λ,μ)-fuzzy ideal of a (λ,μ)-fuzzy subring has nice level-characterizations. In section 4, we will study some basic properties of (λ,μ)-fuzzy ideal of (λ,μ)-fuzzy subring. In particular, it is proved that the considered structures are preserved under homomorphic image and preimage, and are closed under three well-known algebraic operations: intersection, sum, product. Finally, we present some concluding remarks and future research.

2. Preliminaries

In this section, we will recall some notions and notations for later use.

Let $(R,+,\cdot)$ (R for short) be a ring. We assume that:

(1) $\varnothing$ is both a subring and an ideal of R, and $\sup\varnothing=0$;

(2) $x\cdot y$ is simplified as xy for any $x,y\in R$.

A function $A: R \to [0,1]$ is called a fuzzy subset of R. All fuzzy subsets of R are denoted by $\mathcal{F}(R)$. For $A, B \in \mathcal{F}(R)$ and for $\alpha \in [0,1]$, we define:

(1) $\overline{\alpha}$ as the constant value fuzzy subset with $\forall x \in R$, $\overline{\alpha}(x) = \alpha$,

(2) $A \subseteq B$ if and only if $A(x) \le B(x)$ for any $x \in R$,

(3) $A \bigcap B \in \mathcal{F}(R)$ as $(A \bigcap B)(x) = A(x) \wedge B(x)$ for all $x \in R$,

(4) $A \bigcup B \in \mathcal{F}(R)$ as $(A \bigcup B)(x) = A(x) \vee B(x)$ for all $x \in R$,

(5) $A + B \in \mathcal{F}(R)$ as $\forall x \in R$, $(A+B)(x) = \sup\{A(x_1) \wedge B(y_1) \big| x_1 + y_1 = x\}$,

(6) $A \odot B \in \mathcal{F}(R)$ as $\forall x \in R$,

$$(A \odot B)(x) = \sup\{\inf_{1 \le i \le n} (A(y_i) \wedge B(z_i)) \mid x = \sum_{i=1}^{n} y_i z_i, y_i, z_i \in R, 1 \le i \le n, n \in \mathbb{N}\}$$

(7) the α-cut set and strong α-cut of A as respectively

$$A_\alpha = \{x \in R \mid A(x) \ge \alpha\}, A_{(\alpha)} = \{x \in R \mid A(x) \ge \alpha\}.$$

Furthermore, note that

$$A \subseteq B \Leftrightarrow \forall \alpha \in [0,1],\ A_\alpha \subseteq B_\alpha \Leftrightarrow \forall \alpha \in [0,1], A_{(\alpha)} \subseteq B_{(\alpha)}.$$

A function $f: R_1 \to R_2$ between two rings is called a homomorphism of rings if

$$\forall x, y \in R_1, f(x_1 + x_2) = f(x_1) + f(x_2), f(xy) = f(x_1) f(x_2).$$

For $A \in \mathcal{F}(R_1), B \in \mathcal{F}(R_2)$, define $f(A) \in \mathcal{F}(R_2)$ and $f^{-1}(B) \in \mathcal{F}(R_1)$ as

$$\forall y \in R_2, f(A)(y) = \vee\{A(x) \big| f(x) = y\}, \forall x \in R_1, f^{-1}(B)(x) = B(f(x)).$$

Usually, $f(A)$ is called the homomorphic image of A, and $f^{-1}(B)$ called the homomorphic preimage of B, respectively.

In the following, we recall some results about (λ,μ)-fuzzy ideal and (λ,μ)-fuzzy subring from [21], where $0 \le \lambda < \mu \le 1$.

Definition 2.1. ([21]) A fuzzy subset $A \in \mathcal{F}(R)$ is called a (λ,μ)-fuzzy subring of R if, for all $x, y \in R$, it holds that

(F1) $A(x-y) \vee \lambda \ge (A(x) \wedge A(y)) \wedge \mu$, (F2) $A(xy) \vee \lambda \ge (A(x) \wedge A(y)) \wedge \mu$.

Remark 1.1. In [21], it was proved that the condition (F1) is equivalent to the following two conditions:

(F1′)$: A(x+y) \vee \lambda \ge (A(x) \wedge A(y)) \wedge \mu$, (F1″)$: A(-x) \vee \lambda \ge A(x) \wedge \mu$.

Definition 2.2. ([21]) A fuzzy subset $A \in \mathcal{F}(R)$ is called a (λ,μ)-fuzzy ideal of R if for all $x, y \in R$, it holds that

(F1) $A(x-y) \vee \lambda \ge (A(x) \wedge A(y)) \wedge \mu$, (F3) $A(xy) \vee \lambda \ge (A(x) \vee A(y)) \wedge \mu$.

In [21], the author proved that (λ,μ)-fuzzy subrings (resp., (λ,μ)-fuzzy ideals) have good properties. For example, as the following theorems show that (λ,μ)- fuzzy subrings and (λ,μ)-fuzzy ideals have nice $\alpha-$level characterizations, are closed under the operation $\bigcap$,and are preserved under homomorphic image and preimage.

Theorem 2.1. ([21]) A is a (λ,μ)-fuzzy subring (resp., (λ,μ)-fuzzy ideal) of R iff $\forall\alpha\in(\lambda,\mu]$, A_α is a subring (resp., ideal) of R iff $\forall\alpha\in[\lambda,\mu)$, $A_{(\alpha)}$ is a subring (resp., ideal) of R.

