Fuzzy Congruences on MS-algebras

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Abstract. In this paper, we study the fuzzy congruence relation of MS-algebra L and the fuzzy congruence relation generated by a given fuzzy relation on L. We also investigate some properties of the fuzzy congruence relation generated by a given fuzzy relation on L.

Keywords: MS-algebras, fuzzy congruences, fuzzy ideals.

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1. Introduction
In the papers (in particular [7,5,8,1]) have investigated the properties of fuzzy equivalence (congruence) relations of algebras. In particular Yuan and Wangming [9] investigated the relationship between fuzzy ideals and fuzzy congruences on a distributive lattice L. In this paper, we discuss the fuzzy congruence relations in MS-algebras and we study the properties a fuzzy congruence relation generated by μ × μ on L.

2. Preliminaries
In this section, we recall some definitions and basic results on MS-algebras.

Definition 2.1. [3] An MS-algebra is an algebra \((L,∨,∧,0,1)\) of type \((2,2,1,0,0)\), such that \((L,∨,∧,0,1)\) is a bounded distributive lattice and \(a → a^*\) is a unary operation satisfies: 
\[ a \leq a^*, \ (a \land b)^* = a^* \lor b^* \text{ and } 1^* = 0. \]

Lemma 2.2. [3] For any two elements \(a, b\) of an MS-algebra \(L\), we have the following:
1. \(0^* = 1\),
2. \(a \leq b \Rightarrow b^* \leq a^*\),
3. \(a^{***} = a^*\),
4. \((a \lor b)^* = a^* \land b^*\),
5. \((a \lor b)^{**} = a^{**} \lor b^{**}\),
6. \((a \land b)^{**} = a^{**} \land b^{**}\).
Definition 2.3. [4] Let \( L \) be a lattice and let \( H \subseteq L \times L \). We denote by \( \theta(H) \) the smallest congruence relation such that \( a \equiv b \) for all \( (a, b) \in H \), and call it the congruence relation generated by \( H \). If \( H = I \times I \), where \( I \) is an ideal, we write \( \theta[I] \) for all \( \theta(H) \).

Definition 2.4. [2] An equivalence relation \( \theta \) is a congruence relation in MS-algebra \( L \), if it is a lattice congruence and \( (a, b) \in \theta \) implies \((a^*, b^*) \in \theta \) for all \( a, b \in L \).

If we delete the operation \( ^* \), we shall speak the lattice congruence. To distinguish these two types, we shall use the subscript \( \text{'lat'} \) to denote lattice congruence.

Let \( I \) be an ideal of the MS-algebra \( L \). Define \( I_{\text{lat}} = \{ x \in L : i^* \leq x, \text{ for some } i \in I \} \).

Theorem 2.5. [3] Let \( I \) be an ideal of the MS-algebra \( L \). Then \( \theta[I] = \theta[\text{lat}[I_{\text{lat}}]] \vee \theta[\text{lat}[I_{\text{lat}}]] \).

We recall that for any nonempty set \( L \), the characteristic function of \( L \) defined as
\[
\chi_L(x) = \begin{cases} 
1 & \text{if } x \in L, \\
0 & \text{if } x \notin L.
\end{cases}
\]

Let \( \mu \) be a fuzzy subset of \( L \). For any \( \alpha \in [0,1] \), we shall denote the level subset \( \mu^{-1}([\alpha, 1]) \) by simply \( \mu_\alpha \), i.e. \( \mu_\alpha = \{ x \in L : \alpha \leq \mu(x) \} \).

Theorem 2.6. [8] Let \( \mu \) be a fuzzy subset of \( L \). Then \( \mu \) is a fuzzy ideal of \( L \) if and only if any one of the following conditions is satisfied:
1. \( \mu(0) = 1 \) and \( \mu(x \lor y) = \mu(x) \land \mu(y) \) for all \( x, y \in L \),
2. \( \mu(0) = 1 \) and \( \mu(x \lor y) \geq \mu(x) \land \mu(y) \) and \( \mu(x \land y) \geq \mu(x) \lor \mu(y) \) for all \( x, y \in L \).

A fuzzy relations on a set \( X \) are maps \( \theta : X \times X \to [0,1] \). For any \( x, y \in X \) and fuzzy relations \( \theta \land \phi \) and \( \theta \lor \phi \),
\[
(\theta \land \phi)(x, y) = \min\{\theta(x, y), \phi(x, y)\},
(\theta \lor \phi)(x, y) = \max\{\theta(x, y), \phi(x, y)\}.
\]

Definition 2.7. [5] Suppose that \( \theta \) and \( \phi \) are two fuzzy relations on a set \( X \). Then \( (\theta \circ \phi)(x, y) = \sup_{z \in X}((\theta(x, z) \land (\phi(z, y))).

Definition 2.8. [5] A fuzzy relation \( \phi \) on \( X \) is said two be a fuzzy equivalence relation on \( X \) if
1. \( \phi(x, x) = 1 \) for all \( x \in X \) (reflexive),
2. \( \phi(x, y) = \phi(y, x) \) for all \( x, y \in L \) (symmetric),
3. \( \phi(x, z) \geq \phi(x, y) \land \phi(y, z) \) for all \( x, y, z \in L \) (transitive).

Throughout the next sections, \( L \) stands for MS-algebra.

3. Fuzzy congruences on MS-algebras
In this section, we give various characterization for fuzzy congruences on MS-algebra \( L \).

Definition 3.1. A fuzzy equivalence relation \( \phi \) on an MS-algebra \( L \) is called a fuzzy congruence relation on \( L \) if the following are satisfied:
1. \( \phi(x \land z, y \land w) \land \phi(x \lor z, y \lor w) \geq \phi(x, y) \land \phi(z, w) \) for all \( x, y, z, w \in L \),
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2. \( \phi(x^+, y^+) \geq \phi(x, y) \) for all \( x, y \in L \).

Example 3.2. Consider the MS-algebra \( L \) described as in following figure

![Figure 1]

Define a fuzzy relation \( \phi \) on \( L \) as follows: \( \phi(0,0) = \phi(1,1) = \phi(x, x) = \phi(y, y) = 1 \), \( \phi(x, y) = \phi(y, x) = \phi(0, y) = \phi(y, 0) = \phi(x, 0) = \phi(0, x) = 0.6 \). Then it can be easily verify that \( \phi \) is a fuzzy congruence relation on \( L \).

Theorem 3.3. A fuzzy equivalence relation \( \phi \) is fuzzy congruence on \( L \) if and only if

\[
\phi(x, y) \leq \phi(x \land z, y \land z) \land \phi(x \lor z, y \lor z) \land \phi(x^+, y^+) \quad \text{for all} \quad x, y, z \in L.
\]

Proof: The forward proof is straightforward. Conversely, suppose that \( \phi \) is a fuzzy equivalence relation on \( L \) that satisfies \( \phi(x, y) \leq \phi(x \land z, y \land z) \land \phi(x \lor z, y \lor z) \land \phi(x^+, y^+) \) for all \( x, y, z \in L \). This implies \( \phi(x, y) \leq \phi(x \land z, y \land z) \land \phi(x \lor z, y \lor z) \) and \( \phi(x, y) \leq \phi(x^+, y^+) \) for all \( x, y, z \in L \).

\[
\phi(x, y) \land \phi(z, w) \leq \phi(x \land z, y \land z) \land \phi(y \land z, y \land w) \leq \phi(x \land z, y \land w) \quad \text{for all} \quad x, y, z, w \in L.
\]

Similarly \( \phi(x, y) \land \phi(z, w) \leq \phi(x \lor z, y \lor w) \) for all \( x, y, z, w \in L \). Thus \( \phi \) is a fuzzy congruence relation on \( L \).

Theorem 3.4. A relation \( \phi \) on \( L \) is a fuzzy congruence on \( L \) if and only if every level subset \( \phi_\alpha \) of \( \phi \) at \( \alpha \in [0,1] \) is congruence relation on \( L \).

Theorem 3.5. A congruence relation \( \phi \) is a congruence relation on \( L \) if and only if its characteristic function \( \chi_\phi \) is a fuzzy congruence on \( L \).

Theorem 3.6. If \( \{ \phi_i: i \in A \} \) is a family of fuzzy congruence of \( L \), then \( \bigcap_{i \in A} \phi_i \) is a fuzzy congruence on \( L \).

We denoted that the set of all fuzzy congruences of \( L \) by \( \mathcal{F}(L) \) and the set of all congruences of \( L \) by \( \mathcal{C}(L) \). \( \omega = \{(x, y) \in L \times L: x = y\} \) is the smallest and \( \iota = L \times L \) is the largest element of \( \mathcal{C}(L) \).

\[
\chi_\omega(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \omega \\ 0 & \text{if otherwise} \end{cases}
\]

for all \( x, y \in L \) is the smallest and \( \chi_\iota(x, y) = 1 \) for all \( x, y \in L \) is the largest elements of \( \mathcal{F}(L) \).