Theorem 2.2. ([21]) If A,B are (λ,μ) fuzzy subrings (resp., (λ,μ)-fuzzy ideals) of R, then so is $A\bigcap B$.

Theorem 2.3. ([21]) Let $f:R_1\to R_2$ be a homomorphism of rings.

(1) If A is a (λ,μ)-fuzzy subring of R_1 then $f(A)$ is a (λ,μ)-fuzzy subring of R_2.

(2) If f is onto and A is a (λ,μ)-fuzzy ideal of R_1 then $f(A)$ is a (λ,μ)-fuzzy ideal of R_2.

(3) If B is a (λ,μ)-fuzzy subring (resp., (λ,μ)-fuzzy ideal) of R_2 then $f^{-1}(B)$ is a (λ,μ)-fuzzy subring (resp., (λ,μ)-fuzzy ideal) of R_1.

3. (λ,μ)-fuzzy ideal on (λ,μ)-fuzzy subring : definition and characterizations

As we have seen, the notion of (λ,μ)-fuzzy ideal was defined on a crisp ring R. In this section, we shall extend the corresponding notion from a crisp ring R to a (λ,μ)-fuzzy subring B of R ,that is,we will develop a theory of (λ,μ)-fuzzy ideal of a (λ,μ)-fuzzy subring. We will verify that when $B=\bar{1}$, our notion for B reduces to that for R. Moreo ver, we also shall prove that our notion has nice $\alpha-$level characterizations. All the menti oned results indicate that our notion is a reasonable generalization of crisp ideal.

Definition 3.1. Let $A\in\mathcal{F}(R)$ and B be a (λ,μ)-fuzzy subring of R such that $A\bigcap\bar{\mu}\subseteq B\bigcup\bar{\lambda}$. Then

(1) A is called a (λ,μ)-fuzzy left ideal of B if, for all $x,y\in R$,

(F1) $A(x-y)\vee\lambda\geq A(x)\wedge A(y)\wedge\mu$, (FF2) $A(xy)\vee\lambda\geq A(y)\wedge B(x)\wedge\mu$.

(2) A is called a (λ,μ)-fuzzy right ideal of B if, for all $x,y\in R$,

(F1) $A(x-y)\vee\lambda\geq A(x)\wedge A(y)\wedge\mu$, (FF3) $A(xy)\vee\lambda\geq A(x)\wedge B(y)\wedge\mu$.

(3) A is called a (λ,μ)-fuzzy ideal of B if A is both a (λ,μ)-fuzzy left ideal and a (λ,μ)-fuzzy right ideal of B.

Remark 3.1. When $\lambda=0,\mu=1$, the condition $A\bigcap\bar{\mu}\subseteq B\bigcup\bar{\lambda}$ in Definition 3 reduces to $A \subseteq B$. Hence the mentioned condition can be regarded as an interpretation of that A is a subset of B.

Proposition 3.1. Let $A \in \mathcal{F}(R)$ and B be a (λ,μ)-fuzzy subring of R such that $A \cap \overline{\mu} \subseteq B \cup \overline{\lambda}$. Then A is a (λ,μ)-fuzzy ideal of B iff for all $x, y \in R$,

(F1) $A(x-y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$,

(FF4) $A(xy) \vee \lambda \geq (A(x) \vee A(y)) \wedge (B(x) \wedge B(y)) \wedge \mu$.

Proof: We need only check that (FF2)+(FF3) $\Leftrightarrow$ (FF4).

(FF2)+(FF3) $\Rightarrow$ (FF4). Indeed, for any $x, y \in R$, by (FF2) and (FF3) we get

$$\begin{aligned} A(xy) \vee \lambda &\geq [A(y) \wedge B(x) \wedge \mu] \vee [A(x) \wedge B(y) \wedge \mu] \\ &= \big[(A(y) \vee A(x)) \wedge (A(y) \vee B(y)) \wedge (B(x) \vee A(x)) \wedge (B(x) \vee B(y))\big] \wedge \mu \\ &\geq (A(y) \vee A(x)) \wedge B(y) \wedge B(x) \wedge B(x) \wedge \mu \\ &= (A(x) \vee A(y)) \wedge (B(x) \wedge B(y)) \wedge \mu, \end{aligned}$$

i.e., (FF4) holds.

(FF4) $\Rightarrow$ (FF2). Indeed, for any $x, y \in R$, by (FF4) we get

$$\begin{aligned} A(xy) \vee \lambda &\geq [(A(x) \vee A(y)) \wedge (B(x) \wedge B(y)) \wedge \mu] \vee \lambda \\ &\geq [(A(y) \wedge B(x) \wedge \mu) \wedge B(y)] \vee \lambda \\ &= [(A(y) \wedge B(x) \wedge \mu) \vee \lambda] \wedge [B(y) \vee \lambda], \text{ by } A \cap \overline{\mu} \subseteq B \cup \overline{\lambda} \\ &\geq (A(y) \wedge B(x) \wedge \mu) \wedge (A(y) \wedge \mu) = A(y) \wedge B(x) \wedge \mu. \end{aligned}$$

(FF4) $\Rightarrow$ (FF3). It is similar to (FF4) $\Rightarrow$ (FF2).