Definition 3.7. Let \( \phi \) and \( \varphi \) be any two fuzzy congruence relation of \( L \). Then define \( \phi \lor \varphi = \bigcap \{ \theta \in \mathcal{F}(L): \phi \leq \theta \land \varphi \leq \theta \} \), i.e \( \phi \lor \varphi \) is the fuzzy congruence generated by \( \phi \lor \varphi \).
Theorem 3.8. \((\mathbb{FC}(L), \sqsubseteq)\) is complete lattice.

Proof: We note that both fuzzy congruences relations \(\chi_0\) and \(\chi_1\) are the least and the greatest elements of \(\mathbb{FC}(L)\), respectively. Clearly \((\mathbb{FC}(L), \sqsubseteq)\) is poset and \(\wedge\) is lower bound of any family \(\{\phi_i : i \in \Delta\}\) of fuzzy congruences of \(L\). Let \(\emptyset\) be any a lower bound of \(\{\phi_i : i \in \Delta\}\). Then \(\emptyset \sqsubseteq \phi_i\) for all \(i \in \Delta\) and so \(\emptyset \subseteq \bigwedge_{i \in \Delta} \phi_i\). This implies \(\bigwedge_{i \in \Delta} \phi_i\) is a greatest lower bound of \(\{\phi_i : i \in \Delta\}\). Hence \((\mathbb{FC}(L), \sqsubseteq)\) is a complete lattice.

Theorem 3.9. Let \(\phi\) and \(\varphi\) be any fuzzy congruence relations on an MS-algebra \(L\). Then \(\phi \lor \varphi = \bigcup_{n=1}^{\infty} \Theta_n\), where \(\Theta_1 = \phi \circ \varphi \circ \varphi\), \(\Theta_2 = \phi \circ \varphi \circ \varphi \circ \phi\), \(\Theta_3 = \phi \circ \varphi \circ \varphi \circ \varphi \circ \phi \circ \varphi \circ \varphi\). We note that both fuzzy congruence relations \(\Theta_1\) and \(\Theta_2\) are the least and the greatest elements of \(\mathbb{FC}(L)\), respectively. Clearly \((\mathbb{FC}(L), \sqsubseteq)\) is poset and \(\wedge\) is lower bound of any family \(\{\phi_i : i \in \Delta\}\) of fuzzy congruences of \(L\). Let \(\emptyset\) be any a lower bound of \(\{\phi_i : i \in \Delta\}\). Then \(\emptyset \sqsubseteq \phi_i\) for all \(i \in \Delta\) and so \(\emptyset \subseteq \bigwedge_{i \in \Delta} \phi_i\). This implies \(\bigwedge_{i \in \Delta} \phi_i\) is a greatest lower bound of \(\{\phi_i : i \in \Delta\}\). Hence \((\mathbb{FC}(L), \sqsubseteq)\) is a complete lattice.
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Definition 3.10. Let \( L \) be an MS-algebra. The smallest fuzzy congruence generated by the fuzzy relation \( \phi \) on \( L \) is defined by \( \overline{\Theta}(\phi) = \cap \{ \theta \in FC(L) : \phi \subseteq \theta \} \). If \( \phi = \mu \times \mu \) is the product of fuzzy ideal \( \mu \) by itself, where \( \mu \times \mu(x, y) = \mu(x) \land \mu(y) \) for all \( (x, y) \in L \times L \). We write \( \overline{\Theta}[\mu] \) instead of \( \overline{\Theta}(\phi) \).

Theorem 3.11. Let \( \phi \) be a fuzzy relation of \( L \). Then \( \overline{\Theta}(\phi)(x, y) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} \) for any \( (x, y) \in L \times L \) and \( \alpha \in [0, 1] \).

Proof: Let \( \phi(x, y) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} \) for any \( (x, y) \in L \times L \) and \( \alpha \in [0, 1] \). We see that \( \phi = \overline{\Theta}(\phi) \). First we prove that \( \phi \) is a fuzzy congruence of \( L \).