Proposition 3.2. Let $A \in \mathcal{F}(R)$. Then the following statements hold.

(1) A is a (λ,μ)-fuzzy ideal of R iff A is a (λ,μ)-fuzzy ideal of $\bar{1}$.

(2) If A is a (λ,μ)-fuzzy subring of R then A is a (λ,μ)-fuzzy ideal of A.

(3) Let A, B, C be (λ,μ)-fuzzy subrings of R such that $A \cap \overline{\mu} \subseteq B \cup \overline{\lambda}$ and $B \cap \overline{\mu} \subseteq C \cup \overline{\lambda}$. If A is a (λ,μ)-fuzzy ideal of C, then A is a (λ,μ)-fuzzy ideal of B.

Proof: (1) At first, note that $B = \bar{1}$ satisfies the condition $A \cap \overline{\mu} \subseteq B \cup \overline{\lambda}$ since

$$\forall x \in R, (A \cap \overline{\mu})(x) \leq 1 = (B \cup \overline{\lambda})(x).$$

At second, we observe easily that (FF4) $\Leftrightarrow$ (F3) when $B = \bar{1}$.

(2) At first, the desired condition $A \cap \overline{\mu} \subseteq A \cup \overline{\lambda}$ holds obviously. At second, note

$$(A(x) \wedge A(y)) \wedge \mu = (A(x) \vee A(y)) \wedge (A(x) \wedge A(y)) \wedge \mu,$$

it follows by (F2) that

$$A(xy) \vee \lambda \geq (A(x) \vee A(y)) \wedge \mu = (A(x) \vee A(y)) \wedge (A(x) \wedge A(y)) \wedge \mu.$$

(3) It has been known that C is a (λ,μ)-fuzzy subring of R such that $A \cap \overline{\mu} \subseteq B \cup \overline{\lambda}$. Since A is (λ,μ)-fuzzy ideal of C, then A satisfies (F1). Therefore, we need only prove that A, B satisfy (FF4). Indeed, for any $x, y \in R$, by A, C satisfying (FF4)

$$A(xy)\vee\lambda \ge [(A(x)\vee A(y))\wedge(C(x)\wedge C(y))\wedge\mu]\vee\lambda$$
$$\ge (A(x)\vee A(y))\wedge\mu\wedge(C(x)\vee\lambda)\wedge(C(y)\vee\lambda)\text{, by } B\cap\overline{\mu}\subseteq C\cup\overline{\lambda}$$
$$\ge (A(x)\vee A(y))\wedge\mu\wedge(B(x)\vee\lambda)\wedge(B(y)\wedge\mu)$$
$$=(A(x)\vee A(y))\wedge(B(x)\wedge B(y))\wedge\mu$$

Remark 3.2. (1) Proposition 3.2 (1) tells us that the notion of (λ,μ)-fuzzy ideal of a (λ,μ)-fuzzy subring is an extension of the corresponding notion of a ring.
(2) From Remark 3.1 we note easily that Proposition 2 (3) is an extension of the crisp result:

$$A\subseteq B\subseteq C \text{ and } A \text{ is a ideal of } C \Rightarrow A \text{ is a ideal of } B.$$

The next two theorems tell us that (λ,μ)-fuzzy ideals have nice $\alpha-$level character izations.

Lemma 3.1. Let $A,B\in\mathcal{F}(R)$.

(1) If A is a (λ,μ)-fuzzy left ideal of B then A is a (λ,μ)-fuzzy subring of R.
(2) If A is a (λ,μ)-fuzzy right ideal of B then A is a (λ,μ)-fuzzy subring of R.
(3) If A is a (λ,μ)-fuzzy ideal of B then A is a (λ,μ)-fuzzy subring of R.

Proof: We only verify (1). The proofs of (2) and (3) are similar to that of (1). Let A be a (λ,μ)-fuzzy subring of B. Then A satisfies (F1) naturally. We check below that A satisfies (F2). In fact, for any $x,y\in R$, by A,B fulfill (FF2) we get

$$A(xy)\vee\lambda \ge (A(y)\wedge B(x)\wedge\mu)\vee\lambda \ge (A(y)\wedge\mu)\wedge(B(x)\vee\lambda)\text{, by } A\cap\overline{\mu}\subseteq B\cup\overline{\lambda}$$
$$\ge (A(y)\wedge\mu)\wedge(A(x)\wedge\mu) = A(x)\wedge A(y)\wedge\mu.$$

Theorem 3.1. Let $A,B\in\mathcal{F}(R)$. Then
(1) A is a (λ,μ)-fuzzy left ideal of B iff A_α is a left ideal of B_α for all $\alpha\in(\lambda,\mu]$.
(2) A is a (λ,μ)-fuzzy right ideal of B iff A_α is a right ideal of B_α for all $\alpha\in(\lambda,\mu]$.
(3) A is a (λ,μ)-fuzzy ideal of B iff A_α is a ideal of B_α for all $\alpha\in(\lambda,\mu]$.
Proof: We only verify (1). The proofs of (2) and (3) are similar to that of (1).
Necessity. Let A be a (λ,μ)-fuzzy left ideal of B. Then from Definition 3.1 and Lemma 3.1 (1) we get that both A and B are (λ,μ)-fuzzy subrings of R. Then it follows by Theorem 2.1 that both A_α and B_α are subrings of R for all $\alpha\in(\lambda,\mu]$.
Further from $A\cap\overline{\mu}\subseteq B\cup\overline{\lambda}$, we get

$$A_\alpha = A_\alpha\cap\overline{\mu}_\alpha = (A\cap\overline{\mu})_\alpha \subseteq (B\cup\overline{\lambda})_\alpha = B_\alpha\cup\overline{\lambda} = B_\alpha,$$

since $\overline{\mu}_\alpha = R$, $\overline{\lambda}_\alpha=\varnothing$. It shows that A_α is a subring of B_α.
Additionally, for all $x\in A_\alpha$, $y\in B_\alpha$, we have $A(x)\ge\alpha$, $B(y)\ge\alpha$, and hence

$$A(yx)\vee\lambda \ge A(x)\wedge B(y)\wedge\mu \ge \alpha\wedge\mu=\alpha.$$

Since $\lambda<\mu$, it follows that $A(yx)\ge\alpha$, i.e., $yx\in A_\alpha$, which means that A_α is left ideal of B_α.

Sufficiency. Let A_α be a left ideal of B_α for all $\alpha\in(\lambda,\mu]$. Then $A_\alpha\subseteq B_\alpha$ and B_α is a subring of R for all $\alpha\in(\lambda,\mu]$. It follows by Theorem 2.1 that B is a (λ,μ)-fuzzy subring of R.

At first, we prove $A\bigcap\overline{\mu}\subseteq B\bigcup\overline{\lambda}$. We need only verify that $(A\bigcap\overline{\mu})_\alpha\subseteq(B\bigcup\overline{\lambda})_\alpha$ for any $\alpha\in[0,1]$.

If $\alpha\in[0,\lambda]$ then $(A\bigcap\overline{\mu})_\alpha\subseteq R=(B\bigcup\overline{\lambda})_\alpha$.

If $\alpha\in(\lambda,\mu]$ then $(A\bigcap\overline{\mu})_\alpha=A_\alpha\subseteq B_\alpha=(B\bigcup\overline{\lambda})_\alpha$.

If $\mu=1$ then the proof has been completed since the above two cases have ensured that α take all the values from 0 to 1.

If $\mu<1$ and $\alpha\in(\mu,1]$ then $(A\bigcap\overline{\mu})_\alpha=\varnothing\subseteq(B\bigcup\overline{\lambda})_\alpha$.

At second, we prove that A,B satisfy (F1). That is, for any $x,y\in R$,

$$A(x-y)\vee\lambda\ge A(x)\wedge A(y)\wedge\mu$$

Take $\alpha=A(x)\wedge A(y)\wedge\mu$, then $A(x),A(y)\ge\alpha$, i.e., $x,y\in A_\alpha$, and $\mu\ge\alpha$. we divide into two cases:

Case 1: $\lambda\ge\alpha$. Then $A(x-y)\vee\lambda\ge A(x)\wedge A(y)\wedge\mu$ holds obviously.

Case 2: $\lambda<\alpha$. Then $\alpha\in(\lambda,\mu]$. Since A_α is a left ideal of B_α then from $x,y\in A_\alpha$ we get $x-y\in A_\alpha$, i.e., $A(x-y)\ge\alpha$. It follows that

$$A(x-y)\vee\lambda\ge\alpha=A(x)\wedge A(y)\wedge\mu.$$

At third, we prove that A,B satisfy (FF2). That is,

$$A(xy)\vee\lambda\ge A(y)\wedge B(x)\wedge\mu,\ \forall x,y\in R.$$

Indeed, if we assume that there exist $x_0,y_0\in R$ such that

$$A(x_0y_0)\vee\lambda\ge A(y_0)\wedge B(x_0)\wedge\mu.$$

Take $\alpha=A(y_0)\wedge B(x_0)\wedge\mu$, then $\lambda<\alpha\le\mu$, $y_0\in A_\alpha$, $x_0\in B_\alpha$ and $x_0y_0\notin A_\alpha$. So A_α is not a left ideal of B_α. This is a contradiction! Hence the desired inequality holds.

From the above three steps we get that A is a (λ,μ)-fuzzy left ideal of B.

Similar to Theorem 3.1, we can describe (λ,μ)-fuzzy ideal by strong α-cut set.

Theorem 3.2. Let $A,B\in\mathcal{F}(R)$. Then

(1) A is a (λ,μ)-fuzzy left ideal of B iff $A_{(\alpha)}$ is a left ideal of $B_{(\alpha)}$ for all $\alpha\in[\lambda,\mu)$.

(2) A is a (λ,μ)-fuzzy right ideal of B iff $A_{(\alpha)}$ is a right ideal of $B_{(\alpha)}$ for all $\alpha\in[\lambda,\mu)$.

(3) A is a (λ,μ)-fuzzy ideal of B iff $A_{(\alpha)}$ is a ideal of $B_{(\alpha)}$ for all $\alpha\in[\lambda,\mu)$.

4. Basic properties

In this section, we shall verify that (λ,μ)-fuzzy ideals of a (λ,μ)-fuzzy subring have nice properties. Precisely, (λ,μ)-fuzzy ideals of a (λ,μ)-fuzzy subring are preserved under homomorphic image and preimage, and are closed under the operations $\bigcap, +, \odot$.