1. \( \phi(x, y) = \sup\{\alpha : (x, x) \in \Theta(\phi_\alpha)\} = 1 \).
2. \( \phi(x, y) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} = \sup\{\alpha : (y, x) \in \Theta(\phi_\alpha)\} = \phi(y, x) \).
3. \( \phi(x, y) \land \phi(y, z) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} \land \sup\{\lambda : (y, z) \in \Theta(\phi_\lambda)\} = \sup\{\alpha \land \lambda : (x, y) \in \Theta(\phi_{\alpha \land \lambda}), (y, z) \in \Theta(\phi_{\alpha \land \lambda})\} \subseteq \sup\{\alpha \lor \lambda : (x, z) \in \Theta(\phi_{\alpha \lor \lambda})\} = \phi(x, z) \).
4. \( \phi(x, y) \land \phi(w, z) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} \land \sup\{\lambda : (w, z) \in \Theta(\phi_\lambda)\} = \sup\{\alpha \lor \lambda : (x, y) \in \Theta(\phi_{\alpha \lor \lambda}), (w, z) \in \Theta(\phi_{\alpha \lor \lambda})\} \subseteq \sup\{\alpha \land \lambda : (x \land w, y \land z) \in \Theta(\phi_{\alpha \land \lambda})\} = \phi(x \land w, y \land z) \).

Similarly \( \phi(x, y) \land \phi(y, z) \leq \phi(x \lor w, y \lor z) \).
5. \( \phi(x, y) = \sup\{\alpha : (x, y) \in \Theta(\phi_\alpha)\} \leq \sup\{\alpha : (x^*, y^*) \in \Theta(\phi_\alpha)\} = \phi(x^*, y^*) \). Thus \( \phi \) is a fuzzy congruence of MS-algebra \( L \). Next, we prove that \( \phi \subseteq \phi \). Now \( \phi(x, y) = \{\alpha : (x, y) \in \phi_\alpha\} \leq \{\alpha : (x, y) \in \Theta(\phi_\alpha)\} = \phi(x, y) \). Hence \( \phi \subseteq \phi \).

Finally, let \( \tau \) be any fuzzy congruence of an MS-algebra \( L \) such that \( \phi \subseteq \tau \). We see that \( \phi \subseteq \tau \).

If \( \phi \subseteq \tau \), then \( \phi_\alpha \subseteq \tau_\alpha \) and so \( \Theta(\phi_\alpha) \subseteq \Theta(\tau_\alpha) = \tau_\alpha \). Now \( \phi(x, y) = \{\alpha : (x, y) \in \phi_\alpha\} \leq \{\alpha : (x, y) \in \Theta(\phi_\alpha)\} = \phi(x, y) \). Thus \( \phi = \overline{\Theta}(\phi) \).

4. Fuzzy ideals and congruences of MS-algebras

In this section, we study the relation of fuzzy ideals and fuzzy congruences of MS-algebras.

Let \( \mu \) be a fuzzy ideal of an MS-algebra \( L \). Define \( \mu^*(x) = \sup\{\mu(i) : i^* \leq x, i \in L\} \) for all \( x \in L \) and \( \mu_{\sup}(x) = \sup\{\mu(i) : x \leq i^*\} \). Easily verify that \( \mu^* \) is a fuzzy filter of \( L \) and \( \mu_{\sup} \) is a fuzzy ideal of \( L \).

The following is due to Yuan Bo and Wu Wangming [9].

Definition 4.1. Let \( \mu \) be a fuzzy ideal of a distributive lattice \( L \). A fuzzy relation \( \Theta_{\sup}[\mu] \) on \( L \) defined by setting \( \Theta_{\sup}[\mu](x, y) = \{\mu(i) : x \lor i = y \lor i, i \in L\} \) for all \( x, y \in L \) is called the fuzzy relation induced by \( \mu \).

It can be easily verify that it is the smallest lattice fuzzy congruence containing the product fuzzy ideals \( \mu \times \mu \) of \( L \times L \). Using this definition the smallest lattice fuzzy congruence contains \( \mu_{\sup} \times \mu_{\sup} \) defined as \( \Theta[\mu_{\sup}] \) on \( L \) by setting \( \Theta_{\sup}[\mu_{\sup}](x, y) = \{\mu_{\sup}(i) : x \lor i = y \lor i, i \in L\} \) for all \( x, y \in L \).
Lemma 4.2. Let $\mu$ be a fuzzy filter of a distributive lattice $L$. Let us define fuzzy relation $\Theta_{\text{lat}}[\mu]$ on $L$ by, $\Theta_{\text{lat}}[\mu](x,y) = \{\mu(i) : x \land i = y \land i, \: i \in L\}$ for all $x,y \in L$. Then $\Theta_{\text{lat}}[\mu]$ is the smallest fuzzy congruence on $L$ containing the product of fuzzy filter $\mu \times \mu$ of $L \times L$.