The following two theorems tell us that (λ,μ)-fuzzy ideals of a (λ,μ)-fuzzy subring are preserved under homomorphic image and preimage.

Theorem 4.1. Let $f: R_1 \to R_2$ be a homomorphism of rings and $A, B \in \mathcal{F}(R_1)$.

(1) If A is a (λ,μ)-fuzzy left ideal of B then $f(A)$ is a (λ,μ)-fuzzy left ideal of $f(B)$.

(2) If A is a (λ,μ)-fuzzy right ideal of B then $f(A)$ is a (λ,μ)-fuzzy right ideal of $f(B)$.

(3) If A is a (λ,μ)-fuzzy ideal of B then $f\left(A\right)$ is a (λ,μ)-fuzzy ideal of $f(B)$.

Proof: We only verify (1). The proofs of (2) and (3) are similar to that of (1).

Let A be a (λ,μ)-fuzzy left ideal of B. From Definition 3.1 we have that B is a (λ,μ)-fuzzy subring of R_1 and $A \bigcap \overline{\mu} \subseteq B \bigcup \overline{\lambda}$. It follows from Theorem 2.3 (1) that $f(B)$ is a (λ,μ)-fuzzy subring of R_2. And from $A \bigcap \overline{\mu} \subseteq B \bigcup \overline{\lambda}$ we have

$$f(A) \bigcap \overline{\mu} = f(A \bigcap \overline{\mu}) \subseteq f(B \bigcup \overline{\lambda}) = f(B) \bigcup \overline{\lambda}$$

Furthermore, for any $y_1, y_2 \in R_2$, we get

$$\begin{aligned}
f(A)(y_1 - y_2) \vee \lambda &= \sup\{A(x) \mid f(x) = y_1 - y_2\} \vee \lambda \\
&\geq \sup\{A(x_1 - x_2) \mid f(x_1 - x_2) = y_1 - y_2\} \vee \lambda \\
&\geq \sup\{A(x_1 - x_2) \vee \lambda \mid f(x_1) = y_1, f(x_2) = y_2\} \\
\text{by } A \text{ satisfies (F1)} \quad &\geq \sup\{A(x_1) \wedge A(x_2) \wedge \mu \mid f(x_1) = y_1, f(x_2) = y_2\} \\
&= \sup\left\{A(x_1) \mid f(x_1) = y_1\right\} \wedge \sup\left\{A(x_2) \mid f(x_2) = y_2\right\} \wedge \mu \\
&= f(A)(y_1) \wedge f(A)(y_2) \wedge \mu
\end{aligned}$$

$$\begin{aligned}
f(A)(y_1 y_2) \vee \lambda &= \sup\{A(x) \mid f(x) = y_1 y_2\} \vee \lambda \\
&\geq \sup\{A(x_1 x_2) \mid f(x_1 x_2) = y_1 y_2\} \vee \lambda \\
&\geq \sup\{A(x_1 x_2) \vee \lambda \mid f(x_1) = y_1, f(x_2) = y_2\} \\
\text{by } A, B \text{ satisfy (FF2)} \quad &\geq \sup\{A(x_2) \wedge B(x_1) \wedge \mu \mid f(x_1) = y_1, f(x_2) = y_2\} \\
&= \sup\{A(x_2) \mid f(x_2) = y_2\} \wedge \sup\{B(x_1) \mid f(x_1) = y_1\} \wedge \mu \\
&= f(A)(y_2) \wedge f(B)(y_1) \wedge \mu
\end{aligned}$$

Hence, $f(A), f(B)$ satisfy (F1) and (FF2), and so $f(A)$ is a (λ,μ)-fuzzy left ideal of $f(B)$.

Theorem 4.2. Let $f: R_1 \to R_2$ be a homomorphism of rings and $A, B \in \mathcal{F}(R_2)$.

(1) If A is a (λ,μ)-fuzzy left ideal of B then $f^{-1}(A)$ is a (λ,μ)-fuzzy left ideal of $f^{-1}(B)$.

(2) If A is a (λ,μ)-fuzzy right ideal of B then $f^{-1}(A)$ is a (λ,μ)-fuzzy right ideal of $f^{-1}(B)$.

(3) If A is a (λ,μ)-fuzzy ideal of B then $f^{-1}(A)$ is a (λ,μ)-fuzzy ideal of $f^{-1}(B)$.

Proof: We only verify (1). The proofs of (2) and (3) are similar to that of (1).
Let A be a (λ,μ)-fuzzy left ideal of B. From Definition 3.1 we get that B is a (λ,μ)-fuzzy subring of R_2 and $A\cap\overline{\mu}\subseteq B\cup\overline{\lambda}$. It follows from Theorem 2.3 (3) that $f^{-1}(B)$ is a (λ,μ)-fuzzy subring of R_2. And from $A\cap\overline{\mu}\subseteq B\cup\overline{\lambda}$ we have

$$f^{-1}(A)\cap\overline{\mu}=f^{-1}(A\cap\overline{\mu})\subseteq f^{-1}(B\cup\overline{\lambda})=f^{-1}(B)\cup\overline{\lambda}$$

Furthermore, for any $x_1,x_2\in R_2$, we get

$$f^{-1}(A)(x_1-x_2)\vee\lambda=A(f(x_1-x_2))\vee\lambda=A(f(x_1)-f(x_2))\vee\lambda$$

by A satisfying (F1)
$$\geq A(f(x_1))\wedge A(f(x_2))\wedge\mu$$
$$=f^{-1}(A)(x_1)\wedge f^{-1}(A)(x_2)\wedge\mu.$$

$$f^{-1}(A)(x_1x_2)\vee\lambda=A(f(x_1x_2))\vee\lambda=A(f(x_1)f(x_2))\vee\lambda$$

by A,B satisfying (FF2)
$$\geq A(f(x_2))\wedge B(f(x_1))\wedge\mu$$
$$=f^{-1}(A)(x_2)\wedge f^{-1}(B)(x_1)\wedge\mu.$$

Hence, $f^{-1}(A)$, $f^{-1}(B)$ satisfy (F1) and (FF2), and so $f^{-1}(A)$ is a (λ,μ)-fuzzy left ideal of $f^{-1}(B)$.

Now, we consider the $\cap$ operation of (λ,μ)-fuzzy ideals of a (λ,μ)-fuzzy subring .

Theorem 4.3. Let $A,B\in\mathcal{F}(R)$ and C,D be (λ,μ)-fuzzy subrings of R.

(1) If A is a (λ,μ)-fuzzy left ideal of C and B is a (λ,μ)-fuzzy left ideal of D, then $A\cap B$ is a (λ,μ)-fuzzy left ideal of $C\cap D$.

(2) If A is a (λ,μ)-fuzzy right ideal of C and B is a (λ,μ)-fuzzy right ideal of D, then $A\cap B$ is a (λ,μ)-fuzzy right ideal of $C\cap D$.

(3) If A is a (λ,μ)-fuzzy ideal of C and B is a (λ,μ)-fuzzy ideal of D, then $A\cap B$ is a (λ,μ)-fuzzy ideal of $C\cap D$.

Proof: We only verify (1). The proofs of (2) and (3) are similar to that of (1).

Let A be a (λ,μ)-fuzzy left ideal of C and let B be a (λ,μ)-fuzzy left ideal of D. From Definition 3.1 we have that C,D are (λ,μ)-fuzzy subrings of R and

$$A\cap\overline{\mu}\subseteq C\cup\overline{\lambda},\ B\cap\overline{\mu}\subseteq D\cup\overline{\lambda}.$$

Then from Theorem 2.2 we get that $C\bigcap D$ is a (λ,μ)-fuzzy subring of R, and

$$(A\bigcap B)\bigcap\overline{\mu}\subseteq(C\bigcap D)\bigcup\overline{\lambda}.$$

Moreover, for all $x,y\in R$, we have

$$\begin{aligned}(A\bigcap B)(x-y)\vee\lambda &= (A(x-y)\vee\lambda)\wedge(B(x-y)\vee\lambda), \text{ by } A,B \text{ satisfying (F1)}\\ &\geq A(x)\wedge A(y)\wedge B(x)\wedge B(y)\wedge\mu\\ &= (A\bigcap B)(x)\wedge(A\bigcap B)(y)\wedge\mu,\end{aligned}$$

$$\begin{aligned}(A\bigcap B)(xy)\vee\lambda &= (A(xy)\vee\lambda)\wedge(B(xy)\vee\lambda), \text{ by } A,C \text{ and } B,D \text{ satisfying (FF2)}\\ &\geq A(y)\wedge C(x)\wedge B(y)\wedge D(x)\wedge\mu\\ &= (A\bigcap B)(y)\wedge(C\bigcap D)(x)\wedge\mu.\end{aligned}$$

Hence, $A\bigcap B$, $C\bigcap D$ satisfy (F1) and (FF2), and so $A\bigcap B$ is a (λ,μ)-fuzzy left ideal of $C\bigcap D$.

The following Corollary 4.1 (3) exhibits us that (λ,μ)-fuzzy ideals of a (λ,μ)-fuzzy subring are closed under the operation $\bigcap$.

Corollary 4.1. Let $A,B\in\mathcal{F}(R)$ and C,D be (λ,μ)-fuzzy subrings of R.

(1) If A,B are (λ,μ)-fuzzy left ideal of C then so is $A\bigcap B$.

(2) If A,B are (λ,μ)-fuzzy right ideal of C then so is $A\bigcap B$.

(3) If A,B are (λ,μ)-fuzzy ideal of C then so is $A\bigcap B$.

Proof: It follows by taking $C=D$ in Theorem 4.3.

Next, we consider the operations $+$ and $\odot$ of (λ,μ)-fuzzy ideals of a (λ,μ)-fuzzy subring.

Lemma 4.1. Let $A,B\in\mathcal{F}(R)$ and C be a (λ,μ)-fuzzy subring of R s.t. $A\bigcap\overline{\mu}\subseteq C\bigcup\overline{\lambda}$, $B\bigcap\overline{\mu}\subseteq C\bigcup\overline{\lambda}$. Then

(1) $(A+B)\bigcap\overline{\mu}\subseteq C\bigcup\overline{\lambda}$. (2) $(A\odot B)\bigcap\overline{\mu}\subseteq C\bigcup\overline{\lambda}$.