By Lemma 4.2, let $\mu$ be a fuzzy ideal of an MS-algebra $L$. The smallest fuzzy relation containing the product of fuzzy filter $\mu \times \mu$ is $\Theta[\mu^2](x,y) = \{\mu^2(i) : x \land i = y \land i, \: i \in L\}$ for all $x,y \in L$. Then $\Theta[\mu^2]$ is the smallest lattice fuzzy congruence containing the product of fuzzy filter $\mu \times \mu$.

Lemma 4.3. Let $\mu$ be a fuzzy ideal of an MS-algebra $L$. Then the following conditions are hold.

1. $\Theta_{\text{lat}}[\mu^2](x,y) \leq \Theta_{\text{lat}}[\mu_\prec](x^*,y^*)$ for all $x,y \in L$
2. $\Theta_{\text{lat}}[\mu^2](x,y) \leq \Theta_{\text{lat}}[\mu\succ](x^*,y^*)$ for all $x,y \in L$

Proof: (1) $\Theta_{\text{lat}}[\mu^2](x,y) = \sup\{(\mu^2)(i) : x \land i = y \land i, \: i,j \in L\} = \sup\{(\mu)(j), \: j \leq i : x \land i = y \land i, \: i,j \in L\} \leq \sup\{(\mu)(j), \: i \leq \sup^* : x^* \lor i^* = y^* \lor i^*, \: i,j \in L\}$

Similarly (2) holds.

Theorem 4.4. For every fuzzy ideal $\mu$ of an MS-algebra $L$. Then $\Theta[\mu] = \Theta_{\text{lat}}[\mu^2] \lor \Theta[\mu\prec]$.

Proof: Let $\varphi = \Theta_{\text{lat}}[\mu^2] \lor \Theta[\mu\prec]$. We prove that $\varphi$ is the smallest fuzzy congruence containing $\mu \times \mu$. Easily verify that $\mu \times \mu \subseteq \varphi$. Now we prove that $\varphi$ is a fuzzy congruence of $L$. Since $\Theta_{\text{lat}}[\mu^2]$ and $\Theta[\mu\prec]$ are lattice fuzzy congruences, $\varphi$ is a lattice fuzzy congruence of $L$.

By Theorem 3.9 and Lemma 4.3, $\varphi(x,y) = \sup_{z_1,z_2,\ldots,z_n}(\Theta_{\text{lat}}[\mu^2](x,z_1) \land \Theta_{\text{lat}}[\mu\prec](z_1,z_2) \land \Theta_{\text{lat}}[\mu^2](z_2,z_3) \land \ldots \land \Theta_{\text{lat}}[\mu^2](z_{2n},y)) \leq \sup_{z_1,z_2,\ldots,z_{2n}}(\Theta_{\text{lat}}[\mu\prec](x^*,z_1) \land \Theta_{\text{lat}}[\mu^2](z_1,z_2) \land \Theta[\mu\prec](z_2,z_3) \land \Theta_{\text{lat}}[\mu^2](z_3,z_4) \land \ldots \land \Theta_{\text{lat}}[\mu\prec](z_{2n},y^*)) = \varphi(x^*,y^*)$.

Thus $\varphi$ is fuzzy congruence of MS-algebra $L$. Finally, we see that $\varphi$ is the smallest fuzzy congruence containing $\mu \times \mu$. Since $\varphi$ is a fuzzy congruence containing $\mu \times \mu$, $\Theta[\mu] \subseteq \varphi$.

By Theorem 2.5 and Theorem 3.12, $\Theta_{\text{lat}}[\mu\prec](x,y) = \sup\{a : (x,y) \in \Theta[\mu\prec]\} \leq \sup\{a : (x,y) \in \Theta[\mu]\} = \Theta[\mu](x,y)$.

Similarly $\Theta_{\text{lat}}[\mu^2] \subseteq \Theta[\mu]$. Thus $\Theta[\mu^2] \lor \Theta[\mu\prec] \subseteq \Theta[\mu]$. Hence $\varphi \subseteq \Theta[\mu]$.

Therefore $\varphi = \Theta[\mu]$.

The following Theorem is other useful description of $\Theta[\mu]$.

Theorem 4.5. For every fuzzy ideal $\mu$ of an MS-algebra $L$, $\Theta[\mu](x,y) = \sup\{\mu_\prec(i) \land \mu^2(j) : (x \lor i) \land j = (y \lor i) \land j, \: i,j \in L\}$ for any $x,y \in L$.