Proof: (1) For all $x\in R$, if there exist $y,z\in R$ such that $x=y+z$, then

$$\begin{aligned}(C\bigcup\overline{\lambda})(x) &= C(x)\vee\lambda = C(y+z)\vee\lambda, \text{by } C \text{ satisfying (F1')}\\ &\geq (C(y)\wedge C(z)\wedge\mu)\vee\lambda\\ &\geq (C(y)\vee\lambda)\wedge(C(z)\vee\lambda)\wedge\mu,\\ &\geq A(y)\wedge\mu\wedge B(z)\wedge\mu\wedge\mu = A(y)\wedge B(z)\wedge\mu.\end{aligned}$$

So,
$$\begin{aligned}(C\bigcup\overline{\lambda})(x) &\geq \sup\{A(y)\wedge B(z)\wedge\mu \mid x=y+z\}\\ &= \sup\{A(y)\wedge B(z)\mid x=y+z\}\wedge\mu\\ &= (A+B)(x)\wedge\mu = ((A+B)\bigcap\overline{\mu})(x).\end{aligned}$$

Hence, $(A+B)\bigcap\overline{\mu}\subseteq C\bigcup\overline{\lambda}$.

(2) For all $x\in R$, if there exist $y_i,z_i\in R(1\leq i\leq n)$, such that $x=\sum_{i=1}^{n}y_iz_i$, then

$$(C\bigcup\overline{\lambda})(x)=C(x)\vee\lambda=C(\sum_{i=1}^{n}y_iz_i)\vee\lambda \text{,by } C \text{ satisfying (F1'),(F2)}$$
$$\geq\{\inf_{1\leq i\leq n}(C(y_i)\wedge C(z_i))\wedge\mu\}\vee\lambda$$
$$\geq\inf_{1\leq i\leq n}\{(C(y_i)\vee\lambda)\wedge(C(z_i)\vee\lambda)\wedge\mu\}$$
$$\geq\inf_{1\leq i\leq n}\{(A(y_i)\wedge\mu)\wedge(B(z_i)\wedge\mu)\wedge\mu\}$$
$$=\inf_{1\leq i\leq n}\{A(y_i)\wedge B(z_i)\}\wedge\mu.$$

So,

$$(C\bigcup\overline{\lambda})(x)\geq\sup\{\inf_{1\leq i\leq n}(A(y_i)\wedge B(z_i))\wedge\mu\mid x=\sum_{i=1}^{n}y_iz_i,n\in\mathbb{N}\}$$
$$=\sup\{\inf_{1\leq i\leq n}(A(y_i)\wedge B(z_i))\mid x=\sum_{i=1}^{n}y_iz_i,n\in\mathbb{N}\}\wedge\mu$$
$$=(A\odot B)(x)\wedge\mu=((A\odot B)\bigcap\overline{\mu})(x).$$

Hence, $(A\odot B)\bigcap\overline{\mu}\subseteq C\bigcup\overline{\lambda}$.

The following Theorem 4.4 (3) exhibits us that (λ,μ)-fuzzy ideals of a (λ,μ)-fuzzy subring are closed under the operation $+$.

Theorem 4.4. Let A,B,C be a (λ,μ)-fuzzy subring of R.

(1) If A,B are (λ,μ)-fuzzy left ideals of C then so is $A+B$.

(2) If A,B are (λ,μ)-fuzzy right ideals of C then so is $A+B$.

(3) If A,B are (λ,μ)-fuzzy ideals of C then so is $A+B$.

Proof: We only verify (1). The proofs of (2) and (3) are similar to that of (1).

At first, it is observed from Lemma 4.1 (1) that $(A+B)\bigcap\overline{\mu}\subseteq C\bigcup\overline{\lambda}$.

At second, for all $x,y\in R$, we get

$$(A+B)(x-y)\vee\lambda$$
$$=\sup\{A(z)\wedge B(w)\mid z+w=x-y\}\vee\lambda$$
$$\geq\sup\{A(x_1-y_1)\wedge B(x_2-y_2)\mid x_1+x_2=x,y_1+y_2=y\}\vee\lambda$$
$$=\sup\left\{(A(x_1-y_1)\vee\lambda)\wedge(B(x_2-y_2)\vee\lambda)\mid x_1+x_2=x,y_1+y_2=y\right\} \text{ by (F1)}$$
$$\geq\sup\{A(x_1)\wedge A(y_1)\wedge B(x_2)\wedge B(y_2)\wedge\mu\mid x_1+x_2=x,y_1+y_2=y\}$$
$$=(A+B)(x)\wedge(A+B)(y)\wedge\mu.$$

$$(A+B)(xy)\vee\lambda=\sup\{A(z)\wedge B(w)\mid z+w=xy\}\vee\lambda$$
$$\geq\sup\{A(xy_1)\wedge B(xy_2)\mid y_1+y_2=y\}\vee\lambda$$
$$=\sup\{(A(xy_1)\vee\lambda)\wedge(B(xy_2)\vee\lambda)\ \ y_1+y_2=y\}\text{,by } A,B,C \text{ satisfying (FF2)}$$
$$\geq\sup\{A(y_1)\wedge C(x)\wedge B(y_2)\wedge C(x)\wedge\mu\mid y_1+y_2=y\}$$
$$=\sup\{A(y_1)\wedge B(y_2)\mid y_1+y_2=y\}\wedge C(x)\wedge\mu$$

$= (A+B)(y) \wedge C(x) \wedge \mu$

So, $A+B$ and C fulfill (F1) and (FF2), and then $A+B$ is a (λ,μ)-fuzzy left ideal of C.

Theorem 4.5. Let $B, C \in \mathcal{F}(R)$ such that $B \cap \overline{\mu} \subseteq C \cup \overline{\lambda}$.

(1) If $A \in \mathcal{F}(R)$ is a (λ,μ)-fuzzy left ideal of C then so is $A \odot B$.

(2) If $A \in \mathcal{F}(R)$ is a (λ,μ)-fuzzy right ideal of C then so is $A \odot B$.

Proof: We only verify (1). The proof of (2) is similar to that of (1).

At first, it is observed from Lemma 4.1 (2) that $(A \odot B) \cap \overline{\mu} \subseteq C \cup \overline{\lambda}$.

At second, for all $x, y \in R$, we get

$(A \odot B)(xy) \vee \lambda$, by taking $\sum_{i=1}^{n} x(z_i w_i) = xy$

$\geq \sup\{\inf_{1\leq i\leq n}(A(xz_i) \wedge B(w_i)) \mid xy = \sum_{i=1}^{n} x(z_i w_i), n \in \mathbb{N}\} \vee \lambda$

$\geq \sup\{\inf_{1\leq i\leq n}((A(xz_i) \vee \lambda) \wedge B(w_i)) \mid xy = \sum_{i=1}^{n} x(z_i w_i), n \in \mathbb{N}\}$, by A, C satisfying (FF2)

$\geq \sup\{\inf_{1\leq i\leq n}(A(z_i) \wedge B(w_i)) \mid y = \sum_{i=1}^{n} z_i w_i, n \in \mathbb{N}\} \wedge C(x) \wedge \mu$

$= (A \odot B)(y) \wedge C(x) \wedge \mu$.

$(A \odot B)(x-y) \vee \lambda$, by taking $\sum_{i=1}^{n} x_i z_i + \sum_{j=1}^{m} (-y_j) w_j = x - y$

$$\geq \sup\{\inf_{1\leq i\leq n}(A(x_i) \wedge B(z_i)) \wedge \inf_{1\leq i\leq n}(A(-y_i) \wedge B(w_j)) \mid x = \sum_{i=1}^{n} x_i z_i, y = \sum_{j=1}^{m} y_j w_j, n, m \in \mathbb{N}\} \vee \lambda$$

$$\geq \sup\{\inf_{1\leq i\leq n}(A(x_i) \wedge B(z_i)) \mid x = \sum_{i=1}^{m} x_i z_i, n \in \mathbb{N}\} \wedge \sup\{[\inf_{1\leq i\leq n}(A(-y_j) \wedge B(w_j))] \vee \lambda \mid y = \sum_{j=1}^{m} y_j w_j, m \in \mathbb{N}\}$$

$$\geq (A \odot B)(x) \wedge \sup\{\inf_{1\leq i\leq n}((A(-y_j) \vee \lambda) \wedge B(w_j)) \mid y = \sum_{j=1}^{m} y_j w_j, m \in \mathbb{N}\}$$

by A satisfying (F2")

$$\geq (A \odot B)(x) \wedge \sup\{\inf_{1\leq i\leq n}(A(y_j) \wedge \mu \wedge B(w_j)) \mid y = \sum_{j=1}^{m} y_j w_j, m \in \mathbb{N}\}$$

$= (A \odot B)(x) \wedge (A \odot B)(y) \wedge \mu$, Hence, $A \odot B$ and C fulfill (F1) and (FF2), and so $(A \odot B)$ is a (λ, μ)-fuzzy left ideal of C.

The following Corollary 4.2 (2) tells us that (λ, μ)-fuzzy ideals of a (λ, μ)-fuzzy subring are closed under the operation $\odot$.

Corollary 4.2. Let $A, B, C \in \mathcal{F}(R)$.

(1) If A is a (λ, μ)-fuzzy left ideal of C and B is a (λ, μ)-fuzzy right ideal of C, then $(A \odot B)$ is a (λ, μ)-fuzzy ideal of C.

(2) If A, B are (λ, μ)-fuzzy ideal of C then so is $(A \odot B)$.

Proof: (1) Since B is a (λ, μ)-fuzzy right ideal of C, then we have $B \cap \overline{\mu} \subseteq C \cap \overline{\lambda}$. It follows by Theorem 10 (1) that $(A \odot B)$ is a (λ, μ)-fuzzy left ideal of C. Similar, we can prove that $(A \odot B)$ is also a (λ, μ)-fuzzy right ideal of C. Therefore, $(A \odot B)$ is a (λ, μ)-fuzzy ideal of C.

(2) It is a straightforward conclusion of (1).

5. Conclusions

In this paper, we developed a theory of (λ, μ)-fuzzy ideal of a (λ, μ)-fuzzy subring of a ring R. This theory is a natural extension of that of (λ, μ)-fuzzy ideal of a ring R. We proved that the considered notion has nice level characterizations and algebraic properties. In the future work, we will discuss the (λ, μ)-fuzzy quotient subring of a (λ, μ)-fuzzy subring concerning a (λ, μ)-fuzzy ideal, and then establish some isomorphism theorems about (λ, μ)-fuzzy quotient subring.

Acknowledgement. This work is supported by National Natural Science Foundation of China (11801248, 11471152). We are also thankful to the reviewers for their valuable comments.

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