Proof: Put $\varphi(x,y) = \sup\{\mu_\prec(i) \land \mu^2(j) : (x \lor i) \land j = (y \lor i) \land j, \: i,j \in L\}$ for any $x,y \in L$. We show that $\varphi$ is fuzzy congruence of lattice $L$. For any $x,y,z,i,j,a,b,c,d \in L$
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1. \( \varphi(x, x) = \text{sup}\{\mu_{\circ}(i) \land \mu_{\circ}^2(j); (x \lor i) \land j = (x \lor i) \land j, \ i, j \in L\} \geq \mu_{\circ}(0) \land \mu_{\circ}^2(1) = 1. \) Since in particular \((x \lor 0) \land 1 = (x \lor 0) \land 1. \) Hence \( \varphi(x, x) = 1 \) for any \( x \in L. \)

2. \( \varphi(x, y) = \text{sup}\{\mu_{\circ}(i) \land \mu_{\circ}^2(j); (x \lor i) \land j = (x \lor i) \land j, \ i, j \in L\} = \text{sup}\{\mu_{\circ}(i) \land \mu_{\circ}^2(j); (y \lor i) \land j = (x \lor i) \land j, \ i, j \in L\} = \varphi(y, x) \)

3. If \((x \lor a) \land b = (y \lor a) \land b, (y \lor c) \land d = (z \lor c) \land d, \) then
   \[(x \lor (a \lor c)) \land (b \lor d) = \{(x \lor a) \lor (b \lor d)\} \land (c \lor d) = \{\{x \lor a\} \lor \{b \lor d\}\} \land (c \lor d) = \{(x \lor a) \lor (b \lor d)\} = \{(z \lor c) \lor (d \lor d)\} \land (c \lor d) = \{(z \lor c) \lor (b \lor d)\} \land (c \lor d) = \{(z \lor c) \lor (d \lor d)\} \land (c \lor d) = \{(z \lor c) \lor (d \lor d)\} \land (c \lor d) \land (d \lor d) = \{\{z \lor c\} \lor \{d \lor d\}\} \land (c \lor d) \land (d \lor d) = \{(z \lor c) \lor (d \lor d)\} \land (c \lor d) \land (d \lor d) \] \( \varphi(x, y) \land \varphi(y, z) = \text{sup}\{\mu_{\circ}(a) \land \mu_{\circ}^2(b); (x \lor a) \land b = (y \lor a) \land b, a, b \in L\} = \text{sup}\{\mu_{\circ}(a) \land \mu_{\circ}^2(b); (x \lor a) \land b = (y \lor a) \land b, a, b \in L\} \]

4. \( \varphi(x, y) \land \varphi(y, z) = \text{sup}\{\mu_{\circ}(a) \land \mu_{\circ}^2(b); (x \lor a) \land b = (y \lor a) \land b, a, b \in L\} \land \text{sup}\{\mu_{\circ}(c) \land \mu_{\circ}^2(d); (y \lor c) \land d = (z \lor c) \land d, c, d \in L\} = \text{sup}\{\mu_{\circ}(a) \land \mu_{\circ}^2(b); (x \lor a) \land b = (y \lor a) \land b, a, b \in L\} \land \text{sup}\{\mu_{\circ}(c) \land \mu_{\circ}^2(d); (y \lor c) \land d = (z \lor c) \land d, c, d \in L\} \] \( \varphi(x, y) \land \varphi(y, z) = \text{sup}\{\mu_{\circ}(a) \land \mu_{\circ}^2(b); (x \lor a) \land b = (y \lor a) \land b, a, b \in L\} \land \text{sup}\{\mu_{\circ}(c) \land \mu_{\circ}^2(d); (y \lor c) \land d = (z \lor c) \land d, c, d \in L\} = \text{sup}\{\mu_{\circ}(a) \land \mu_{\circ}^2(b); (x \lor a) \land b = (y \lor a) \land b, a, b \in L\} \land \text{sup}\{\mu_{\circ}(c) \land \mu_{\circ}^2(d); (y \lor c) \land d = (z \lor c) \land d, c, d \in L\} \]

5. \( \varphi(x, y) \land \varphi(y, z) \leq \varphi(x, z). \)

6. \( \text{after routine work we have got, } ((x \land z) \lor (a \lor c)) \land (b \lor d) = ((y \land w) \lor (a \lor c)) \land (b \lor d) \)
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